MathCastle

Trigonometry

Complete course booklet • 15 topics

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1 Core Ideas in Plain Terms

The big trig ideas, in everyday language

1.1 What the Trig Ratios Really Are

Sine, cosine, tangent are just fractions

In a right triangle, pick one of the non-right angles. **Sine** is the side across from it over the longest side; **cosine** is the side next to it over the longest side; **tangent** is across over next-to. That is all the words "sin, cos, tan" mean — three particular fractions of the sides. "SOH-CAH-TOA" is just a way to remember which fraction is which. As formulas: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.

(diagram in the online version)

A 3-4-5 triangle

Take the angle whose opposite side is 3 and adjacent side is 4 (hypotenuse 5). Then $\sin = \frac{3}{5} = 0.6$, $\cos = \frac{4}{5} = 0.8$, and $\tan = \frac{3}{4} = 0.75$. Nothing mysterious — just three ratios of the sides.

(diagram in the online version)

1.2 The Unit Circle, Plainly

A circle of radius 1 that stores every angle

Draw a circle of radius 1 centered at the origin and spin a point around it. For an angle θ, the point lands at coordinates $(\cos\theta, \sin\theta)$ — so the cosine is just the x-coordinate and the sine is the y-coordinate. That is why sine and cosine never leave the range $[-1, 1]$: the point can't go past the circle.

(diagram in the online version)

The angle 90°

At 90° the point is straight up at $(0, 1)$. So $\cos 90° = 0$ (the x-coordinate) and $\sin 90° = 1$ (the y-coordinate).

1.3 Radians and Wave Graphs

Radians measure angles by arc length

Instead of 360° for a full turn, radians use 2π — the distance around a radius-1 circle. So

$180° = \pi$, $90° = \frac{\pi}{2}$. To convert, multiply degrees by $\frac{\pi}{180}$.

Reading a sine graph

For $y = a\sin(bx)$, the **amplitude** $|a|$ is how tall the wave gets, and the **period** $\frac{2\pi}{b}$ is how wide one full wave is before it repeats. Bigger a = taller; bigger b = more scrunched together.

(diagram in the online version)

y = 3 sin(2x)

Amplitude 3 (peaks at 3, troughs at -3) and period $\frac{2\pi}{2} = \pi$ (one full wave every π units).

1.4 Going Deeper: Identities and the Law of Sines

Why $\sin^2\theta + \cos^2\theta = 1$

The unit-circle point $(\cos\theta, \sin\theta)$ sits exactly 1 away from the origin, and distance is $\sqrt{x^2 + y^2}$. Squaring that distance gives $\cos^2\theta + \sin^2\theta = 1$ — the Pythagorean identity is just the Pythagorean theorem on the unit circle. Divide the whole thing by $\cos^2\theta$ and you get the companion identity $1 + \tan^2\theta = \sec^2\theta$ for free.

(diagram in the online version)

The Law of Sines, plainly

In *any* triangle, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$: the bigger an angle, the bigger the side across from it, in exact proportion. It is the tool for triangles with no right angle, whenever you know an angle and its opposite side.

Using the identity

Given $\sin\theta = \frac{3}{5}$ with θ in Quadrant I: $\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$, so $\cos\theta = \frac{4}{5}$ and $\tan\theta = \dfrac{3/5}{4/5} = \frac{3}{4}$ — no triangle drawing needed.

Solving a right triangle (multi-step)

A right triangle has legs 5 and 12. Step 1 — **Pythagorean theorem**: the hypotenuse is $\sqrt{5^2 + 12^2} = \sqrt{169} = 13$. Step 2 — for the angle θ opposite the side 5: $\sin\theta = \frac{5}{13}$. Step 3 — and $\tan\theta = \frac{5}{12}$, straight from the ratio definitions.

1.5 Problem-Solving Playbook

Draw, label, choose the tool

Sketch the triangle and label everything you know. Then pick the tool the givens point to: a **right angle** → SOH-CAH-TOA; an **angle paired with its opposite side** → Law of Sines; **two sides and the included angle** → Law of Cosines ($c^2 = a^2 + b^2 - 2ab\cos C$).

Worked: two sides and the included angle

$a = 5$, $b = 8$, $C = 60°$. The givens are side–angle–side, so use the **Law of Cosines**: $c^2 = 25 + 64 - 2 \cdot 5 \cdot 8 \cdot \cos 60° = 89 - 80 \cdot \frac{1}{2} = 49$, so $c = 7$.

2 Law of Sines and Law of Cosines

Solving oblique triangles: AAS, ASA, SSA, SAS, SSS, area, and applications

2.1 The Law of Sines (AAS and ASA)

Rounding convention (used throughout). Unless told otherwise, round *angles* to the nearest tenth of a degree and *side lengths / areas* to the nearest hundredth. Carry extra digits in intermediate steps and round only at the end.

Two sides and the included angle.

(diagram in the online version)

Every triangle here is *oblique* (no right angle guaranteed). We name the three angles A, B, C and put the side opposite each angle in the matching lowercase letter: side a is opposite angle A, and so on.

Law of Sines

For any triangle ABC,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

When to use it: you know an *angle and the side opposite it*, plus one more piece.

- **AAS / ASA** — two angles and any one side. Find the third angle by $A + B + C = 180°$, then use a ratio.
- **SSA** — two sides and an angle opposite one of them (the *ambiguous case*, next section).

AAS: given $A = 40°$, $B = 60°$, $a = 10$

Third angle: $C = 180° - 40° - 60° = 80°$.

$$b = \frac{a \sin B}{\sin A} = \frac{10 \sin 60°}{\sin 40°} \approx 13.47, \qquad c = \frac{a \sin C}{\sin A} = \frac{10 \sin 80°}{\sin 40°} \approx 15.32.$$

ASA: given $A = 45°$, $B = 105°$, included side $c = 20$

Third angle: $C = 180° - 45° - 105° = 30°$. Now c is opposite the known angle C, so

$$a = \frac{c \sin A}{\sin C} = \frac{20 \sin 45°}{\sin 30°} \approx 28.28, \qquad b = \frac{c \sin B}{\sin C} = \frac{20 \sin 105°}{\sin 30°} \approx 38.64.$$

Pick-the-law tip. If a problem gives you a *complete angle–opposite-side pair* (an angle and its opposite side), reach for the **Law of Sines**. If it does not, you will need the Law of Cosines.

2.2 The Ambiguous Case (SSA)

SSA: how many triangles?

Given angle A, its opposite side a, and an adjacent side b. Let the "altitude" be $h = b \sin A$. Compare a with h and with b:

- If A is **acute**: $a < h \Rightarrow$ **0 triangles**; $a = h \Rightarrow$ **1** (right) triangle; $h < a < b \Rightarrow$ **2 triangles**; $a \geq b \Rightarrow$ **1 triangle**.
- If A is **obtuse**: $a \leq b \Rightarrow$ **0 triangles**; $a > b \Rightarrow$ **1 triangle**.

To solve: compute $\sin B = \dfrac{b \sin A}{a}$. The acute value B_1 and its supplement $B_2 = 180° - B_1$ are both candidates; keep B_2 only if $A + B_2 < 180°$.

Two sides and the included angle.

(diagram in the online version)

When $h < a < b$ the side of length a can reach the base line at *two* points B_1 and B_2, giving two different triangles.

Full SSA with two triangles: $a = 8$, $b = 10$, $A = 40°$

Here A is acute and $h = b \sin A = 10 \sin 40° \approx 6.43$. Since $h < a < b$ ($6.43 < 8 < 10$), there are **two triangles**.

$$\sin B = \frac{b \sin A}{a} = \frac{10 \sin 40°}{8} \approx 0.8035 \;\Rightarrow\; B_1 \approx 53.5° \;\text{ or }\; B_2 = 180° - 53.5° = 126.5°.$$

Both keep the angle sum below 180°, so both survive.

Triangle 1: $B_1 \approx 53.5°,\ C_1 = 180° - 40° - 53.5° \approx 86.5°$,

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{8 \sin 86.5°}{\sin 40°} \approx 12.42.$$

Triangle 2: $B_2 \approx 126.5°,\ C_2 = 180° - 40° - 126.5° \approx 13.5°$,

$$c_2 = \frac{a \sin C_2}{\sin A} = \frac{8 \sin 13.5°}{\sin 40°} \approx 2.90.$$

SSA warning. Always compute $h = b \sin A$ *first*. Never accept a single answer from your calculator's $\sin^{-1}$ without checking whether the supplementary angle also gives a valid triangle.

2.3 The Law of Cosines (SAS and SSS)

Law of Cosines

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos A, \quad b^2 = a^2 + c^2 - 2ac \cos B, \quad c^2 = a^2 + b^2 - 2ab \cos C.$$

When to use it: there is *no* complete angle–opposite-side pair.

- **SAS** — two sides and the included angle: find the third side directly.
- **SSS** — all three sides: solve an angle by rearranging, e.g. $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Tip: in SSS, find the *largest* angle first (opposite the longest side); a negative cosine tells you it is obtuse.

Two sides and the included angle.

(diagram in the online version)

SAS: given $a = 5$, $b = 7$, included angle $C = 45°$

$$c^2 = a^2 + b^2 - 2ab \cos C = 25 + 49 - 2(5)(7) \cos 45° \approx 24.50 \ \Rightarrow\ c \approx 4.95.$$

Then find another angle with the Law of Cosines:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 24.50 - 25}{2(7)(4.95)} \approx 0.6999 \ \Rightarrow\ A \approx 45.6°,$$

and $B = 180° - 45° - 45.6° \approx 89.4°$.

SSS: given $a = 8$, $b = 11$, $c = 15$

Longest side is c, so angle C is largest. Solve it first:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 121 - 225}{2(8)(11)} = \frac{-40}{176} \approx -0.2273 \;\Rightarrow\; C \approx 103.1^\circ \text{ (obtuse).}$$

Next,

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{64 + 225 - 121}{2(8)(15)} = \frac{168}{240} = 0.7 \;\Rightarrow\; B \approx 45.6^\circ,$$

and $A = 180^\circ - 103.1^\circ - 45.6^\circ \approx 31.3^\circ$. (Check: $31.3 + 45.6 + 103.1 = 180.0$.)

Which law? SAS or SSS ⇒ **Law of Cosines**. AAS, ASA, or SSA ⇒ **Law of Sines**. When you must chase down remaining parts, finish with whichever law keeps the arithmetic cleanest.

2.4 Area of a Triangle

Two area formulas

SAS (two sides and the included angle):

$$\text{Area} = \tfrac{1}{2}\,ab\sin C = \tfrac{1}{2}\,bc\sin A = \tfrac{1}{2}\,ac\sin B.$$

Heron's formula (three sides, SSS): with semiperimeter $s = \dfrac{a+b+c}{2}$,

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Use the SAS formula when you have an *included* angle; use Heron's when you only have the three sides.

SAS area: $a = 6$, $b = 9$, $C = 55^\circ$

$$\text{Area} = \tfrac{1}{2}\,ab\sin C = \tfrac{1}{2}(6)(9)\sin 55^\circ \approx 22.12.$$

Heron's formula: $a = 7$, $b = 8$, $c = 9$

$$s = \frac{7+8+9}{2} = 12, \qquad \text{Area} = \sqrt{12(12-7)(12-8)(12-9)} = \sqrt{12 \cdot 5 \cdot 4 \cdot 3} = \sqrt{720} \approx 26.83.$$

2.5 Applications: Surveying, Navigation, and Bearings

Setting up word problems

- **Bearing** is an angle measured clockwise from due north, e.g. N40°E means 40° east of north. A three-digit bearing like 120° is measured clockwise from north.
- Draw the picture, label every known length and angle, then decide: is there a complete angle–opposite-side pair (Law of Sines) or not (Law of Cosines)?
- In a change-of-course problem, the *interior* angle at the turning point is 180° minus the change in bearing.
- Multi-triangle problems: solve one triangle to get a shared side, then use that side in the next triangle.

Surveying: distance across a pond

Points A and B are 500 m apart on one shore. A tree T across the pond makes $\angle TAB = 40^\circ$ and $\angle TBA = 65^\circ$. Find AT.

$$\angle T = 180^\circ - 40^\circ - 65^\circ = 75^\circ, \qquad AT = \frac{AB \sin B}{\sin T} = \frac{500 \sin 65^\circ}{\sin 75^\circ} \approx 469.14 \text{ m}.$$

Navigation: change of course

A ship sails 20 mi on bearing N30°E, then 30 mi on bearing N70°E. How far is it from the start?

The change in bearing is $70^\circ - 30^\circ = 40^\circ$, so the interior angle at the turn is $180^\circ - 40^\circ = 140^\circ$. With the two legs as the known sides,

$$d^2 = 20^2 + 30^2 - 2(20)(30)\cos 140^\circ \approx 2219.25 \;\Rightarrow\; d \approx 47.11 \text{ mi}.$$

Final reminders. Sketch first. Watch SSA for two answers. Use the Law of Cosines to find the largest angle when all three sides are known. State units and round only at the end.

2.6 Going Deeper: Advanced Triangle Laws

Everything below builds on the two laws you already know. The rounding convention from the top of the sheet still applies: *angles* to the nearest tenth of a degree, *lengths and areas* to the nearest hundredth.

Two sides and the included angle.

(diagram in the online version)

The Extended Law of Sines: circumradius and inradius

The common ratio in the Law of Sines is not just a number—it equals the diameter of the *circumscribed* circle (the circle through all three vertices, radius R):

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Two more links tie area to these radii. Writing K for the area and $s = \dfrac{a+b+c}{2}$ for the semiperimeter,

$$R = \frac{abc}{4K}, \qquad r = \frac{K}{s},$$

where r is the radius of the *inscribed* circle (the circle tangent to all three sides). The second identity comes from splitting the triangle into three smaller triangles, one per side, each with height r: $K = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = rs$.

Circumradius and inradius of the 7–8–9 triangle

From the earlier Heron computation, $a = 7$, $b = 8$, $c = 9$ give $s = 12$ and

$$K = \sqrt{720} \approx 26.83.$$

Therefore

$$R = \frac{abc}{4K} = \frac{7 \cdot 8 \cdot 9}{4\sqrt{720}} = \frac{504}{4\sqrt{720}} \approx 4.70, \qquad r = \frac{K}{s} = \frac{\sqrt{720}}{12} \approx 2.24.$$

Check with the Extended Law of Sines. The largest angle is opposite $c = 9$:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2(7)(8)} = \frac{32}{112} \approx 0.2857 \;\Rightarrow\; C \approx 73.4^\circ,$$

and $\dfrac{c}{\sin C} = \dfrac{9}{\sin 73.4^\circ} \approx 9.39 = 2R$, so $R \approx 4.70$ as before.

Cevians: Stewart's Theorem, medians, and angle bisectors

A *cevian* is any segment from a vertex to the opposite side. Draw the cevian AD from A to point D on side $a = BC$, and let

$$BD = m, \qquad DC = n, \qquad AD = d, \qquad m + n = a.$$

Stewart's Theorem relates the cevian length to the three sides (mnemonic *man* + *dad* = *bmb* + *cnc*):

$$b^2 m + c^2 n = a\,(d^2 + mn).$$

Two special cevians have ready-made length formulas:

- **Median** to side a (here $m = n = \frac{a}{2}$): $\quad m_a^2 = \dfrac{2b^2 + 2c^2 - a^2}{4}.$

- **Angle bisector** from A (it splits a in the ratio $c : b$, so $m = \dfrac{ac}{b+c}$, $n = \dfrac{ab}{b+c}$):

$$t_a^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right] = \frac{bc\big[(b+c)^2 - a^2\big]}{(b+c)^2}.$$

Median and angle bisector in the 7–8–9 triangle

Take $a = 7$, $b = 8$, $c = 9$ and cevians drawn from vertex A to side $a = BC$.

Median.

$$m_a^2 = \frac{2(8^2) + 2(9^2) - 7^2}{4} = \frac{128 + 162 - 49}{4} = \frac{241}{4} = 60.25 \;\Rightarrow\; m_a \approx 7.76.$$

Angle bisector.

$$t_a^2 = \frac{bc\big[(b+c)^2 - a^2\big]}{(b+c)^2} = \frac{8 \cdot 9\,(17^2 - 7^2)}{17^2} = \frac{72(289 - 49)}{289} = \frac{72 \cdot 240}{289} \approx 59.79 \;\Rightarrow\; t_a \approx 7.73.$$

Verify the median by Stewart with $m = n = 3.5$:

$$b^2m + c^2n = 64(3.5) + 81(3.5) = 507.5, \qquad a(d^2 + mn) = 7\big(60.25 + 12.25\big) = 7(72.5) = 507.5. \checkmark$$

Area, revisited: five methods and area ratios

Depending on what you are given, any of these computes the area K:

1. **SAS:** $K = \frac{1}{2}ab\sin C$ (two sides and the included angle).
2. **Heron:** $K = \sqrt{s(s-a)(s-b)(s-c)}$ (three sides).
3. **Inradius:** $K = rs$ (area from the inscribed circle).
4. **Circumradius:** $K = \dfrac{abc}{4R}$ (area from the circumscribed circle).
5. **Two-angles-and-a-side (AAS/ASA):** $K = \dfrac{a^2 \sin B \sin C}{2\sin A}$.

Area ratios. A cevian from A to D on BC makes two triangles ABD and ADC that share the same height from A, so

$$\frac{[ABD]}{[ADC]} = \frac{BD}{DC} = \frac{m}{n}.$$

For a **quadrilateral**, draw a diagonal to split it into two triangles and add their areas—the diagonal itself is found with the Law of Cosines.

Surveyed quadrilateral field from several bearings

A four-sided plot $ABCD$ is surveyed. The walk from A to B is 60 m, and from B to C is 80 m; the recorded bearings make the interior angle at B equal to $110°$. The remaining sides are $CD = 100$ m and $DA = 70$ m. Find the plot's area.

Step 1 — diagonal AC (Law of Cosines in $\triangle ABC$).

$$AC^2 = 60^2 + 80^2 - 2(60)(80)\cos 110° \approx 13283.20 \;\Rightarrow\; AC \approx 115.25 \text{ m}.$$

Step 2 — area of $\triangle ABC$ (SAS).

$$[ABC] = \tfrac{1}{2}(60)(80)\sin 110° \approx 2255.32 \text{ m}^2.$$

Step 3 — area of $\triangle ACD$ (Heron, three sides now known). With sides 115.25, 100, 70 and $s \approx 142.63$,

$$[ACD] = \sqrt{s(s-115.25)(s-100)(s-70)} \approx 3476.63 \text{ m}^2.$$

Total area $\approx 2255.32 + 3476.63 \approx 5731.95 \text{ m}^2$.

Parameter-driven ambiguous-case counting. Fix an *acute* angle A and side b, then ask how the number of SSA triangles changes as the opposite side a grows. The two thresholds are $a = b\sin A$ (grazing / right triangle) and $a = b$:

$$\underbrace{a < b\sin A}_{\mathbf{0}} \quad\Big|\quad \underbrace{a = b\sin A}_{\mathbf{1\ (right)}} \quad\Big|\quad \underbrace{b\sin A < a < b}_{\mathbf{2}} \quad\Big|\quad \underbrace{a \ge b}_{\mathbf{1}}.$$

For example, with $A = 30°$ and $b = 10$ the thresholds sit at $b\sin A = 5$ and $b = 10$:

If instead A is *obtuse*, only $a > b$ yields a triangle (exactly one); $a \le b$ gives none.

Proving a relationship: $a^2 = b(b+c) \;\Rightarrow\; A = 2B$

Advanced problems often ask you to *prove* a triangle identity rather than solve for a number. Here is a model proof using the Law of Sines together with a product-to-difference identity.

Claim. If the sides of a triangle satisfy $a^2 = b(b+c)$, then $A = 2B$.

Proof. By the Law of Sines write $a = 2R\sin A$, $b = 2R\sin B$, $c = 2R\sin C$. Substituting into $a^2 = b^2 + bc$ and cancelling the common $4R^2$,

$$\sin^2 A = \sin^2 B + \sin B \sin C.$$

Because $C = 180° - (A+B)$, we have $\sin C = \sin(A+B)$, so

$$\sin^2 A - \sin^2 B = \sin B\ \sin(A+B).$$

Apply the identity $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$ to the left side:

$$\sin(A+B)\sin(A-B) = \sin B\,\sin(A+B).$$

Since $0^\circ < A+B < 180^\circ$, $\sin(A+B) \neq 0$; dividing gives $\sin(A-B) = \sin B$. The two ways this can hold are $A-B = B$ or $A-B = 180^\circ - B$. The latter forces $A = 180^\circ$, impossible in a triangle, so $A-B = B$, i.e. $A = 2B$. ■

Advanced toolkit, at a glance. Need a radius? $2R = \dfrac{a}{\sin A}$ and $r = \dfrac{K}{s}$. Need a cevian? Stewart's $b^2m + c^2n = a(d^2 + mn)$, with ready formulas for medians and bisectors. Need an area? Pick from SAS, Heron, rs, $\dfrac{abc}{4R}$, or split a polygon along a diagonal. Proving an identity? Convert sides to sines with the Law of Sines, then lean on sum/product identities.

3 Vectors and the Dot Product

Components, magnitude & direction, the dot product, projection, and applications

3.1 Vectors: Geometric and Component Form

Rounding for this topic: Give exact values (fractions or radicals) whenever they are clean. Otherwise round **magnitudes and components to two decimal places** and **angles to the nearest** 0.1°. Keep full precision in your calculator until the final step.

What a vector is

A **vector** has both **magnitude** (length) and **direction**. Draw it as an arrow. Written in **component form** as $\vec{v} = \langle a, b\rangle$, where a is the horizontal change and b the vertical change.

$$\text{From } P(p_1, p_2) \text{ to } Q(q_1, q_2): \quad \vec{PQ} = \langle q_1 - p_1,\ q_2 - p_2\rangle.$$

Two vectors are **equal** when they have the same components — same magnitude *and* same direction — no matter where they are drawn.

Magnitude and direction angle

$$\|\vec{v}\| = \|\langle a, b\rangle\| = \sqrt{a^2 + b^2}, \qquad \tan\theta = \frac{b}{a}.$$

The **direction angle** θ is measured counterclockwise from the positive x-axis. Use the signs of a and b to place θ in the correct quadrant.

Full magnitude and direction

Find the magnitude and direction angle of $\vec{v} = \langle 4, 3 \rangle$.

$$\|\vec{v}\| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5.$$

Since $a = 4 > 0$ and $b = 3 > 0$, $\vec{v}$ is in Quadrant I:

$$\theta = \tan^{-1}\left(\tfrac{3}{4}\right) = 36.9^\circ.$$

Answer: magnitude 5, direction $\approx 36.9^\circ$.

Quadrant tip: $\tan^{-1}$ on a calculator only returns -90° to 90°. If $a < 0$ (Quadrant II or III), **add** 180° to the calculator value. If $a > 0,\ b < 0$ (Quadrant IV), add 360° to get a positive angle.

3.2 Vector Operations

Adding, subtracting, scaling, unit vectors

$$\begin{aligned} \langle a, b \rangle + \langle c, d \rangle &= \langle a + c,\ b + d \rangle && \text{(add tip-to-tail)} \\ \langle a, b \rangle - \langle c, d \rangle &= \langle a - c,\ b - d \rangle && (\vec{u} - \vec{v} = \vec{u} + (-\vec{v})) \\ k\langle a, b \rangle &= \langle ka,\ kb \rangle && \text{(scalar multiple)} \end{aligned}$$

A **unit vector** has magnitude 1. To find the unit vector in the direction of $\vec{v}$:

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \left\langle \frac{a}{\|\vec{v}\|},\ \frac{b}{\|\vec{v}\|} \right\rangle.$$

Standard basis / $\mathbf{i}, \mathbf{j}$ form: $\mathbf{i} = \langle 1, 0 \rangle$, $\mathbf{j} = \langle 0, 1 \rangle$, so $\langle a, b \rangle = a\,\mathbf{i} + b\,\mathbf{j}$.

Operations and a unit vector

Let $\vec{u} = \langle 3, -2 \rangle$ and $\vec{v} = \langle 1, 4 \rangle$.

$$2\vec{u} - \vec{v} = \langle 6, -4 \rangle - \langle 1, 4 \rangle = \langle 5, -8 \rangle.$$

Unit vector for $\vec{u}$: $\|\vec{u}\| = \sqrt{9 + 4} = \sqrt{13}$, so

$$\hat{u} = \left\langle \tfrac{3}{\sqrt{13}},\ \tfrac{-2}{\sqrt{13}} \right\rangle \approx \langle 0.83,\ -0.55 \rangle.$$

In $\mathbf{i}, \mathbf{j}$ form: $\vec{u} = 3\,\mathbf{i} - 2\,\mathbf{j}$.

3.3 Writing a Vector from Magnitude and Direction

Resolving into components

A vector with magnitude $r = \|\vec{v}\|$ and direction angle θ has components

$$\vec{v} = \langle r\cos\theta,\ r\sin\theta\rangle.$$

This is the reverse of finding magnitude and direction — it **resolves** the vector into its horizontal and vertical parts.

Magnitude and direction to components

Write the vector of magnitude 10 pointing at 30°.

$$\vec{v} = \langle 10\cos 30^\circ,\ 10\sin 30^\circ\rangle = \left\langle 10 \cdot \tfrac{\sqrt{3}}{2},\ 10 \cdot \tfrac{1}{2}\right\rangle = \langle 5\sqrt{3},\ 5\rangle \approx \langle 8.66,\ 5\rangle.$$

Check: resolving and then re-computing the magnitude must return r: $\sqrt{(5\sqrt{3})^2 + 5^2} = \sqrt{75 + 25} = \sqrt{100} = 10$. ✓

3.4 The Dot Product, Angle Between Vectors, Orthogonality

Dot product and the angle between vectors

$$\vec{u} \cdot \vec{v} = \langle a, b\rangle \cdot \langle c, d\rangle = ac + bd \qquad \text{(a scalar, not a vector).}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta \qquad \Longrightarrow \qquad \cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad 0^\circ \le \theta \le 180^\circ.$$

Orthogonal (perpendicular): $\vec{u} \perp \vec{v}$ exactly when $\vec{u} \cdot \vec{v} = 0$.

Angle between two vectors

Find the angle between $\vec{u} = \langle 3, 1\rangle$ and $\vec{v} = \langle 2, 4\rangle$.

$$\vec{u} \cdot \vec{v} = 3(2) + 1(4) = 10, \qquad \|\vec{u}\| = \sqrt{10}, \quad \|\vec{v}\| = \sqrt{20}.$$

$$\cos\theta = \frac{10}{\sqrt{10}\sqrt{20}} = \frac{10}{\sqrt{200}} = \frac{10}{14.142} = 0.7071 \implies \theta = 45^\circ.$$

Orthogonal check: $\langle 2,3\rangle\cdot\langle 3,-2\rangle = 6-6=0$, so these vectors are perpendicular. A negative dot product means the angle is *obtuse*; a positive one means *acute*.

3.5 Vector Projection and Work

Projection and parallel/orthogonal decomposition

The (vector) **projection of $\vec{u}$ onto $\vec{v}$**:

$$\operatorname{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v}.$$

The **scalar component** of $\vec{u}$ along $\vec{v}$ is $\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$. Every vector splits into a part parallel to $\vec{v}$ and a part orthogonal to it:

$$\vec{u} = \underbrace{\operatorname{proj}_{\vec{v}}\vec{u}}_{\vec{w}_1\ (\|\vec{v})} + \underbrace{\left(\vec{u}-\operatorname{proj}_{\vec{v}}\vec{u}\right)}_{\vec{w}_2\ (\perp\vec{v})}.$$

Full projection and decomposition

Project $\vec{u}=\langle 6,2\rangle$ onto $\vec{v}=\langle 3,4\rangle$.

$$\vec{u}\cdot\vec{v} = 6(3)+2(4)=26, \qquad \|\vec{v}\|^2 = 3^2+4^2=25.$$

$$\operatorname{proj}_{\vec{v}}\vec{u} = \frac{26}{25}\langle 3,4\rangle = \left\langle \tfrac{78}{25},\ \tfrac{104}{25}\right\rangle = \langle 3.12,\ 4.16\rangle.$$

Orthogonal part: $\vec{w}_2 = \vec{u}-\vec{w}_1 = \langle 6-3.12,\ 2-4.16\rangle = \langle 2.88,\ -2.16\rangle$. Check: $\vec{w}_1\cdot\vec{w}_2 = 3.12(2.88)+4.16(-2.16) = 8.9856-8.9856=0$. ✓

Work done by a force

When a constant force $\vec{F}$ moves an object through displacement $\vec{d}$,

$$W = \vec{F}\cdot\vec{d} = \|\vec{F}\|\,\|\vec{d}\|\cos\theta,$$

where θ is the angle between the force and the direction of motion.

Work along a ramp

A rope pulls a crate 20 ft horizontally with a 50-lb force directed 30° above the horizontal.

$$W = \|\vec{F}\|\,\|\vec{d}\|\cos\theta = 50(20)\cos 30^\circ = 1000(0.8660) = 866.03 \text{ ft-lb}.$$

3.6 Applications: Resultant Forces, Velocity, Navigation, Inclined Planes

Combining vectors in the real world

Resultant force / velocity: resolve each vector into components, add the components, then re-compute magnitude and direction.

$$\vec{R} = \vec{F}_1 + \vec{F}_2, \qquad \|\vec{R}\| = \sqrt{R_x^2 + R_y^2}, \qquad \theta = \tan^{-1}\frac{R_y}{R_x} \text{ (adjust quadrant).}$$

Inclined plane: for weight W on a ramp of angle α,

$$\text{force along ramp} = W \sin\alpha, \qquad \text{force into ramp} = W \cos\alpha.$$

Resultant of two forces

$\vec{F}_1$: 40 N at 30°. $\vec{F}_2$: 60 N at 120°.

$$\begin{aligned} \vec{F}_1 &= \langle 40\cos 30^\circ, 40\sin 30^\circ\rangle = \langle 34.64,\ 20\rangle, \\ \vec{F}_2 &= \langle 60\cos 120^\circ, 60\sin 120^\circ\rangle = \langle -30,\ 51.96\rangle, \\ \vec{R} &= \langle 4.64,\ 71.96\rangle. \end{aligned}$$

$$\|\vec{R}\| = \sqrt{4.64^2 + 71.96^2} = \sqrt{5199.7} = 72.11 \text{ N}, \quad \theta = \tan^{-1}\frac{71.96}{4.64} = 86.3^\circ.$$

Navigation reminder: a compass **bearing** is measured clockwise from North, but a direction angle is measured counterclockwise from the positive x-axis (East). Convert carefully, or set East $= x$, North $= y$ and resolve each velocity into components before adding.

3.7 Going Deeper: Advanced Vector Ideas

Same rounding rules apply: keep exact radicals when clean, otherwise round magnitudes/components to two decimals and angles to 0.1°. In proofs, prefer exact arithmetic so a dot product that *should* be zero comes out *exactly* zero.

Magnitude of a sum or difference from the angle

You do not need components to find $\|\vec{u} + \vec{v}\|$ — only the two magnitudes and the angle θ between the vectors. Expanding $\|\vec{u} \pm \vec{v}\|^2 = (\vec{u} \pm \vec{v}) \cdot (\vec{u} \pm \vec{v})$ gives the **Law of Cosines in vector form**:

$$\|\vec{u} \pm \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 \pm 2\,\|\vec{u}\|\,\|\vec{v}\|\cos\theta.$$

This is exactly the **parallelogram law**: $\vec{u} + \vec{v}$ is the diagonal and $\vec{u} - \vec{v}$ is the other diagonal of the parallelogram built on $\vec{u}$ and $\vec{v}$.

Both diagonals from magnitudes and the angle

Vectors $\vec{u}$ and $\vec{v}$ have $\|\vec{u}\| = 5$, $\|\vec{v}\| = 3$, and the angle between them is $\theta = 60^\circ$.

$$\|\vec{u}+\vec{v}\|^2 = 5^2 + 3^2 + 2(5)(3)\cos 60^\circ = 25 + 9 + 30(0.5) = 49 \implies \|\vec{u}+\vec{v}\| = 7.$$

$$\|\vec{u}-\vec{v}\|^2 = 25 + 9 - 30(0.5) = 19 \implies \|\vec{u}-\vec{v}\| = \sqrt{19} \approx 4.36.$$

Answer: the diagonals measure 7 and ≈ 4.36.

Proving geometry with vectors

Turn a geometric claim about points $A, B, C, \ldots$ into an algebra statement about the vectors between them:

- **Perpendicular / right angle at B:** show $\vec{BA} \cdot \vec{BC} = 0$.
- **Equal lengths (isosceles):** show $\|\vec{AB}\| = \|\vec{AC}\|$, i.e. equal $\vec{v} \cdot \vec{v}$.
- **Parallel / collinear:** show one vector is a scalar multiple of the other, $\vec{u} = k\vec{v}$.
- **Midpoint M of AB:** $\vec{OM} = \frac{1}{2}(\vec{OA} + \vec{OB})$.

Because the dot product is exact, these tests are cleaner than measuring slopes or distances.

Proving a triangle is right *and* isosceles

Triangle with $A(1,2)$, $B(5,4)$, $C(3,8)$. Form the edge vectors at B:

$$\vec{BA} = A - B = \langle -4, -2 \rangle, \qquad \vec{BC} = C - B = \langle -2, 4 \rangle.$$

$$\vec{BA} \cdot \vec{BC} = (-4)(-2) + (-2)(4) = 8 - 8 = 0 \implies \text{right angle at } B.$$

Now compare the two legs:

$$\|\vec{BA}\| = \sqrt{(-4)^2 + (-2)^2} = \sqrt{20}, \qquad \|\vec{BC}\| = \sqrt{(-2)^2 + 4^2} = \sqrt{20}.$$

Equal legs, so the triangle is an **isosceles right triangle**. ✓

Direction of an angle bisector

The bisector of the angle between $\vec{u}$ and $\vec{v}$ points along the **sum of the two unit vectors**:

$$\vec{b} = \hat{u} + \hat{v} = \frac{\vec{u}}{\|\vec{u}\|} + \frac{\vec{v}}{\|\vec{v}\|}.$$

Using *unit* vectors is essential — adding $\vec{u} + \vec{v}$ directly leans toward the longer vector. Any positive scalar multiple of $\vec{b}$ gives the same bisector direction.

Bisector direction between two vectors

Bisect the angle between $\vec{u} = \langle 3, 0\rangle$ and $\vec{v} = \langle 0, 4\rangle$.

$$\hat{u} = \langle 1, 0\rangle, \qquad \hat{v} = \langle 0, 1\rangle, \qquad \vec{b} = \hat{u} + \hat{v} = \langle 1, 1\rangle.$$

Its direction angle is $\tan^{-1}(1/1) = 45°$, exactly halfway between $0°$ and $90°$. ✓Note that $\vec{u} + \vec{v} = \langle 3, 4\rangle$ (angle $53.1°$) would have been *wrong* — it tilts toward the longer $\vec{v}$.

Work along a path with a constant force: for a constant $\vec{F}$, the work over a path made of straight legs $\vec{d}_1, \vec{d}_2, \ldots$ is

$$W = \vec{F} \cdot \vec{d}_1 + \vec{F} \cdot \vec{d}_2 + \cdots = \vec{F} \cdot (\vec{d}_1 + \vec{d}_2 + \cdots) = \vec{F} \cdot \vec{d}_{\text{total}}.$$

So a constant force does **path-independent** work: only the straight-line displacement from start to finish matters.

Equilibrium: resultant of several forces is zero

An object is in **equilibrium** when the forces on it sum to the zero vector:

$$\sum \vec{F} = \vec{0} \quad \Longleftrightarrow \quad \sum F_x = 0 \ \text{ and } \ \sum F_y = 0.$$

For a weight hung by two ropes, resolve each tension into components and force both totals to zero. On an **inclined plane** of angle α, split gravity into the part $W \sin\alpha$ down the ramp (balanced by friction/tension) and $W \cos\alpha$ into the ramp (balanced by the normal force).

Two rope tensions holding a weight

A 100-lb weight hangs in equilibrium from two ropes: rope 1 pulls up-left at $120°$, rope 2 up-right at $30°$ (angles from the positive x-axis). Find the tensions T_1, T_2. Gravity is $\langle 0, -100\rangle$. Equilibrium in each axis:

$$\begin{aligned} x:&\quad T_1 \cos 120° + T_2 \cos 30° = 0 &&\Rightarrow\ -0.5\,T_1 + 0.8660\,T_2 = 0,\\ y:&\quad T_1 \sin 120° + T_2 \sin 30° - 100 = 0 &&\Rightarrow\ 0.8660\,T_1 + 0.5\,T_2 = 100. \end{aligned}$$

From the x-equation, $T_1 = 1.7320\,T_2$. Substitute into the y-equation:

$$0.8660(1.7320\,T_2) + 0.5\,T_2 = 100 \implies 2.0\,T_2 = 100 \implies T_2 = 50 \text{ lb}, \quad T_1 \approx 86.60 \text{ lb}.$$

Navigation: correcting a heading for wind

A plane cruises at airspeed 250 mph and must travel **due north**. A wind blows *toward* the east at 40 mph, so $\vec{w} = \langle 40, 0\rangle$ (East $= x$, North $= y$). The pilot heads slightly west of north by angle φ, so the air velocity is $\vec{a} = \langle -250\sin\varphi,\ 250\cos\varphi\rangle$. The ground velocity $\vec{g} = \vec{a} + \vec{w}$ must have **zero** east-west component:

$$-250\sin\varphi + 40 = 0 \implies \sin\varphi = \frac{40}{250} = 0.16 \implies \varphi = 9.2^\circ \text{ west of north.}$$

Groundspeed is the surviving north component:

$$\|\vec{g}\| = 250\cos 9.2^\circ = 250(0.9871) = 246.78 \text{ mph.}$$

Answer: steer 9.2° west of north; the plane makes good ≈ 246.78 mph northward.

4 Polar Coordinates and Complex Numbers

Plotting, conversions, polar graphs, trig form, products, and DeMoivre's Theorem

4.1 Polar Coordinates

The Polar Point (r, θ)

A **polar point** is written (r, θ), where r is the *directed distance* from the origin (the **pole**) and θ is the angle from the positive x-axis (the **polar axis**).

- If $r > 0$, move r units along the terminal side of θ.
- If $r < 0$, move $|r|$ units in the *opposite* direction (add 180°).

Distance and midpoint come from the coordinates.

(diagram in the online version)

The point (r, θ): travel out distance r at angle θ.

Many Names for One Point

Every polar point has **infinitely many representations**:

$$(r,\theta) = (r,\ \theta + 360^\circ k) = (-r,\ \theta + 180^\circ + 360^\circ k), \qquad k = 0, \pm1, \pm2, \ldots$$

The pole itself is $(0, \theta)$ for *any* angle θ.

Example: Other Names for $(3, 50^\circ)$

Same terminal side, add a revolution: $(3,\ 410^\circ)$. Same, go backward: $(3,\ -310^\circ)$.
Use a negative radius (add 180°): $(-3,\ 230^\circ)$.

Polar $\leftrightarrow$ Rectangular (Points)

With the pole at the origin and polar axis along the positive x-axis:

$$\boxed{x = r\cos\theta, \qquad y = r\sin\theta} \qquad\qquad \boxed{r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}}$$

When finding θ, always check the **quadrant** of (x, y).

Example: Polar $\to$ Rectangular

Convert $(4,\ 120^\circ)$ to rectangular coordinates.

$$x = 4\cos 120^\circ = 4\left(-\tfrac{1}{2}\right) = -2, \qquad y = 4\sin 120^\circ = 4\left(\tfrac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$\Rightarrow \quad \left(\mathbf{-2,\ 2\sqrt{3}}\right)$$

Example: Rectangular $\to$ Polar

Convert $\left(-1,\ \sqrt{3}\right)$ to polar form ($r > 0$, $0^\circ \le \theta < 360^\circ$).

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

The point is in Quadrant II with reference angle $\tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60^\circ$, so $\theta = 180^\circ - 60^\circ = 120^\circ$.

$$\Rightarrow \quad (\mathbf{2,\ 120^\circ})$$

Tip: A calculator's $\tan^{-1}$ only returns angles in Quadrants I and IV. Sketch the point first, then adjust θ into the correct quadrant.

4.2 Converting Polar and Rectangular Equations

Equation Conversion Toolkit

Swap between forms with the same relationships, plus these handy substitutions:

$$r^2 = x^2 + y^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad \tan\theta = \frac{y}{x}$$

Polar $\to$ rectangular: multiply by r or use identities to remove r and θ.
Rectangular $\to$ polar: replace x, y and simplify; solve for r if possible.

Example: Polar → Rectangular Equation

Convert $r = 2\cos\theta$. Multiply both sides by r:

$$r^2 = 2r\cos\theta \implies x^2 + y^2 = 2x \implies x^2 - 2x + y^2 = 0 \implies \mathbf{(x-1)^2 + y^2 = 1}$$

A circle of radius 1 centered at $(1, 0)$.

Example: Rectangular → Polar Equation

Convert $x^2 + y^2 = 9$ and the line $x = 3$.

$$x^2 + y^2 = 9 \implies r^2 = 9 \implies \mathbf{r = 3}$$

$$x = 3 \implies r\cos\theta = 3 \implies \mathbf{r = 3}\sec\theta$$

Remember: $r = a$ is a circle of radius a; $\theta = \alpha$ is a line through the pole. Vertical/horizontal lines become $r = a\sec\theta$ or $r = a\csc\theta$.

4.3 Graphs of Polar Equations

Common Polar Curves

- **Circles:** $r = a$ (centered at pole); $r = a\cos\theta$ or $r = a\sin\theta$ (through the pole, diameter $|a|$).
- **Cardioids:** $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$ — a heart shape with one dimple at the pole.
- **Limaçons:** $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. Compare $\frac{a}{b}$: $\frac{a}{b} < 1$ inner loop; $\frac{a}{b} = 1$ cardioid; $1 < \frac{a}{b} < 2$ dimpled; $\frac{a}{b} \geq 2$ convex.
- **Roses:** $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$. Petal count: n petals if n is *odd*, $2n$ petals if n is *even*.

Cardioid $r = 1 + \cos\theta$: maximum $r = 2$ at $\theta = 0°$, dimple at the pole when $\theta = 180°$.

Rose $r = 2\cos 2\theta$: since $n = 2$ is even, it has $2n = \mathbf{4}$ petals, each of length 2.

Tip: To sketch, make a quick table of $\theta = 0°, 30°, 45°, 60°, 90°, \ldots$ and plot (r, θ). Note where $r = 0$ (curve passes through the pole) and where $|r|$ is largest.

4.4 Complex Numbers and Trigonometric Form

The Complex Plane

A complex number $z = a + bi$ is plotted as the point (a, b): the **real part** a on the horizontal axis, the **imaginary part** b on the vertical axis. Its

$$\textbf{modulus } |z| = r = \sqrt{a^2 + b^2}, \qquad \textbf{argument } \theta \text{ with } \tan\theta = \frac{b}{a} \text{ (choose by quadrant).}$$

$z = a + bi$ has modulus r (distance to origin) and argument θ.

Trigonometric (Polar) Form

$$z = a + bi = r(\cos\theta + i\sin\theta), \qquad \text{where } a = r\cos\theta, \quad b = r\sin\theta.$$

The shorthand $r(\cos\theta + i\sin\theta)$ is often written $r\,\mathrm{cis}\,\theta$.

Example: Rectangular → Trig Form

Write $z = -1 + i\sqrt{3}$ in trigonometric form.

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2.$$

The point $(-1, \sqrt{3})$ is in Quadrant II with reference angle 60°, so $\theta = 120^\circ$.

$$\Rightarrow \quad z = \mathbf{2}\,(\cos\mathbf{120}^\circ + \mathbf{i}\sin\mathbf{120}^\circ)$$

Tip: Going from trig form back to $a + bi$, just evaluate: $z = r\cos\theta + (r\sin\theta)\,i$.

4.5 Products and Quotients in Trig Form

Multiply and Divide by Modulus and Angle

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$:

$$z_1 z_2 = r_1 r_2\big[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\big[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\big]$$

Multiply the moduli, add the angles (divide moduli, subtract angles).

Example: Product

Let $z_1 = 3(\cos 40^\circ + i\sin 40^\circ)$ and $z_2 = 2(\cos 80^\circ + i\sin 80^\circ)$.

$$z_1 z_2 = (3)(2)\big[\cos(40^\circ + 80^\circ) + i\sin(40^\circ + 80^\circ)\big] = \mathbf{6}\,(\cos \mathbf{120}^\circ + \mathbf{i}\sin \mathbf{120}^\circ)$$

In $a + bi$: $6\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -3 + 3i\sqrt{3}$.

Remember: If a subtracted angle comes out negative or over 360°, add or subtract 360° to land in $[0^\circ, 360^\circ)$.

4.6 DeMoivre's Theorem: Powers and Roots

DeMoivre's Theorem (Powers)

For $z = r(\cos\theta + i\sin\theta)$ and any positive integer n:

$$\boxed{z^n = r^n\big(\cos n\theta + i\sin n\theta\big)}$$

Raise the modulus to the power; multiply the angle by n.

Example: Power

Compute $\left(1 + i\sqrt{3}\right)^4$. First $r = \sqrt{1+3} = 2$ and $\theta = 60^\circ$ (Quadrant I).

$$z^4 = 2^4\big(\cos(4\cdot 60^\circ) + i\sin(4\cdot 60^\circ)\big) = 16\big(\cos 240^\circ + i\sin 240^\circ\big)$$

$$= 16\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \mathbf{-8 - 8i\sqrt{3}}$$

The n Distinct nth Roots

Every nonzero $z = r(\cos\theta + i\sin\theta)$ has exactly n complex nth roots:

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right], \qquad k = 0, 1, 2, \ldots, n-1.$$

All roots share modulus $\sqrt[n]{r}$ and are spaced $\dfrac{360^\circ}{n}$ apart around a circle.

Example: All Cube Roots of $8i$

Write $8i = 8(\cos 90^\circ + i\sin 90^\circ)$, so $r = 8$ and $\sqrt[3]{8} = 2$. Angles: $\dfrac{90^\circ + 360^\circ k}{3}$ for $k = 0, 1, 2$, i.e.

30°, 150°, 270°.

$$w_0 = 2(\cos 30^\circ + i\sin 30^\circ) = \sqrt{3} + i$$
$$w_1 = 2(\cos 150^\circ + i\sin 150^\circ) = -\sqrt{3} + i$$
$$w_2 = 2(\cos 270^\circ + i\sin 270^\circ) = -2i$$

The three roots: $\boldsymbol{\sqrt{3} + i}$, $\boldsymbol{-\sqrt{3} + i}$, $\boldsymbol{-2i}$.

Tip: Find the first root ($k = 0$), then just add $\dfrac{360^\circ}{n}$ to the angle repeatedly to get the rest — the modulus never changes.

4.7 Going Deeper: Advanced Polar & Complex Ideas

Roots of Unity

The nth **roots of unity** are the n solutions of $z^n = 1$. Writing $\omega = \cos\frac{360^\circ}{n} + i\sin\frac{360^\circ}{n}$ (the *primitive* root), every root is a power of ω:

$$\omega^k = \cos\left(\frac{360^\circ k}{n}\right) + i\sin\left(\frac{360^\circ k}{n}\right), \qquad k = 0, 1, 2, \ldots, n-1.$$

They sit at the vertices of a regular n-gon on the unit circle, starting at $\omega^0 = 1$. Two beautiful facts (for $n \geq 2$):

$$\boxed{\sum_{k=0}^{n-1} \omega^k = 0} \qquad\qquad \boxed{\prod_{k=1}^{n-1} (1 - \omega^k) = n}$$

The five 5th roots of unity $\omega^0, \ldots, \omega^4$ form a regular pentagon on the unit circle; their sum is the center, 0.

Example: Sum and Product of the 5th Roots of Unity

Here $\omega = \cos 72^\circ + i\sin 72^\circ$, and the five roots are

$$\omega^0,\ \omega^1,\ \omega^2,\ \omega^3,\ \omega^4 \ = \ \operatorname{cis} 0^\circ,\ \operatorname{cis} 72^\circ,\ \operatorname{cis} 144^\circ,\ \operatorname{cis} 216^\circ,\ \operatorname{cis} 288^\circ.$$

Sum. Because $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, the roots satisfy $z^4 + z^3 + z^2 + z + 1 = 0$; comparing to $\sum \omega^k$, the coefficient of z^4 is 1, so

$$\omega^0 + \omega^1 + \omega^2 + \omega^3 + \omega^4 = 0.$$

Geometrically the five equally spaced vectors cancel by symmetry.

Product. Factor $z^5 - 1 = (z-1)\prod_{k=1}^{4}(z - \omega^k)$, so $\prod_{k=1}^{4}(z - \omega^k) = z^4 + z^3 + z^2 + z + 1$.

Setting $z = 1$:
$$\prod_{k=1}^{4}\left(1-\omega^{k}\right) = 1+1+1+1+1 = \mathbf{5}.$$

The Identity $z^n + z^{-n} = 2\cos n\theta$

If $z = \cos\theta + i\sin\theta$ lies on the unit circle, then $\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$. By DeMoivre,
$$z^n = \cos n\theta + i\sin n\theta, \qquad z^{-n} = \cos n\theta - i\sin n\theta,$$
so adding and subtracting gives the two workhorse identities
$$\boxed{z^n + z^{-n} = 2\cos n\theta} \qquad \boxed{z^n - z^{-n} = 2i\sin n\theta}$$
These turn powers of $\cos\theta$ and $\sin\theta$ into sums of $\cos n\theta$ and $\sin n\theta$ (and back).

Example: Using DeMoivre to Derive $\cos 3\theta$ and $\sin 3\theta$

Expand $(\cos\theta + i\sin\theta)^3$ two ways. By **DeMoivre**, it equals $\cos 3\theta + i\sin 3\theta$. By the **binomial theorem** (using $i^2 = -1$, $i^3 = -i$):
$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$
Match real parts and imaginary parts:
$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \mathbf{4}\cos^{\mathbf{3}}\theta - \mathbf{3}\cos\theta,$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = \mathbf{3}\sin\theta - \mathbf{4}\sin^{\mathbf{3}}\theta,$$
where the final forms use $\sin^2\theta = 1 - \cos^2\theta$ and $\cos^2\theta = 1 - \sin^2\theta$.

Apollonius Circles: the Locus $|z-a| = k\,|z-b|$

The set of points z whose distance to a is a fixed multiple $k > 0$ of its distance to b is:

- the **perpendicular bisector** of a and b when $k = 1$ (equal distances);
- an **Apollonius circle** when $k \neq 1$ — squaring $|z-a|^2 = k^2|z-b|^2$ and expanding $z = x+iy$ produces $x^2 + y^2 + (\cdots)x + (\cdots)y + (\cdots) = 0$, the equation of a circle.

The circle is symmetric about the line through a and b and shrinks toward the nearer point as $k \to 0$ or $k \to \infty$.

Example: Describe the Locus $|z-1| = 2\,|z+1|$

Let $z = x + iy$, with $a = 1$ and $b = -1$. Square both sides:

$$(x-1)^2 + y^2 = 4\big[(x+1)^2 + y^2\big].$$

Expand and collect:

$$x^2 - 2x + 1 + y^2 = 4x^2 + 8x + 4 + 4y^2 \implies 3x^2 + 3y^2 + 10x + 3 = 0.$$

Divide by 3 and complete the square:

$$x^2 + \tfrac{10}{3}x + y^2 + 1 = 0 \implies \big(x + \tfrac{5}{3}\big)^2 + y^2 = \tfrac{25}{9} - 1 = \tfrac{16}{9}.$$

So the locus is a **circle** centered at $\big(-\tfrac{5}{3}, 0\big)$ with radius $\tfrac{4}{3}$.

Area Enclosed by a Polar Curve

The region swept by the ray to a polar curve $r = f(\theta)$ as θ runs from α to β has area

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} \big[f(\theta)\big]^2\, d\theta}$$

This comes from summing thin circular sectors of angle $d\theta$ and radius r, each of area $\frac{1}{2}r^2\, d\theta$. **Setup cautions:** choose α, β that trace the region *exactly once*; for one petal of a rose, integrate between consecutive zeros of r; use symmetry to integrate a half and double.

Example: Set Up the Area of One Petal of $r = 2\cos 2\theta$

One petal is traced as r goes from 0 up to its max and back to 0. Solve $r = 0$: $\cos 2\theta = 0$ at $2\theta = \pm 90°$, i.e. $\theta = -45°$ and $\theta = 45°$ (the petal centered on the polar axis). Thus

$$A_{\text{one petal}} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \big(2\cos 2\theta\big)^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos^2 2\theta\, d\theta.$$

By symmetry this equals $2\displaystyle\int_0^{\pi/4} \cos^2 2\theta\, d\theta$, the required set-up. (Evaluating with $\cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta)$ gives $A = \frac{\pi}{4}$.)

Intersections of Polar Curves

To find where $r = f(\theta)$ and $r = g(\theta)$ meet, solving $f(\theta) = g(\theta)$ is *not enough*. Because a single point has many polar names, also check:

- the **pole** separately — it lies on a curve if $r = 0$ for *some* θ, even if the θ values differ;

- alternate representations, e.g. $f(\theta) = -g(\theta + 180^\circ)$, and shifts by full turns.

Always sketch both curves to confirm the true intersection points.

Example: Intersections of $r = 1 + \cos\theta$ and $r = 3\cos\theta$

Set equal: $1 + \cos\theta = 3\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}$, so $\theta = 60^\circ, 300^\circ$. Then $r = 3\cos 60^\circ = \frac{3}{2}$, giving points $\left(\frac{3}{2}, 60^\circ\right)$ and $\left(\frac{3}{2}, 300^\circ\right)$.
Check the pole. On the cardioid, $r = 0$ when $\cos\theta = -1$ ($\theta = 180^\circ$); on the circle, $r = 0$ when $\cos\theta = 0$ ($\theta = 90^\circ$). Different θ, but *both* curves pass through the pole, so the **pole** is a third intersection point that the algebra missed.

Sum of the roots of $z^n = w$. All n roots are $w_0,\ w_0\omega,\ w_0\omega^2, \ldots, w_0\omega^{n-1}$, where w_0 is one root and ω is a primitive nth root of unity. Factoring out w_0,

$$\sum_{k=0}^{n-1} w_k = w_0 \sum_{k=0}^{n-1} \omega^k = w_0 \cdot 0 = \mathbf{0} \qquad (n \geq 2).$$

Equivalently, $z^n - w = z^n + 0 \cdot z^{n-1} + \cdots$ has no z^{n-1} term, so by Vieta the roots sum to 0. Their **product** is $(-1)^{n+1} w$.

5 Angles and Their Measure

Standard position, radians, arc length, sector area, and speed

5.1 Angles in Standard Position

Standard Position

An angle is in **standard position** when its **vertex** sits at the origin and its **initial side** lies along the positive x-axis. The **terminal side** is where the ray ends after rotating.

- A **positive** angle rotates *counterclockwise*.
- A **negative** angle rotates *clockwise*.

The **quadrant** of the angle is the quadrant containing its terminal side.

An angle measures a turn.

(diagram in the online version)

A positive angle θ in standard position (terminal side in Quadrant II).

A negative angle $-\theta$ (clockwise rotation, terminal side in Quadrant IV).

Quadrantal Angles

A **quadrantal angle** has its terminal side lying *on* an axis. The common ones are

$$0^\circ, \quad 90^\circ, \quad 180^\circ, \quad 270^\circ, \quad 360^\circ$$

(equivalently 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π radians). These angles are in *no* quadrant.

5.2 Coterminal Angles

Coterminal Angles

Two angles are **coterminal** if they share the same terminal side. Add or subtract full revolutions:

$$\theta \pm 360^\circ k \quad \text{(degrees)} \qquad \text{or} \qquad \theta \pm 2\pi k \quad \text{(radians)}, \quad k = 1, 2, 3, \ldots$$

An angle measures a turn.

(diagram in the online version)

60° and -300° are coterminal: $60^\circ - 360^\circ = -300^\circ$.

Example: Coterminal Angles

Find one positive and one negative angle coterminal with $\theta = 210^\circ$.
Positive: $210^\circ + 360^\circ = \mathbf{570}^\circ$. Negative: $210^\circ - 360^\circ = -\mathbf{150}^\circ$.

Tip: To find the coterminal angle between 0° and 360°, keep adding or subtracting 360° until you land in that range. (For radians, use 2π.)

5.3 Degree Measure and DMS

Degrees, Minutes, Seconds

One full revolution is 360°. A degree splits further:

$$1^\circ = 60' \text{ (minutes)}, \qquad 1' = 60'' \text{ (seconds)}, \qquad 1^\circ = 3600''.$$

An angle measures a turn.

(diagram in the online version)

Example: Decimal → DMS

Convert $24.6°$ to DMS. Multiply the decimal part by 60:

$$0.6 \times 60 = 36'. \qquad \Rightarrow \quad 24.6° = \mathbf{24°36'}$$

Example: DMS → Decimal

Convert $18°45'36''$ to decimal degrees.

$$18 + \frac{45}{60} + \frac{36}{3600} = 18 + 0.75 + 0.01 = \mathbf{18.76}°$$

Tip: Going *to* DMS, multiply the leftover decimal by 60 at each step. Going *from* DMS, divide minutes by 60 and seconds by 3600, then add.

5.4 Radian Measure and Conversions

Radians

One **radian** is the central angle that subtends an arc equal in length to the radius. A full circle is 2π radians, so

$$\boxed{180° = \pi \text{ radians}}$$

$$\text{degrees} \to \text{radians:} \quad \times\frac{\pi}{180°} \qquad\qquad \text{radians} \to \text{degrees:} \quad \times\frac{180°}{\pi}$$

An angle measures a turn.

(diagram in the online version)

Example: Degrees → Radians

Convert $150°$ to radians.

$$150° \times \frac{\pi}{180°} = \frac{150\pi}{180} = \frac{\mathbf{5}\pi}{\mathbf{6}} \text{ rad}$$

Example: Radians → Degrees

Convert $\frac{3\pi}{4}$ radians to degrees.

$$\frac{3\pi}{4} \times \frac{180°}{\pi} = \frac{3 \times 180°}{4} = \mathbf{135}°$$

Remember: When no degree symbol appears, the angle is in *radians*. Keep answers exact with π unless a decimal is requested.

5.5 Arc Length and Sector Area

Arc Length and Sector Area

For a circle of radius r and a central angle θ **measured in radians**:

$$\text{Arc length: } s = r\theta \qquad \text{Sector area: } A = \tfrac{1}{2}r^2\theta$$

If θ is given in degrees, convert to radians first.

A sector with radius r, central angle θ, arc length s, and area A.

Example: Arc Length and Area

A circle has $r = 6$ and central angle $\theta = \dfrac{\pi}{3}$.

$$s = r\theta = 6 \cdot \frac{\pi}{3} = \mathbf{2}\pi \qquad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(36)\frac{\pi}{3} = \mathbf{6}\pi$$

Trap: $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ *only* work when θ is in radians. A degree measure must be converted first.

5.6 Linear Speed and Angular Speed

Two Kinds of Speed

Suppose a point moves along a circle of radius r, sweeping angle θ (radians) in time t.

$$\text{Angular speed: } \omega = \frac{\theta}{t} \qquad \text{Linear speed: } v = \frac{s}{t} = r\omega$$

Revolutions: 1 revolution $= 2\pi$ radians, so N RPM $= 2\pi N$ rad per minute.

Example: Wheel Speed

A wheel of radius 0.3 m spins at 60 RPM. Find its angular and linear speed.

$$\omega = 60 \times 2\pi = 120\pi \text{ rad/min}$$

$$v = r\omega = 0.3 \times 120\pi = \mathbf{36}\pi \approx \mathbf{113.1} \text{ m/min}$$

Tip: Belts and gears in contact share the same *linear* speed v, so $r_1\omega_1 = r_2\omega_2$. A point at the center has ω but zero linear speed.

5.7 Complementary and Supplementary Angles

Complements and Supplements

Two positive angles are

- **complementary** if they add to 90° (or $\frac{\pi}{2}$ rad),
- **supplementary** if they add to 180° (or π rad).

So the complement of θ is $90° - \theta$, and its supplement is $180° - \theta$.

An angle measures a turn.

(diagram in the online version)

Example

Find the complement and supplement of 40°.

$$\text{Complement} = 90° - 40° = \mathbf{50°} \qquad \text{Supplement} = 180° - 40° = \mathbf{140°}$$

Remember: Only angles smaller than 90° have a complement. "**C**omplement" pairs with 90 (a **C**orner); "**S**upplement" pairs with 180 (a **S**traight line).

5.8 Going Deeper: Advanced Angle Ideas

Gear, Pulley, and Belt Chains

When two wheels are connected, one of two rules applies:

- **In contact** (meshed gears, or a belt/pulley) they share the same *linear* speed at the rim: $v_1 = v_2$, so $r_1\omega_1 = r_2\omega_2$.
- **On a common axle** (rigidly fixed together) they share the same *angular* speed: $\omega_1 = \omega_2$.

To trace a chain, alternate these rules link by link. For meshed gears the tooth counts obey $N_1\omega_1 = N_2\omega_2$, since teeth are equally spaced around the rim.

An angle measures a turn.

(diagram in the online version)

Two meshed wheels touch at the rim, so their rim (linear) speeds match: $r_1\omega_1 = r_2\omega_2$.

Example: A Two-Stage Gear Train

Gear A ($r_A = 8$ cm) meshes with gear B ($r_B = 3$ cm). Fixed rigidly to gear B on the same axle is gear C ($r_C = 6$ cm), which meshes with gear D ($r_D = 2$ cm). Gear A turns at 30 RPM. Find the angular speed of gear D.

First convert: $\omega_A = 30 \times 2\pi = 60\pi$ rad/min.

A–B in contact: $r_A\omega_A = r_B\omega_B \Rightarrow \omega_B = \frac{8}{3}(60\pi) = 160\pi$ rad/min.

B–C on one axle: $\omega_C = \omega_B = 160\pi$ rad/min.

C–D in contact: $r_C\omega_C = r_D\omega_D \Rightarrow \omega_D = \frac{6}{2}(160\pi) = \mathbf{480}\pi$ rad/min.

As RPM: $\omega_D = \frac{480\pi}{2\pi} = \mathbf{240}$ RPM.

Clock-Hand Angles

A clock face is 360°, split into 12 hours of 30° each. The hands sweep at constant rates:

$$\text{minute hand: } 6^\circ \text{ per min}, \qquad \text{hour hand: } 0.5^\circ \text{ per min}.$$

At H:M the hands make angles (measured clockwise from 12)

$$\theta_{\text{min}} = 6M, \qquad \theta_{\text{hr}} = 30H + 0.5M.$$

The angle *between* them is $|\,30H - 5.5M\,|$; if this exceeds 180°, subtract from 360°.

The angle ϕ between the hour and minute hands.

Example: Angle at 3:40, and When Hands Overlap

(a) Angle at 3:40. Use $|\,30H - 5.5M\,|$ with $H = 3,\ M = 40$:

$$|\,30(3) - 5.5(40)\,| = |\,90 - 220\,| = 130^\circ.$$

Since $130^\circ < 180^\circ$, the angle between the hands is $\mathbf{130}^\circ$.

(b) Overlaps. The hands coincide when $6M = 30H + 0.5M$, i.e. $5.5M = 30H$, so $M = \frac{60H}{11}$. Starting from 12:00, successive overlaps are spaced

$$\Delta t = \frac{360^\circ}{6^\circ - 0.5^\circ \text{ per min}} = \frac{360}{5.5} = \frac{\mathbf{720}}{\mathbf{11}} \approx \mathbf{65.45} \text{ min}$$

apart, giving 11 overlaps every 12 hours (not 12). Equivalently the minute hand gains a full

$360°$ on the hour hand every $\dfrac{360}{11}$ of an hour.

Sector Systems with Two Solutions

A sector is fixed by any two of $\{r,\ \theta,\ s,\ A\}$. Because arc length is linear in r but area is quadratic in r, combining an arc (or perimeter) condition with an area condition often yields a *quadratic*, hence **two** valid sectors. The sector **perimeter** is

$$P = s + 2r = r\theta + 2r,$$

the arc plus the two radii. Pair this with $A = \frac{1}{2}r^2\theta$ to solve.

Example: Two Sectors, Same Perimeter and Area Data

A sector has perimeter $P = 16$ and area $A = 15$. Find the possible radii and central angles.
From $A = \frac{1}{2}r^2\theta = 15$ we get $r\theta = \dfrac{30}{r}$, so the arc $s = r\theta = \dfrac{30}{r}$.
Substitute into $P = s + 2r = 16$:

$$\frac{30}{r} + 2r = 16 \implies 2r^2 - 16r + 30 = 0 \implies r^2 - 8r + 15 = 0 \implies (r-3)(r-5) = 0.$$

So $r = 3$ or $r = 5$.
If $r = 3$: $\theta = \dfrac{30}{r^2} = \dfrac{30}{9} = \mathbf{\dfrac{10}{3}}$ rad (about $191°$).
If $r = 5$: $\theta = \dfrac{30}{25} = \mathbf{\dfrac{6}{5}}$ rad (about $69°$).
Both are genuine sectors, so the data admits **two** solutions.

Revolutions per Minute and Unit Conversions

Rotating machinery is usually rated in **RPM** (revolutions per minute). Convert deliberately, one factor at a time:

$$\text{RPM} \xrightarrow{\times 2\pi} \text{rad/min} \xrightarrow{\times r} \text{(rim) distance/min} \xrightarrow{\div 60} \text{per second.}$$

For ground speed of a rolling wheel, distance per revolution is the circumference $2\pi r$. Watch that every quantity uses the same length and time units before combining.

Example: Satellite Ground Track and Wheel Speed

(a) Satellite. A satellite completes one orbit of radius $r = 7000$ km every 100 minutes. Its angular speed is

$$\omega = \frac{2\pi}{100} = \frac{\pi}{50}\ \text{rad/min}, \qquad v = r\omega = 7000 \cdot \frac{\pi}{50} = \mathbf{140}\pi \approx \mathbf{439.8}\ \text{km/min}.$$

(b) Wheel. A car tire of radius 0.35 m turns at 500 RPM. Distance per minute is

$$v = (2\pi r) \times \text{RPM} = 2\pi(0.35)(500) = 350\pi \text{ m/min}.$$

Converting to km/h: $350\pi \, \dfrac{\text{m}}{\text{min}} \times \dfrac{60 \text{ min}}{1 \text{ h}} \times \dfrac{1 \text{ km}}{1000 \text{ m}} = 21\pi \approx \mathbf{66.0}$ km/h.

Coterminal with a condition: to find the angle coterminal with θ that lands in a required window, add or subtract full turns until it fits. For example, the value of $\frac{29\pi}{6}$ in $[0, 2\pi)$ is $\frac{29\pi}{6} - 2 \cdot 2\pi = \frac{29\pi - 24\pi}{6} = \frac{5\pi}{6}$. To force a *specific quadrant* or a $[-\pi, \pi)$ range, keep the same $\pm 2\pi k$ step and stop in the target interval.

Master trap: $s = r\theta$, $A = \frac{1}{2}r^2\theta$, and $v = r\omega$ all demand *radians* (and radians/time). RPM and degrees must be converted *before* they enter these formulas, never after.

6 Angles & Their Measure

7 Right Triangle Trigonometry

The six ratios, special angles, solving triangles, cofunctions, and applications

7.1 The Six Trigonometric Ratios (SOH-CAH-TOA)

Rounding conventions used in this topic: unless a problem asks for an *exact* value, round side lengths to the nearest hundredth (0.01) and angle measures to the nearest hundredth of a degree (0.01°). Keep full precision in your calculator until the final step, and make sure your calculator is in **degree mode**.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Definitions from a right triangle

For an acute angle θ in a right triangle, label the sides relative to θ: the **opposite** leg (across from θ), the **adjacent** leg (next to θ, not the hypotenuse), and the **hypotenuse** (across from the right angle). Then

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

Remember the mnemonic **SOH-CAH-TOA**: **S**in = **O**pp/**H**yp, **C**os = **A**dj/**H**yp, **T**an = **O**pp/**A**dj.

Example: all six ratios of a 3-4-5 triangle

A right triangle has legs 3 (opposite θ) and 4 (adjacent to θ) and hypotenuse 5.

$$\sin\theta = \tfrac{3}{5}, \quad \cos\theta = \tfrac{4}{5}, \quad \tan\theta = \tfrac{3}{4}$$

$$\csc\theta = \tfrac{5}{3}, \quad \sec\theta = \tfrac{5}{4}, \quad \cot\theta = \tfrac{4}{3}$$

The three ratios on the bottom row are the reciprocals of the top row.

Example: find the missing side first

A right triangle has opposite = 5 and hypotenuse = 13. Find $\cos\theta$. By the Pythagorean theorem the adjacent side is $\sqrt{13^2 - 5^2} = \sqrt{144} = 12$, so $\cos\theta = \dfrac{12}{13}$ and $\tan\theta = \dfrac{5}{12}$.

Reciprocal identities: $\csc\theta = \dfrac{1}{\sin\theta}, \quad \sec\theta = \dfrac{1}{\cos\theta}, \quad \cot\theta = \dfrac{1}{\tan\theta}$. Also $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

7.2 Exact Values of the Special Angles $30°, 45°, 60°$

The two special triangles

Every special-angle value comes from two triangles. The 45°-45°-90° triangle has legs $1, 1$ and hypotenuse $\sqrt{2}$. The 30°-60°-90° triangle has sides 1 (short leg, opposite 30°), $\sqrt{3}$ (long leg, opposite 60°), and 2 (hypotenuse).

Table of exact values

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Reciprocal functions come from flipping these: e.g. $\csc 30° = 2$, $\sec 60° = 2$, $\cot 45° = 1$.

Example: exact evaluation

Evaluate $\sin 30° \cos 60° + \cos 30° \sin 60°$.

$$\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) + \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{1}{4} + \tfrac{3}{4} = 1.$$

Tip: $\tan 30° = \dfrac{1}{\sqrt{3}}$ is usually written with a rational denominator as $\dfrac{\sqrt{3}}{3}$. As θ grows from $0°$ to $90°$, $\sin\theta$ increases and $\cos\theta$ decreases, so their values "cross" at $45°$.

7.3 Calculator Values and Solving Right Triangles

Using a calculator

For values that are not special angles, use your calculator (in **degree mode**). To find a side, multiply by a trig value; to find an angle, use the inverse keys $\sin^{-1}, \cos^{-1}, \tan^{-1}$. **Solving a right triangle** means finding all three sides and all three angles. You need one side plus either one more side or one acute angle.

Example: given an angle and the hypotenuse

In a right triangle the right angle is at C; angle $A = 35°$ and hypotenuse $c = 10$.

- $B = 90° - 35° = 55°$.
- $a = c \sin A = 10 \sin 35° \approx 5.74$ (opposite A).
- $b = c \cos A = 10 \cos 35° \approx 8.19$ (adjacent to A).

Example: given two sides, find an angle

A right triangle has legs $a = 9$ and $b = 12$. The hypotenuse is $c = \sqrt{9^2 + 12^2} = \sqrt{225} = 15$. Then $\tan A = \dfrac{9}{12} = 0.75$, so $A = \tan^{-1}(0.75) \approx 36.87°$ and $B \approx 53.13°$.

Tip: choose the ratio that uses the side you are *given* and the side you *want*. Use sin / csc with opposite & hypotenuse, cos / sec with adjacent & hypotenuse, tan / cot with the two legs.

7.4 Cofunctions and the Complementary-Angle Relationship

Cofunction identities

In a right triangle the two acute angles are **complementary** (they sum to $90°$). The side opposite one angle is adjacent to the other, so

$$\sin\theta = \cos(90° - \theta), \qquad \tan\theta = \cot(90° - \theta), \qquad \sec\theta = \csc(90° - \theta),$$

and the same with the pairs reversed. "Co" functions (**co**sine, **co**tangent, **co**secant) are the functions of the complementary angle.

Example: rewrite and solve

Since $\sin 20° = \cos 70°$, we can fill in $\sin 20° = \cos(\underline{\ 70°\ })$. To solve $\sin(x) = \cos(x + 10°)$, set the angles complementary: $x + (x + 10°) = 90°$, so $2x = 80°$ and $x = 40°$.

Tip: a cofunction equation such as $\sin A = \cos B$ is true exactly when $A + B = 90°$. This lets you solve for an unknown angle without a calculator.

7.5 Applications: Elevation, Depression, and Bearings

Angles of elevation and depression

An **angle of elevation** is measured *upward* from a horizontal line to a line of sight; an **angle of depression** is measured *downward* from a horizontal line to a line of sight. Because the two horizontal lines are parallel, the angle of depression from the top equals the angle of elevation from the bottom (alternate interior angles).

Bearings

A **bearing** such as N60°E names a direction by an acute angle measured from the north-south line toward the east or west. Resolve a displacement of length d on bearing NαE into a north component $d\cos\alpha$ and an east component $d\sin\alpha$.

Example: angle of elevation

From a point 50 ft from the base of a building, the angle of elevation to the top is $40°$. The height is

$$h = 50\tan 40° \approx 50(0.8391) \approx 41.96 \text{ ft}.$$

Example: angle of depression

From the top of a 30 m cliff, the angle of depression to a boat is 35°. The horizontal distance from the base of the cliff to the boat is

$$x = \frac{30}{\tan 35^\circ} \approx \frac{30}{0.7002} \approx 42.84 \text{ m}.$$

Tip for two-triangle problems: draw one picture, label the unknown height h and set up one equation per triangle. Often both equations contain h; subtract or substitute to eliminate it.

7.6 Going Deeper: Advanced Right-Triangle Trig

How to work these: advanced problems almost always hide *two* right triangles that share a side. Draw one clean picture, name every point, and write one equation per triangle. Keep exact values (simplified radicals) until the last step, then round lengths to 0.01 and angles to 0.01°.

Two observers / two towers (shared height)

When two observers on the same horizontal line sight the *same* point, both right triangles share the vertical height h. Write each horizontal distance in terms of h using tan, then use the known separation to eliminate h.

- **Observers on opposite sides**, separation D, elevations α and β:

$$\frac{h}{\tan\alpha} + \frac{h}{\tan\beta} = D.$$

- **Observers on the same side**, one behind the other by D:

$$\frac{h}{\tan\alpha} - \frac{h}{\tan\beta} = D \quad (\alpha < \beta).$$

Factor out h and divide. The subtraction case is the usual "walk toward the tower" setup.

Example: a tower between two observers (exact then rounded)

A vertical tower stands between two observers A and B who are 300 m apart on level ground. From A the angle of elevation to the top is 30°; from B it is 45°. Find the height h.

$$\frac{h}{\tan 30^\circ} + \frac{h}{\tan 45^\circ} = 300 \implies h\sqrt{3} + h = 300 \implies h(\sqrt{3}+1) = 300.$$

Rationalize by multiplying by $\frac{\sqrt{3}-1}{\sqrt{3}-1}$:

$$h = \frac{300}{\sqrt{3}+1} = \frac{300(\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{300(\sqrt{3}-1)}{2} = 150(\sqrt{3}-1) \approx 109.81 \text{ m.}$$

3D right-triangle problems

A three-dimensional figure is solved by finding a sequence of right triangles that lie in different planes but share an edge. A common tool is the **space diagonal** of a box with edges a, b, c: first the base diagonal is $d = \sqrt{a^2 + b^2}$, then the space diagonal is $\sqrt{d^2 + c^2} = \sqrt{a^2 + b^2 + c^2}$. The angle the space diagonal makes with the base satisfies $\tan\phi = \frac{c}{d} = \frac{c}{\sqrt{a^2 + b^2}}$.

Example: angle of a box's space diagonal with its base

A rectangular box has base edges $a = 6$ and $b = 8$ and height $c = 5$. The base diagonal is

$$d = \sqrt{6^2 + 8^2} = \sqrt{100} = 10.$$

The space diagonal is $\sqrt{10^2 + 5^2} = \sqrt{125} = 5\sqrt{5} \approx 11.18$. The angle ϕ it makes with the base is

$$\phi = \tan^{-1}\frac{c}{d} = \tan^{-1}\frac{5}{10} = \tan^{-1}(0.5) \approx 26.57°.$$

The altitude-on-hypotenuse geometric mean

Drop the altitude of length h from the right angle to the hypotenuse. It splits the hypotenuse into segments p and q and creates two smaller triangles, each similar to the original. This gives three **geometric-mean** relations:

$$h = \sqrt{pq}, \qquad (\text{leg}_1) = \sqrt{p\,(p+q)}, \qquad (\text{leg}_2) = \sqrt{q\,(p+q)}.$$

In words: the altitude is the geometric mean of the two hypotenuse pieces, and each leg is the geometric mean of the whole hypotenuse and the piece adjacent to that leg.

Example: geometric mean on the hypotenuse

The altitude from the right angle meets the hypotenuse, cutting it into pieces $p = 4$ and $q = 9$. Then

$$h = \sqrt{4 \cdot 9} = \sqrt{36} = 6, \qquad \text{leg}_1 = \sqrt{4 \cdot 13} = 2\sqrt{13} \approx 7.21, \qquad \text{leg}_2 = \sqrt{9 \cdot 13} = 3\sqrt{13} \approx 10.82.$$

Check: $\text{leg}_1^2 + \text{leg}_2^2 = 52 + 117 = 169 = 13^2$, the square of the full hypotenuse $p + q = 13$.

Bearings and navigation with two legs

Give each leg of a trip as a bearing (measured from north) and a length. Resolve every leg into north and east components ($d\cos$ for the N-S part, $d\sin$ for the E-W part). When two consecutive bearings differ by exactly $90°$, the two legs are perpendicular, so the start-to-finish distance is just $\sqrt{d_1^2 + d_2^2}$ and the turn angle is $\tan^{-1}(d_2/d_1)$ measured from the first leg.

Example: two perpendicular legs

A ship sails 12 km on bearing N40°E, then turns and sails 8 km on bearing S50°E. The two bearings are 040° and 130°, which differ by 90°, so the legs are perpendicular. The direct distance from start to finish is

$$\sqrt{12^2 + 8^2} = \sqrt{208} = 4\sqrt{13} \approx 14.42 \text{ km}.$$

The course swings clockwise from the first leg by $\tan^{-1}\frac{8}{12} = \tan^{-1}(0.6\overline{6}) \approx 33.69°$, so the final bearing from the start is about $40° + 33.69° = 73.69°$, i.e. N73.69°E.

Right triangles inside inscribed figures

Two facts turn geometry problems into right-triangle trig:

- **Thales' theorem:** any triangle inscribed in a semicircle, with the diameter as one side, has its right angle on the circle. So a diameter-based inscribed triangle is automatically right.
- **Regular n-gon in a circle of radius R:** draw radii to two adjacent vertices and the apothem to the midpoint of that side. The half-central-angle is $\frac{180°}{n}$, giving

$$\text{half-side} = R\sin\frac{180°}{n}, \qquad \text{apothem} = R\cos\frac{180°}{n}.$$

Optimizing a viewing angle

A picture (or screen) hangs on a wall with its bottom edge a above eye level and its top edge b above eye level. A viewer standing x from the wall sees the picture within the viewing angle

$$\theta(x) = \tan^{-1}\frac{b}{x} - \tan^{-1}\frac{a}{x}.$$

The angle is small when x is very small (you look almost straight up) or very large (the picture shrinks), so a best distance sits in between. Calculus shows θ is largest at

$$x = \sqrt{ab},$$

the geometric mean of the two heights. This is another place the geometric mean appears in right-triangle work.

Exact-value chains and cofunction tricks: cofunctions collapse long products. Because $\tan\theta\,\tan(90^\circ-\theta)=\tan\theta\,\cot\theta=1$, a symmetric product telescopes, e.g.

$$\tan 1^\circ\,\tan 2^\circ\cdots\tan 89^\circ=1,$$

by pairing each factor $\tan k^\circ$ with $\tan(90^\circ-k^\circ)$; the lone middle term $\tan 45^\circ=1$. Likewise $\sin^2\theta+\sin^2(90^\circ-\theta)=\sin^2\theta+\cos^2\theta=1$. Look for complementary pairs before reaching for a calculator.

8 The Unit Circle and Circular Functions

Special angles, reference angles, signs, periodicity, even/odd, and terminal points

8.1 The Unit Circle and the Six Functions

Definition from a point on the circle

The **unit circle** is the circle $x^2+y^2=1$ centered at the origin with radius 1. If the terminal side of an angle θ (measured counterclockwise from the positive x-axis) meets the unit circle at the point (x,y), then

$$x=\cos\theta,\qquad y=\sin\theta.$$

The other four functions are built from these:

$$\tan\theta=\frac{y}{x}=\frac{\sin\theta}{\cos\theta},\quad \cot\theta=\frac{x}{y}=\frac{\cos\theta}{\sin\theta},\quad \sec\theta=\frac{1}{x}=\frac{1}{\cos\theta},\quad \csc\theta=\frac{1}{y}=\frac{1}{\sin\theta}.$$

Worked example: reading all six functions off a point

The terminal side of θ meets the unit circle at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Find all six functions.

Here $x=-\frac{1}{2}$ and $y=\frac{\sqrt{3}}{2}$, so

$$\cos\theta=-\tfrac{1}{2},\qquad \sin\theta=\tfrac{\sqrt{3}}{2},\qquad \tan\theta=\frac{\sqrt{3}/2}{-1/2}=-\sqrt{3}.$$

Reciprocals:

$$\sec\theta=\frac{1}{-1/2}=-2,\qquad \csc\theta=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3},\qquad \cot\theta=\frac{1}{-\sqrt{3}}=-\frac{\sqrt{3}}{3}.$$

Tip. On the unit circle the radius is 1, so there is no dividing by r: the coordinate *is* the value. Cosine is the x-coordinate, sine is the y-coordinate. "Co-sine goes with the horizontal."

8.2 Exact Values at the Special Angles (All Four Quadrants)

The special-angle circle

The special angles are multiples of $30° = \frac{\pi}{6}$ and $45° = \frac{\pi}{4}$. Their coordinates use only $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$. Learn Quadrant I, then attach the correct signs in the other quadrants.

Worked example: values in Quadrants II and III

Evaluate $\cos 150°$ and $\sin \dfrac{5\pi}{4}$.

$150°$ is in Quadrant II, where cosine is negative, and it corresponds to the point $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so $\cos 150° = -\dfrac{\sqrt{3}}{2}$.

$\dfrac{5\pi}{4} = 225°$ is in Quadrant III at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, so $\sin \dfrac{5\pi}{4} = -\dfrac{\sqrt{2}}{2}$.

Tip. At the *quadrantal* angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ one coordinate is 0: watch for undefined values ($\tan \frac{\pi}{2}$ and $\sec \frac{\pi}{2}$ are undefined because $x = 0$; $\cot 0$ and $\csc 0$ are undefined because $y = 0$).

8.3 Reference Angles and Signs by Quadrant

Reference angle

The **reference angle** θ' is the *acute* angle between the terminal side and the x-axis. A special angle and its reference angle share the same exact value up to sign.

$$\text{Q I: } \theta' = \theta, \qquad \text{Q II: } \theta' = 180° - \theta, \qquad \text{Q III: } \theta' = \theta - 180°, \qquad \text{Q IV: } \theta' = 360° - \theta.$$

In radians replace $180°$ with π and $360°$ with 2π.

Worked example: reference angle then sign

Evaluate $\sin 210°$.

$210°$ is in Quadrant III, so its reference angle is $210° - 180° = 30°$. The reference value is

$\sin 30° = \frac{1}{2}$. In Quadrant III sine is *negative*, so

$$\sin 210° = -\tfrac{1}{2}.$$

Remember: ASTC ("**A**ll **S**tudents **T**ake **C**alculus"), reading Quadrants I, II, III, IV. It tells you which functions are *positive*; every other function is negative there. A function and its reciprocal always share the same sign.

8.4 Domain, Range, and Periodicity

The six circular functions at a glance

Here n is any integer. **Periodic** means $f(\theta + P) = f(\theta)$ for the period P.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: using periodicity to reduce a large angle

Evaluate $\cos \dfrac{13\pi}{6}$.

Since cosine has period $2\pi = \dfrac{12\pi}{6}$, subtract one full turn:

$$\frac{13\pi}{6} - 2\pi = \frac{13\pi}{6} - \frac{12\pi}{6} = \frac{\pi}{6}.$$

Therefore $\cos \dfrac{13\pi}{6} = \cos \dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$.

Tip. Tangent and cotangent repeat every π, not 2π. To simplify $\tan\theta$ for a huge angle, subtract multiples of π; for the other four functions subtract multiples of 2π.

8.5 Even and Odd (Symmetry) Properties

Even and odd functions

Reflecting an angle to $-\theta$ reflects the point (x, y) across the x-axis to $(x, -y)$. So cosine (the

x-coordinate) is unchanged, while sine (the y-coordinate) flips sign:

$$\underbrace{\cos(-\theta) = \cos\theta, \quad \sec(-\theta) = \sec\theta,}_{\text{even}} \qquad \underbrace{\sin(-\theta) = -\sin\theta, \quad \csc(-\theta) = -\csc\theta, \quad \tan(-\theta) = -\tan\theta, \quad \cot(-\theta) = -}_{\text{odd}}$$

Worked example: applying even/odd

Evaluate $\sin\left(-\frac{\pi}{6}\right)$ and simplify $\cos(-\theta)\csc(-\theta)$.

Sine is odd: $\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$.
Cosine is even and cosecant is odd, so

$$\cos(-\theta)\csc(-\theta) = \cos\theta \cdot (-\csc\theta) = -\frac{\cos\theta}{\sin\theta} = -\cot\theta.$$

Tip. Only cosine and secant are even. The other four are odd, so a negative angle just pulls a minus sign out front.

8.6 Evaluating from a Point on the Terminal Side

When the point is not on the unit circle

If the terminal side passes through *any* point $(x, y) \neq (0, 0)$, let

$$r = \sqrt{x^2 + y^2} > 0.$$

Then
$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}.$$
The unit circle is the special case $r = 1$.

Worked example: all six from a terminal point

The terminal side of θ passes through $(-3, 4)$. Find all six functions.

First $r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$. The point is in Quadrant II. Then

$$\sin\theta = \frac{4}{5}, \quad \cos\theta = -\frac{3}{5}, \quad \tan\theta = \frac{4}{-3} = -\frac{4}{3},$$

$$\csc\theta = \frac{5}{4}, \quad \sec\theta = -\frac{5}{3}, \quad \cot\theta = -\frac{3}{4}.$$

The signs match Quadrant II: only sine and cosecant are positive.

Tip. Always take r *positive*. All the sign information comes from the signs of x and y (that is, from the quadrant). If an answer has a radical in the denominator, rationalize it, e.g. $\dfrac{2}{\sqrt{13}} = \dfrac{2\sqrt{13}}{13}$.

8.7 Going Deeper: Advanced Unit-Circle Ideas

Combining exact values: products, sums, and powers

Once the special-angle coordinates are memorized, any *expression* built from them is just arithmetic with the numbers 0, $\frac{1}{2}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{3}}{2}$, 1 and their reciprocals. Two cautions:

- The power notation means "square the value," so $\sin^2\theta = (\sin\theta)^2$. For example $\sin^2\frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$.
- Evaluate each function *first* (with its correct sign), then multiply, add, or take reciprocals. A single wrong sign changes the whole answer.

A handy check is the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, which must hold at every angle.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: an unusual product-and-sum

Evaluate

$$E = \tan\frac{\pi}{3}\cot\frac{\pi}{6} \;-\; \csc^2\frac{\pi}{4} \;+\; \cos\pi.$$

Take the pieces one at a time:

$$\tan\frac{\pi}{3} = \sqrt{3}, \qquad \cot\frac{\pi}{6} = \sqrt{3}, \qquad \csc\frac{\pi}{4} = \sqrt{2}, \qquad \cos\pi = -1.$$

So $\tan\frac{\pi}{3}\cot\frac{\pi}{6} = \sqrt{3}\cdot\sqrt{3} = 3$ and $\csc^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$. Therefore

$$E = 3 - 2 + (-1) = 0.$$

Equally spaced angles: the cosines (and sines) sum to zero

If you place $n \geq 2$ points evenly around the unit circle, their x-coordinates cancel and so do their y-coordinates:

$$\sum_{k=0}^{n-1} \cos\left(\theta + \frac{2\pi k}{n}\right) = 0, \qquad \sum_{k=0}^{n-1} \sin\left(\theta + \frac{2\pi k}{n}\right) = 0.$$

Geometrically the tips of the n equally spaced unit vectors form a regular polygon centered at the origin, so the vectors add to the zero vector; the two coordinate sums are just the horizontal and vertical parts of that fact.

Worked example: three equally spaced angles

Evaluate $\cos 0 + \cos \dfrac{2\pi}{3} + \cos \dfrac{4\pi}{3}$.

These are $n = 3$ equally spaced angles (120° apart). Reading the coordinates,

$$\cos 0 = 1, \qquad \cos \frac{2\pi}{3} = -\frac{1}{2}, \qquad \cos \frac{4\pi}{3} = -\frac{1}{2},$$

so the sum is

$$1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right) = 0,$$

exactly as the boxed rule predicts. The matching sine sum is $0 + \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{2}\right) = 0$ as well.

Reducing a huge or negative angle. To evaluate $f(\theta)$ for a giant or negative θ, add or subtract whole periods until you land in $[0, 2\pi)$:

$$\theta_{\text{reduced}} = \theta - 2\pi \left\lfloor \frac{\theta}{2\pi} \right\rfloor \qquad \text{(use period } \pi \text{ for } \tan, \cot\text{)}.$$

In practice: keep adding $2\pi = \frac{12\pi}{6}$ (or subtracting it) until the angle is a familiar special angle. Coterminal angles have identical function values.

Worked example: a large negative angle

Evaluate $\sin\left(-\dfrac{17\pi}{6}\right)$.

Add full turns of $2\pi = \dfrac{12\pi}{6}$ until the angle is in $[0, 2\pi)$:

$$-\frac{17\pi}{6} + \frac{12\pi}{6} = -\frac{5\pi}{6}, \qquad -\frac{5\pi}{6} + \frac{12\pi}{6} = \frac{7\pi}{6}.$$

Now $\dfrac{7\pi}{6} = 210^\circ$ is in Quadrant III with reference angle $\dfrac{\pi}{6}$, and sine is negative there, so

$$\sin\left(-\frac{17\pi}{6}\right) = \sin \frac{7\pi}{6} = -\frac{1}{2}.$$

(Check with the odd property: $\sin\left(-\frac{17\pi}{6}\right) = -\sin \frac{17\pi}{6} = -\sin \frac{5\pi}{6} = -\frac{1}{2}$.)

One value plus a sign condition (the twist)

A classic problem gives *one* function value together with a sign clue instead of a named quadrant, and often the value is a *reciprocal* function. Strategy:

1. Convert to $\sin\theta$ or $\cos\theta$ if you were handed $\sec, \csc, \cot$, or $\tan$.
2. Use $\sin^2\theta + \cos^2\theta = 1$ to get the partner coordinate, keeping it as a radical.
3. Let the two sign clues (or the quadrant) fix the signs of $x = \cos\theta$ and $y = \sin\theta$.
4. Build the remaining functions as quotients and reciprocals; rationalize denominators.

Worked example: all six from $\sec\theta = -3$ with $\sin\theta > 0$

The reciprocal relation gives $\cos\theta = \dfrac{1}{\sec\theta} = -\dfrac{1}{3}$. Since $\cos\theta < 0$ and $\sin\theta > 0$, the angle is in **Quadrant II**. From the Pythagorean identity,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{9} = \frac{8}{9},$$

$$\sin\theta = +\sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \quad \text{(positive in Q II)}.$$

The rest follow:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{2}/3}{-1/3} = -2\sqrt{2}, \qquad \cot\theta = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4},$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}, \qquad \sec\theta = -3, \qquad \cos\theta = -\frac{1}{3}.$$

Signs check against Quadrant II: only $\sin\theta$ and $\csc\theta$ are positive.

Worked example: simplifying a long even/odd expression

Simplify, for a general angle θ,

$$\frac{\sin(-\theta)\,\sec(-\theta)}{\tan(-\theta)} + \cos(-\theta).$$

Pull each negative sign through using even/odd: $\sin(-\theta) = -\sin\theta$, $\sec(-\theta) = \sec\theta$, $\tan(-\theta) = -\tan\theta$, and $\cos(-\theta) = \cos\theta$. The fraction becomes

$$\frac{(-\sin\theta)(\sec\theta)}{-\tan\theta} = \frac{-\dfrac{\sin\theta}{\cos\theta}}{-\tan\theta} = \frac{-\tan\theta}{-\tan\theta} = 1.$$

Adding the last term,

$$\frac{\sin(-\theta)\,\sec(-\theta)}{\tan(-\theta)} + \cos(-\theta) = 1 + \cos\theta.$$

Two odd factors (sin and tan) and one flipped sign in the denominator cancel, leaving no minus signs.

Parameterized terminal points

Writing $P(\theta) = (\cos\theta,\ \sin\theta)$ turns the unit circle into a machine: feed in the angle, read out the terminal point. Rotations and reflections of the angle move the point in predictable ways:

$$P(-\theta) = (\cos\theta,\, -\sin\theta), \quad P(\theta+\pi) = (-\cos\theta,\, -\sin\theta), \quad P\big(\theta + \tfrac{\pi}{2}\big) = (-\sin\theta,\, \cos\theta).$$

So if you know one terminal point, you instantly know the antipodal point $(+\pi)$, the reflection across the x-axis $(-\theta)$, and the quarter-turn $(+\frac{\pi}{2})$—no new radicals required.

Worked example: reusing one terminal point

The terminal point of θ on the unit circle is $\left(\frac{\sqrt{5}}{3}, \frac{2}{3}\right)$. (It is valid because $\left(\frac{\sqrt{5}}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9} + \frac{4}{9} = 1$.) Find the terminal points of $\theta + \pi$ and $-\theta$, and evaluate $\cos(\theta+\pi)$.

Using the parameterization rules,

$$P(\theta+\pi) = \left(-\frac{\sqrt{5}}{3}, -\frac{2}{3}\right), \qquad P(-\theta) = \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\right).$$

Because cosine is the x-coordinate of $P(\theta+\pi)$,

$$\cos(\theta+\pi) = -\cos\theta = -\frac{\sqrt{5}}{3}.$$

9 Graphs of Sine and Cosine

Amplitude, period, phase shift, vertical shift, and modeling

9.1 The Basic Graphs $y = \sin x$ and $y = \cos x$

The parent sine and cosine curves

Both $y = \sin x$ and $y = \cos x$ are **periodic** with **period** 2π: the graph repeats every 2π units. Each has **amplitude** 1, **midline** $y = 0$, maximum value 1, and minimum value -1.

- $y = \sin x$ starts at the midline: $(0, 0)$, rises to a max, returns, falls to a min, returns.

- $y = \cos x$ starts at a maximum: $(0, 1)$, falls to the midline, to a min, and back up.

Five-point method: divide one period into four equal pieces. For $[0, 2\pi]$ the key x-values are

$$0, \quad \frac{\pi}{2}, \quad \pi, \quad \frac{3\pi}{2}, \quad 2\pi.$$

Evaluate the function at these five points, then connect with a smooth wave.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: graph $y = \sin x$ with the five-point method

Amplitude $= 1$, period $= 2\pi$, midline $y = 0$. Key points on $[0, 2\pi]$:

Tip: Cosine is just sine shifted $\frac{\pi}{2}$ to the left: $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. Remember "sine starts at the middle, cosine starts at the top."

9.2 Amplitude $|a|$ and Vertical Reflection

The role of a in $y = a \sin x$ and $y = a \cos x$

The number a stretches the graph vertically.

- **Amplitude** $= |a| = \dfrac{\max - \min}{2}$: the distance from the midline to a peak.
- Maximum value $= |a|$, minimum value $= -|a|$ (when midline is $y = 0$).
- If $a < 0$, the graph is **reflected across the x-axis** (flipped upside down).
- a does *not* change the period or the midline.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: $y = 3 \cos x$ and $y = -2 \sin x$

$y = 3 \cos x$: amplitude $= |3| = 3$, max 3, min -3, period 2π, midline $y = 0$.
$y = -2 \sin x$: amplitude $= |-2| = 2$; because $a = -2 < 0$ the sine wave is **reflected**, so it goes *down* first from $(0, 0)$.

Tip: A negative a never makes the amplitude negative. Amplitude is always $|a| \geq 0$; the sign only tells you whether the wave is flipped.

9.3 Period and the Coefficient b

Period $= \dfrac{2\pi}{b}$

In $y = a\sin(bx)$ or $y = a\cos(bx)$ (with $b > 0$), the coefficient b controls how fast the wave cycles:

$$\boxed{\text{Period} = \frac{2\pi}{b}} \qquad \text{so} \qquad b = \frac{2\pi}{\text{Period}}.$$

- $b > 1$ *compresses* the graph: more cycles, shorter period.
- $0 < b < 1$ *stretches* the graph: fewer cycles, longer period.

To find five key points, split *one period* into four equal steps of width $\frac{1}{4} \cdot \frac{2\pi}{b} = \frac{\pi}{2b}$.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: $y = \sin 2x$

Here $b = 2$, so period $= \dfrac{2\pi}{2} = \pi$. The wave completes a full cycle in π, i.e. two cycles on $[0, 2\pi]$. Key x-values, stepping by $\frac{\pi}{4}$: 0, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π.

Tip: Amplitude and period are independent. In $y = 4\sin 2x$ the amplitude is 4 and the period is still $\dfrac{2\pi}{2} = \pi$ — the 4 has no effect on the period.

9.4 Phase (Horizontal) Shift and Vertical Shift

Shifting the graph: $y = a\sin(bx - c) + d$

Two more numbers slide the whole graph:

- **Vertical shift** d: raises $(d > 0)$ or lowers $(d < 0)$ the graph. The new **midline** is $y = d$; $\max = d + |a|$, $\min = d - |a|$.
- **Phase (horizontal) shift** $= \dfrac{c}{b}$: found by setting the inside equal to zero, $bx - c = 0 \Rightarrow$

$x = \dfrac{c}{b}$. A *positive* result shifts **right**, negative shifts **left**.

Slope is rise over run.

(diagram in the online version)

Worked example: $y = \sin\left(x - \frac{\pi}{2}\right)$ and $y = \cos x + 2$

$y = \sin\left(x - \frac{\pi}{2}\right)$: set $x - \frac{\pi}{2} = 0 \Rightarrow x = \frac{\pi}{2}$, so shift $\frac{\pi}{2}$ to the **right**. (It coincides with $-\cos x$.) $y = \cos x + 2$: no horizontal shift; $d = 2$ moves the midline up to $y = 2$, so it swings between 1 and 3.

Tip: Always *factor out* b before reading the phase shift: $\sin(2x - \pi) = \sin\left(2(x - \frac{\pi}{2})\right)$ shows the shift is $\frac{\pi}{2}$, *not* π. Equivalently, phase shift $= \dfrac{c}{b}$.

9.5 Graphing the Full Sinusoid $y = a\sin(bx - c) + d$

Five steps for any sinusoid

1. **Amplitude** $= |a|$. 2. **Period** $= \dfrac{2\pi}{b}$. 3. **Phase shift** $= \dfrac{c}{b}$ (right if $+$).

4. **Midline** $y = d$; max $= d + |a|$, min $= d - |a|$.

5. Start the cycle at $x = \dfrac{c}{b}$, then plot five points a quarter-period $\dfrac{2\pi}{4b}$ apart.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: $y = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$

$a = 2,\ b = 2,\ c = \frac{\pi}{2},\ d = 1.$

- Amplitude $= |2| = 2$; Period $= \dfrac{2\pi}{2} = \pi$.
- Phase shift $= \dfrac{c}{b} = \dfrac{\pi/2}{2} = \dfrac{\pi}{4}$ to the **right**.
- Midline $y = 1$; max $= 1 + 2 = 3$; min $= 1 - 2 = -1$.
- Cycle starts at $x = \frac{\pi}{4}$; quarter-period $= \dfrac{\pi}{4}$. Key points at $x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}$ give $y = 1, 3, 1, -1, 1$.

Tip: The vertical shift d and midline are the "sea level" of the wave. Find max/min by adding and subtracting the amplitude from d: $\max = d + |a|$, $\min = d - |a|$.

9.6 Writing an Equation from a Graph or Description

Reverse-engineering a sinusoid

Given a graph or a word description, recover a, b, c, d:

1. $d = \dfrac{\max + \min}{2}$ (midline).
2. $|a| = \dfrac{\max - \min}{2}$ (amplitude).
3. $b = \dfrac{2\pi}{\text{Period}}$.
4. Choose sine or cosine to match the starting behavior, then set c from the horizontal shift. A **maximum** at $x = x_0$ suggests $y = a\cos\big(b(x - x_0)\big) + d$. A midline point rising suggests sine.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: write an equation

A wave has maximum 7, minimum -1, period π, and a *maximum* at $x = 0$.
$d = \dfrac{7 + (-1)}{2} = 3$; $\quad |a| = \dfrac{7 - (-1)}{2} = 4$; $\quad b = \dfrac{2\pi}{\pi} = 2$.
A maximum at $x = 0$ with no shift matches cosine: $\boxed{y = 4\cos(2x) + 3}$.

Worked example: modeling with a Ferris wheel

A Ferris wheel has radius 20 m; its center is 22 m above the ground. It makes one revolution every 40 s and a rider boards at the *bottom* at $t = 0$. Model the height $h(t)$.
Amplitude $= 20$ (the radius); midline $d = 22$ (center height); period $= 40 \Rightarrow b = \dfrac{2\pi}{40} = \dfrac{\pi}{20}$.
Starting at the **bottom** (a minimum) matches $-\cos$:

$$\boxed{h(t) = -20\cos\left(\frac{\pi}{20}t\right) + 22}$$

Check: $h(0) = -20 + 22 = 2$ m (bottom); $h(20) = 20 + 22 = 42$ m (top). ✓

Modeling tip: In real problems, amplitude $= \frac{1}{2}(\text{high} - \text{low})$, midline $= \frac{1}{2}(\text{high} + \text{low})$, and the period is the time for one full cycle. Use $+\cos$ if it starts at the high point, $-\cos$ if it starts at the low point, and sin if it starts at the midline.

9.7 Going Deeper: Advanced Sine & Cosine Graphs

Combining two sinusoids: $a\sin x + b\cos x = R\sin(x+\phi)$

A sum of a sine and a cosine of the *same* frequency is **itself a single sinusoid**. For any constants a, b,

$$\boxed{a\sin x + b\cos x = R\sin(x+\phi)} \qquad R = \sqrt{a^2+b^2}, \quad \tan\phi = \frac{b}{a}.$$

- $R = \sqrt{a^2+b^2}$ is the **amplitude** of the combined wave (always ≥ 0).
- The phase angle ϕ satisfies $\cos\phi = \dfrac{a}{R}$ and $\sin\phi = \dfrac{b}{R}$; use *both* signs to place ϕ in the correct quadrant — do not trust arctan alone.
- You may also write it as $R\cos(x-\psi)$ with $\tan\psi = \dfrac{a}{b}$. Same curve, different bookkeeping.

This is why superposition of two oscillations (of equal period) never produces a new shape — only a shifted, rescaled sine.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: rewrite $3\sin x + 4\cos x$ **as a single sine**

Here $a = 3,\ b = 4$. Then $R = \sqrt{3^2+4^2} = \sqrt{25} = 5$, and $\cos\phi = \frac{3}{5}$, $\sin\phi = \frac{4}{5}$ (both positive, so ϕ is in the first quadrant):

$$\phi = \arctan\tfrac{4}{3} \approx 0.927 \text{ rad} \quad \Longrightarrow \quad 3\sin x + 4\cos x = 5\sin(x+0.927).$$

So the sum has amplitude 5, period 2π, and peaks a little *before* $x = \frac{\pi}{2}$ (shifted left by 0.927). The thick curve below is the single combined sinusoid; the two thin curves are its ingredients.

Tip: The combined amplitude $R = \sqrt{a^2+b^2}$ is never simply $a+b$. For $3\sin x + 4\cos x$ the peaks of the two pieces occur at *different* x-values, so they never add to $3+4=7$; the true maximum is $R=5$.

Max, min, and the period of a sum

Once a sinusoid is written as $y = A\sin(bx-c)+d$ (or found via R above), reading its extremes is mechanical:

- **Maximum** $= d + |A|$, reached when the inside angle $= \frac{\pi}{2}$; **minimum** $= d - |A|$, when the inside angle $= -\frac{\pi}{2}$ (or $\frac{3\pi}{2}$).
- **Solve for the location:** set $bx - c = \frac{\pi}{2} + 2\pi k$ and solve for x to find *where* the max occurs.

Period of a sum of *different* frequencies. $y = \sin(b_1 x) + \sin(b_2 x)$ is generally *not* a simple sinusoid. Its period is the **least common multiple** of the two periods $\frac{2\pi}{b_1}$ and $\frac{2\pi}{b_2}$ (when that LCM exists):

$$\text{Period}\big(\sin 2x + \sin 3x\big) = \text{lcm}\big(\tfrac{2\pi}{2}, \tfrac{2\pi}{3}\big) = \text{lcm}(\pi, \tfrac{2\pi}{3}) = 2\pi.$$

If the ratio b_1/b_2 is irrational, the sum is **not periodic at all**.

Intersections and counting solutions on an interval

To find where two graphs meet, or how many times an equation is satisfied on an interval, think geometrically:

- **Intersection of two curves** $y = f(x)$ and $y = g(x)$: set $f(x) = g(x)$ and solve. Each solution is one crossing point.
- **Number of solutions of** $\sin x = k$ **on** $[0, 2\pi)$**:** draw the horizontal line $y = k$. If $-1 < k < 1$ there are **2** solutions; if $k = \pm 1$ there is exactly **1** (a tangent touch at the peak/trough); if $|k| > 1$ there are **0**.
- For a compressed wave $\sin(bx) = k$ on $[0, 2\pi)$, the line crosses roughly $2b$ times because the wave completes b full cycles. **Count crossings, do not just solve once.**

Worked example: count solutions of $\cos x = \frac{1}{2}$ vs. $\cos 2x = \frac{1}{2}$ on $[0, 2\pi)$

The line $y = \frac{1}{2}$ meets $y = \cos x$ where $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$ — **2 solutions**. But $y = \cos 2x$ completes *two* full cycles on $[0, 2\pi)$, so the same line cuts it **4 times**: $2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$, giving $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

Doubling the frequency doubled the number of solutions.

Worked example: tide model with a solve-for-time step

At a harbor the water is 6 m deep at high tide and 2 m at low tide; high tide occurs at $t = 0$ hours and the cycle repeats every 12 hours. Model the depth and find when the depth first reaches 5 m.

Build the model. Midline $d = \frac{6+2}{2} = 4$; amplitude $|A| = \frac{6-2}{2} = 2$; period $12 \Rightarrow b = \frac{2\pi}{12} = \frac{\pi}{6}$. It starts at a *maximum*, so use $+\cos$:

$$\boxed{h(t) = 2\cos\big(\tfrac{\pi}{6} t\big) + 4}$$

Solve for time. Set $h(t) = 5$:

$$2\cos\left(\tfrac{\pi}{6}t\right) + 4 = 5 \implies \cos\left(\tfrac{\pi}{6}t\right) = \tfrac{1}{2}$$
$$\tfrac{\pi}{6}t = \tfrac{\pi}{3} \text{ or } \tfrac{5\pi}{3} \implies t = 2 \text{ h or } t = 10 \text{ h}.$$

The depth first hits 5 m at $t = 2$ hours (falling), and again at $t = 10$ hours (rising back up).

Writing a sinusoid from scattered conditions, and identical graphs

From scattered data. You do not need a clean starting point. Given a few facts:

1. Get the midline and amplitude from any known high and low: $d = \frac{1}{2}(\text{high} + \text{low})$, $|A| = \frac{1}{2}(\text{high} - \text{low})$.
2. Get b from the period, or from the *distance between a max and the next min* (that gap is a *half*-period).
3. Pin the horizontal shift with one labeled point — ideally a max, min, or midline crossing — by forcing the model to pass through it.

Different formulas, identical graphs. Because sine and cosine are shifts of each other, many equations describe the *same* curve. All four below are one graph:

$$\sin\left(x + \tfrac{\pi}{2}\right) = \cos x = -\cos(x + \pi) = \sin\left(\tfrac{\pi}{2} - x\right).$$

A reflection $-\sin x$ equals a half-period shift $\sin(x - \pi)$; adding 2π to a phase changes nothing. When two answers look different, test a few points (or compare A, b, d and one peak location) to confirm they match.

Big-picture tip: Every equal-frequency combination, reflection, and phase shift of sine and cosine is *still a single sinusoid* — same midline spacing, same period, just relocated and rescaled. Reach for $R\sin(x + \phi)$ to unify them, and always finish a modeling problem with an explicit **solve-for-t** step.

10 Graphs of the Other Trigonometric Functions

Tangent, cotangent, secant, and cosecant: periods, asymptotes, and transformations

10.1 The Graph of $y = \tan x$

Tangent: key features

Since $\tan x = \dfrac{\sin x}{\cos x}$, the graph has a **vertical asymptote** wherever $\cos x = 0$.

- **Period:** π (the pattern repeats every π units).
- **Vertical asymptotes:** $x = \dfrac{\pi}{2} + n\pi$ for any integer n (i.e. $\ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$).
- **x-intercepts:** $x = n\pi$ (where $\sin x = 0$).
- **Key points on the middle branch:** $\left(-\frac{\pi}{4}, -1\right)$, $(0, 0)$, $\left(\frac{\pi}{4}, 1\right)$.
- Each branch **increases** from $-\infty$ to $+\infty$ between consecutive asymptotes.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Example: asymptotes of $y = \tan x$ near the origin

The two asymptotes closest to the origin come from $\cos x = 0$, which happens at $x = \pm\dfrac{\pi}{2}$. The branch through the origin lives on the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and passes through $(0, 0)$ with $\tan\left(\frac{\pi}{4}\right) = 1$.

Tip: Tangent asymptotes sit where *cosine* is zero. Its zeros sit where *sine* is zero. Sketch the asymptotes first, then draw one increasing branch between each pair.

10.2 The Graph of $y = \cot x$

Cotangent: key features

Since $\cot x = \dfrac{\cos x}{\sin x}$, the graph has a **vertical asymptote** wherever $\sin x = 0$.

- **Period:** π.
- **Vertical asymptotes:** $x = n\pi$ (i.e. $\ldots, -\pi, 0, \pi, 2\pi, \ldots$).
- **x-intercepts:** $x = \dfrac{\pi}{2} + n\pi$ (where $\cos x = 0$).
- **Key points on a branch:** $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{4}, -1\right)$.
- Each branch **decreases** from $+\infty$ to $-\infty$ between consecutive asymptotes.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Example: asymptotes and intercepts of $y = \cot x$

Set $\sin x = 0$: asymptotes at $x = 0,\ \pm\pi,\ \pm 2\pi, \ldots$. Between $x = 0$ and $x = \pi$ the branch falls from $+\infty$ to $-\infty$, crossing the axis at its midpoint $x = \frac{\pi}{2}$.

Tip: Cotangent is the "mirror image" behavior of tangent: same period π, but its asymptotes are at multiples of π and each branch *decreases*.

10.3 The Graphs of $y = \sec x$ and $y = \csc x$

Secant and cosecant as reciprocals

Because $\sec x = \dfrac{1}{\cos x}$ and $\csc x = \dfrac{1}{\sin x}$, you can sketch each one from its "parent" wave:

- An asymptote appears wherever the parent (cosine or sine) equals 0.
- Where the parent has a *maximum* of 1, the reciprocal has a *minimum* of 1; where the parent has a *minimum* of -1, the reciprocal has a *maximum* of -1.
- **Period of each:** 2π. **Range of each:** $(-\infty, -1] \cup [1, \infty)$ (no values strictly between -1 and 1).
- **Secant asymptotes:** $x = \frac{\pi}{2} + n\pi$ (where $\cos x = 0$).
- **Cosecant asymptotes:** $x = n\pi$ (where $\sin x = 0$).

Slope is rise over run.

(diagram in the online version)

Example: reading the secant graph

At $x = 0$, $\cos 0 = 1$, so $\sec 0 = \frac{1}{1} = 1$: the upward U touches its minimum $(0, 1)$. At $x = \pi$, $\cos\pi = -1$, so $\sec\pi = -1$: the downward U touches its maximum $(\pi, -1)$. The asymptotes at $x = \pm\frac{\pi}{2}$ separate these U-shapes.

Tip: Lightly sketch the cosine (for sec) or the sine (for csc) first. Draw asymptotes through its zeros, then draw a U touching each hump. There is never any part of the graph between $y = -1$ and $y = 1$.

10.4 Transformations: $y = a\,\mathrm{f}(bx - c) + d$

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Period, phase shift, and asymptotes

For $b > 0$, write the inside as $b\left(x - \frac{c}{b}\right)$. Then:

- **Phase (horizontal) shift:** $\frac{c}{b}$; **vertical shift:** d; $|a|$ stretches vertically (it does *not* change the period).
- **Period:** tangent & cotangent use $\frac{\pi}{|b|}$; secant & cosecant use $\frac{2\pi}{|b|}$.
- **Asymptotes** come from the inside expression:

Example: $y = 3\tan\left(2x - \frac{\pi}{2}\right)$

Rewrite: $2x - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$, so $b = 2$, phase shift $= \frac{\pi}{4}$ right, $a = 3$.

$$\text{Period} = \frac{\pi}{|b|} = \frac{\pi}{2}.$$

Asymptotes: $2x - \frac{\pi}{2} = \frac{\pi}{2} + n\pi \;\Rightarrow\; 2x = \pi + n\pi \;\Rightarrow\; x = \frac{\pi}{2} + \frac{n\pi}{2}$. The factor $a = 3$ makes the branches steeper but leaves the period unchanged.

Example: $y = \csc\left(\frac{x}{2}\right)$

Here $b = \frac{1}{2}$, so the period is $\frac{2\pi}{|b|} = \frac{2\pi}{1/2} = 4\pi$. Asymptotes come from $\frac{x}{2} = n\pi \Rightarrow x = 2n\pi$ (that is, $x = 0, \pm 2\pi, \pm 4\pi, \ldots$).

Tip: To find asymptotes of a transformed graph, set the *inside* of the function equal to the asymptote condition for the parent ($\frac{\pi}{2} + n\pi$ for tan / sec, or $n\pi$ for cot / csc) and solve for x. The domain is then "all real numbers except those x-values."

10.5 Going Deeper: Advanced Graph Ideas

Full analysis of $y = a\tan(bx - c) + d$

Treat the whole function as one transformed tangent. With $b > 0$, rewrite the inside as $b\left(x - \frac{c}{b}\right)$.

- **Period:** $\frac{\pi}{|b|}$; **phase shift:** $\frac{c}{b}$; **vertical shift:** d.

- **Asymptote family:** solve $bx - c = \dfrac{\pi}{2} + n\pi$, giving $x = \dfrac{c}{b} + \dfrac{\pi}{2b} + \dfrac{n\pi}{b}$ for every integer n. Consecutive asymptotes are exactly one period apart.
- **Range:** all real numbers, $(-\infty, \infty)$ — the vertical stretch $|a|$ and shift d never bound a tangent.
- **Nearest "intercept" (center) point:** the branch crosses its centerline $y = d$ where $bx - c = n\pi$, i.e. $x = \dfrac{c}{b} + \dfrac{n\pi}{b}$. Between two asymptotes, this midpoint is the point of symmetry.

Slope is rise over run.

(diagram in the online version)

Full analysis of $y = a\sec(bx - c) + d$

Sketch the guide wave $y = a\cos(bx - c) + d$ first; the secant hugs its humps.

- **Period:** $\dfrac{2\pi}{|b|}$; **phase shift:** $\dfrac{c}{b}$; **vertical shift:** d.
- **Asymptote family:** solve $bx - c = \dfrac{\pi}{2} + n\pi$, giving $x = \dfrac{c}{b} + \dfrac{\pi}{2b} + \dfrac{n\pi}{b}$ (wherever the guide cosine is zero).
- **Range:** the guide's minima sit at $y = d - |a|$ and maxima at $y = d + |a|$, so the range is $(-\infty,\, d - |a|] \cup [d + |a|,\, \infty)$. Nothing lands in the open gap $\big(d - |a|,\, d + |a|\big)$.
- **No x-intercepts unless** the gap contains 0, i.e. unless $|a| > |d|$; otherwise the graph never crosses the x-axis.

Worked example: full analysis of $y = 2\sec\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) - 1$

First force the "$bx - c$" form: $\dfrac{x}{2} + \dfrac{\pi}{4} = \dfrac{1}{2}\left(x + \dfrac{\pi}{2}\right)$, so $b = \dfrac{1}{2}$, phase shift $= -\dfrac{\pi}{2}$ (left), $a = 2$, $d = -1$.

- **Period:** $\dfrac{2\pi}{|b|} = \dfrac{2\pi}{1/2} = 4\pi$.
- **Asymptotes:** $\dfrac{x}{2} + \dfrac{\pi}{4} = \dfrac{\pi}{2} + n\pi \Rightarrow \dfrac{x}{2} = \dfrac{\pi}{4} + n\pi \Rightarrow x = \dfrac{\pi}{2} + 2n\pi$.
- **Range:** $d \pm |a| = -1 \pm 2$, so minima at $y = 1$, maxima at $y = -3$: range $(-\infty, -3] \cup [1, \infty)$.
- **Intercepts:** since $|a| = 2 > |d| = 1$, the gap $(-3, 1)$ contains 0, so the graph *does* cross the x-axis — solve $2\sec(\cdots) = 1$, i.e. $\sec(\cdots) = \frac{1}{2}$, which is impossible ($|\sec| \geq 1$). Re-checking: $\sec u = \frac{1}{2}$ has no solution, so despite the gap straddling 0, the reciprocal can never take that value. **No x-intercepts.**

This is the subtle point: an x-intercept needs 0 inside the range *and* a genuine secant value there; here $0 \in (-3, 1)$ but the required $\sec u = \frac{1}{2}$ is unreachable.

Worked example: writing an equation from a described graph

A tangent-type curve increases left to right, has consecutive vertical asymptotes at $x = 1$ and $x = 5$, passes through its center point $(3, 4)$, and reaches $y = 7$ one quarter-period to the right of that center. Build $y = a\tan(bx - c) + d$.

- **Period** $= 5 - 1 = 4$, and for tangent period $= \dfrac{\pi}{b}$, so $b = \dfrac{\pi}{4}$.
- **Center** $(3, 4)$: the point of symmetry sits at the midpoint of the asymptotes ($x = 3$, check) and gives the vertical shift $d = 4$.
- **Phase:** center at $x = 3$ means $bx - c = 0$ there: $\dfrac{\pi}{4}(3) - c = 0 \Rightarrow c = \dfrac{3\pi}{4}$.
- **Amplitude factor:** a quarter-period right of center is $x = 4$, where tan of the inside equals $\tan\dfrac{\pi}{4} = 1$. Then $y = a(1) + 4 = 7 \Rightarrow a = 3$.

$$y = 3\tan\left(\frac{\pi}{4}x - \frac{3\pi}{4}\right) + 4.$$

Worked example: where two reciprocal graphs meet

Find where $y = \sec x$ meets $y = \csc x$ on $[0, 2\pi)$. Set $\sec x = \csc x$, i.e. $\dfrac{1}{\cos x} = \dfrac{1}{\sin x}$. Cross-multiplying (both denominators nonzero) gives $\sin x = \cos x$, so $\tan x = 1$.

$$x = \frac{\pi}{4} \quad \text{or} \quad x = \frac{5\pi}{4}.$$

At $x = \dfrac{\pi}{4}$: $\sec x = \sqrt{2}$, so they meet at $\left(\dfrac{\pi}{4}, \sqrt{2}\right)$. At $x = \dfrac{5\pi}{4}$: $\sec x = -\sqrt{2}$, meeting at $\left(\dfrac{5\pi}{4}, -\sqrt{2}\right)$. Always discard any candidate where either $\cos x = 0$ or $\sin x = 0$ (an asymptote of one of the graphs).

Transformations that produce identical graphs

Different-looking formulas can trace the exact same curve. Watch for these coincidences:

- **Cofunction shift:** $\sec\left(x - \dfrac{\pi}{2}\right) = \csc x$ and $\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$; a phase shift can turn one reciprocal into another.
- **Half-period shift for tangent/cotangent:** because the period is π, $\tan(x + \pi) = \tan x$; adding a full period changes nothing.
- **Reflection vs. shift:** $-\tan x = \tan(-x)$, so a sign on a can be absorbed into a horizontal

reflection.

- **Sign of a on secant:** $-\sec x = \sec(x - \pi)$ shifts the U's rather than needing a reflection.

So $y = \csc x$ and $y = \sec\left(x - \frac{\pi}{2}\right)$ are *the same graph*, even though one is built from sine and the other from cosine.

Worked example: range of a shifted cosecant, and a combined domain

(a) For $y = -3\csc(2x) + 5$: the guide is $y = -3\sin(2x) + 5$ with extremes 5 ± 3, so minima 2 and maxima 8. The cosecant fills outside the humps: **range** $(-\infty, 2] \cup [8, \infty)$.

(b) **Domain of a combined expression** $f(x) = \tan x + \csc x$. Exclude everything either piece forbids: $\tan x$ dies where $\cos x = 0$ ($x = \frac{\pi}{2} + n\pi$) and $\csc x$ dies where $\sin x = 0$ ($x = n\pi$). Union of the two forbidden sets:

$$x \neq \frac{n\pi}{2} \quad \text{(all integer multiples of } \tfrac{\pi}{2}\text{)}.$$

So the domain is all reals except $x = \frac{n\pi}{2}$.

Tip: For a *range* question on a shifted secant/cosecant, only d and $|a|$ matter — the answer is always $(-\infty,\ d - |a|] \cup [d + |a|,\ \infty)$. For a *domain* question on a sum of these functions, take the *union* of each piece's forbidden x-values. For an *intersection* question, set the two equal, reduce to a single trig equation, then throw out any solution that lands on an asymptote of either curve.

11 Inverse Trigonometric Functions

Definitions, exact values, compositions, graphs & equations

11.1 Definitions: Domain & Range

An inverse trig function *undoes* a trig function. Because $\sin, \cos, \tan$ are not one-to-one, we first **restrict** their domains so an inverse can exist. Those restrictions become the **ranges** of the inverse functions.

The three inverse trig functions

$y = \arcsin x$ means "y is the angle whose sine is x": $\sin y = x$.
The same idea defines arccos and arctan. Read $\sin^{-1} x = \arcsin x$ (the -1 is *not* an exponent).

The range tells you **which quadrant** the output angle lives in: arcsin and arctan give angles in QI or QIV (right half), while arccos gives angles in QI or QII (upper half).

Sign memory aid. arccos always returns an angle in $[0, \pi]$, so its output is *never negative*. arcsin and arctan return negative angles for negative inputs.

11.2 Evaluating Exactly and with a Calculator

For **special values**, ask "what angle *in the correct range* has this sine/cosine/tangent?"

Exact value with a negative input

Evaluate $\arccos\left(-\dfrac{\sqrt{2}}{2}\right)$.

Need angle y with $\cos y = -\frac{\sqrt{2}}{2}$ and $y \in [0, \pi]$.
Reference angle: $\frac{\pi}{4}$. Cosine is negative in QII.
$y = \pi - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$.

Check: $\frac{3\pi}{4} \in [0, \pi]$, and $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$. ✓

Exact value for arcsin of a negative

Evaluate $\arcsin\left(-\dfrac{1}{2}\right)$.
Need y with $\sin y = -\frac{1}{2}$ and $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Reference angle $\frac{\pi}{6}$; sine negative means QIV (a negative angle): $\arcsin\left(-\frac{1}{2}\right) = -\dfrac{\pi}{6}$.

Calculator. Put the calculator in **radian mode**. Use `SIN`$^{-1}$, `COS`$^{-1}$, `TAN`$^{-1}$. Example: $\arctan(4) \approx 1.326$, $\arccos(-0.72) \approx 2.375$. The calculator only ever reports the *one* value inside the range above.

11.3 Compositions with a Reference Triangle

To turn $\sin(\arccos x)$, $\tan(\arcsin x)$, etc. into an algebraic expression: let θ be the inner inverse, draw a right triangle in which θ has the required ratio, fill in the missing side with the Pythagorean theorem, then read off the outer function.

An angle in standard position.

(diagram in the online version)

Numeric composition

Evaluate $\cos\left(\arcsin\dfrac{3}{5}\right)$.
Let $\theta = \arcsin\frac{3}{5}$, so $\sin\theta = \frac{3}{5} = \frac{\text{opp}}{\text{hyp}}$ with θ in QI.

Missing leg: $\sqrt{5^2 - 3^2} = 4$. Then $\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \boxed{\dfrac{4}{5}}$.

Algebraic composition

Write $\tan(\arccos x)$ as an algebraic expression for $0 < x \le 1$.
Let $\theta = \arccos x$, so $\cos\theta = \dfrac{x}{1} = \dfrac{\text{adj}}{\text{hyp}}$. Take adj $= x$, hyp $= 1$, so opp $= \sqrt{1-x^2}$.

$$\tan(\arccos x) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{1-x^2}}{x}.$$

Likewise $\sin(\arccos x) = \sqrt{1-x^2}$ and $\tan(\arcsin x) = \dfrac{x}{\sqrt{1-x^2}}$.

Tip. For an inner $\arctan\frac{x}{2}$, put opp $= x$, adj $= 2$, so hyp $= \sqrt{x^2+4}$. The number under the square root is always (first leg)^2 + (second leg)^2 or $1 - x^2$.

11.4 Compositions Where the Range Restriction Matters

$\arcsin(\sin\theta) = \theta$ *only* when θ is already inside the inverse's range. Otherwise you must first evaluate the inside, then take the inverse — the answer is the angle in range that has the same value.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

The classic trap

Evaluate $\arcsin\left(\sin\dfrac{5\pi}{6}\right)$.
Tempting (wrong) answer: $\frac{5\pi}{6}$. But $\frac{5\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the functions do *not* simply cancel.

$$\sin\tfrac{5\pi}{6} = \tfrac{1}{2},$$
$$\arcsin\left(\tfrac{1}{2}\right) = \tfrac{\pi}{6} \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

So $\arcsin\left(\sin\frac{5\pi}{6}\right) = \boxed{\dfrac{\pi}{6}}$.

Cosine version

Evaluate $\arccos\left(\cos\dfrac{7\pi}{6}\right)$.

$\frac{7\pi}{6} \notin [0,\pi]$. Compute inside first: $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$. Then $\arccos\left(-\frac{\sqrt{3}}{2}\right) = \dfrac{5\pi}{6}$ (the angle in $[0,\pi]$).

Cancellation rules. $\arcsin(\sin\theta) = \theta$ only if $\theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; $\arccos(\cos\theta) = \theta$ only if $\theta \in [0,\pi]$; $\arctan(\tan\theta) = \theta$ only if $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. In contrast, $\sin(\arcsin x) = x$ always holds for $x \in [-1,1]$.

11.5 Graphs of the Inverse Functions

Each inverse graph is the reflection of the restricted trig graph across the line $y = x$. Domain and range trade places.

Read the graphs. arcsin and arccos have endpoints at $x = \pm 1$ (domain $[-1,1]$). arctan is defined for all x and flattens toward the horizontal asymptotes $y = \pm\frac{\pi}{2}$; it never reaches them.

11.6 Solving Simple Trig Equations

Inverse functions give *one* solution; use symmetry (reference angle + quadrants) or add $2\pi k$ to get the rest.

Two solutions on one revolution

Solve $\cos x = -\dfrac{1}{2}$ on $[0,2\pi)$.

Reference angle: $\arccos\frac{1}{2} = \frac{\pi}{3}$. Cosine is negative in QII and QIII:

$$x = \pi - \tfrac{\pi}{3} = \tfrac{2\pi}{3} \quad \text{and} \quad x = \pi + \tfrac{\pi}{3} = \tfrac{4\pi}{3}.$$

Solving recipe. (1) Isolate the trig function. (2) Take the inverse to get the reference angle. (3) Use the sign to decide the quadrants. (4) List every solution in the requested interval; for "all solutions" add $+2\pi k$ (sine/cosine) or $+\pi k$ (tangent).

11.7 Going Deeper: Advanced Inverse Trig

This section pushes past single evaluations into **identities** that combine inverse functions, **quadrant traps** hidden inside the arctangent sum formula, and the **domain bookkeeping** needed when inverses are composed.

The complementary identity

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

$\arcsin x + \arccos x = \dfrac{\pi}{2}$

For every $x \in [-1, 1]$,

$$\arcsin x + \arccos x = \frac{\pi}{2}.$$

Why: let $\alpha = \arcsin x$, so $\sin\alpha = x$ with $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha = x$, and $\frac{\pi}{2} - \alpha \in [0, \pi]$ — exactly the range of arccos. So $\frac{\pi}{2} - \alpha = \arccos x$. The identity is just "sine and cosine of complementary angles agree."

Use it to dodge triangles. Any sum $\arcsin x + \arccos x$ collapses to $\frac{\pi}{2}$ *instantly*. For example $\arccos x = \frac{\pi}{2} - \arcsin x$, so $\sin(\arccos x) = \sin\left(\frac{\pi}{2} - \arcsin x\right) = \cos(\arcsin x) = \sqrt{1 - x^2}$, matching the triangle method with no drawing.

Algebraic compositions with double angles

Simplify $\sin(2\arctan x)$

Let $\theta = \arctan x$, so $\tan\theta = \dfrac{x}{1} = \dfrac{\text{opp}}{\text{adj}}$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Build the triangle: $\text{opp} = x$, $\text{adj} = 1$, $\text{hyp} = \sqrt{1 + x^2}$.

Apply the double-angle formula $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\begin{aligned} \sin(2\arctan x) &= 2 \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \\ &= \boxed{\frac{2x}{1+x^2}}. \end{aligned}$$

The same triangle gives $\cos(2\arctan x) = \cos^2\theta - \sin^2\theta = \dfrac{1 - x^2}{1 + x^2}$.

Pattern to remember. These are the tangent half-angle substitutions: with $t = \arctan x$, $\sin 2t = \dfrac{2x}{1+x^2}$ and $\cos 2t = \dfrac{1-x^2}{1+x^2}$. Valid for *all* real x because arctan has domain $(-\infty, \infty)$ and always lands in QI/QIV where the triangle sides carry the right signs.

The arctangent sum formula and the $ab > 1$ trap

Adding two arctangents

The tangent addition formula gives

$$\tan(\arctan a + \arctan b) = \frac{a+b}{1-ab},$$

so it is *tempting* to write $\arctan a + \arctan b = \arctan\dfrac{a+b}{1-ab}$. But arctan only returns values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, while the true sum can land outside it. The correct statement carries a quadrant correction:

$$\arctan a + \arctan b = \arctan\frac{a+b}{1-ab} + k\pi,$$

where $k = 0$ if $ab < 1$, $k = +1$ if $ab > 1$ and $a > 0$, and $k = -1$ if $ab > 1$ and $a < 0$.

Safe case ($ab < 1$) vs. the trap ($ab > 1$)

(a) Safe. Evaluate $\arctan \frac{1}{2} + \arctan \frac{1}{3}$. Here $ab = \frac{1}{6} < 1$, so no correction:

$$\arctan \tfrac{1}{2} + \arctan \tfrac{1}{3} = \arctan \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = \arctan \frac{5/6}{5/6} = \arctan 1 = \boxed{\tfrac{\pi}{4}}.$$

(b) Trap. Evaluate $\arctan 2 + \arctan 3$. Now $ab = 6 > 1$, so the raw formula lies. Compute the fraction first:

$$\frac{2+3}{1-2\cdot 3} = \frac{5}{-5} = -1, \qquad \arctan(-1) = -\tfrac{\pi}{4}.$$

That negative answer is impossible: $\arctan 2$ and $\arctan 3$ are *both* bigger than $\frac{\pi}{4}$, so their sum exceeds $\frac{\pi}{2}$. Since $ab > 1$ and $a > 0$ we add π:

$$\arctan 2 + \arctan 3 = -\tfrac{\pi}{4} + \pi = \boxed{\tfrac{3\pi}{4}}.$$

Check: $\frac{3\pi}{4}$ is in QII where tangent is -1. ✓

Sanity test before trusting the formula. Estimate the size of each arctangent. If both inputs are positive and their sum of angles clearly exceeds $\frac{\pi}{2}$, you are in the $ab > 1$ regime and must add π. When in doubt, add angles numerically on a calculator to pick the right branch.

Telescoping (Machin-like) sums

Turning a sum into a difference

The difference version of the formula is often exact with $k = 0$:

$$\arctan\frac{1}{n} - \arctan\frac{1}{n+1} = \arctan\frac{1}{1+n(n+1)} = \arctan\frac{1}{n^2+n+1}.$$

(The product of the two small inputs is $\frac{1}{n(n+1)} < 1$, so no correction is needed.) A long sum of the right-hand terms then **telescopes**: consecutive pieces cancel and only the endpoints survive.

Evaluate a telescoping arctangent sum

Find $S = \arctan\frac{1}{3} + \arctan\frac{1}{7} + \arctan\frac{1}{13} + \arctan\frac{1}{21}$.
Notice $3 = 1^2+1+1,\ \ 7 = 2^2+2+1,\ \ 13 = 3^2+3+1,\ \ 21 = 4^2+4+1$. Rewrite each term as a difference:

$$\begin{aligned} S &= \left(\arctan\tfrac{1}{1} - \arctan\tfrac{1}{2}\right) + \left(\arctan\tfrac{1}{2} - \arctan\tfrac{1}{3}\right) \\ &\quad + \left(\arctan\tfrac{1}{3} - \arctan\tfrac{1}{4}\right) + \left(\arctan\tfrac{1}{4} - \arctan\tfrac{1}{5}\right). \end{aligned}$$

Every interior term cancels, leaving only the ends:

$$S = \arctan 1 - \arctan\tfrac{1}{5} = \tfrac{\pi}{4} - \arctan\tfrac{1}{5} \approx 0.588.$$

The same idea underlies *Machin's formula* $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$, historically used to compute π.

Nested compositions with range traps

$\arccos(\cos\theta)$ is a "fold," not the identity

Because arccos must output a value in $[0, \pi]$, the composition $\arccos(\cos\theta)$ returns the *unique* number in $[0, \pi]$ having the same cosine as θ:

$$\arccos(\cos\theta) = \begin{cases} \theta, & 0 \le \theta \le \pi, \\ 2\pi - \theta, & \pi \le \theta \le 2\pi, \\ -\theta, & -\pi \le \theta \le 0. \end{cases}$$

Geometrically it **folds** the angle back into $[0, \pi]$ using the even symmetry $\cos\theta = \cos(-\theta)$ and periodicity.

Same input, three wrappers

Take $\theta = \dfrac{4\pi}{3}$ (which is *not* in $[0,\pi]$).

$$\begin{aligned}
\arccos\bigl(\cos\tfrac{4\pi}{3}\bigr) &= 2\pi - \tfrac{4\pi}{3} = \boxed{\tfrac{2\pi}{3}} && \text{(fold into } [0,\pi]\text{)},\\
\arcsin\bigl(\sin\tfrac{4\pi}{3}\bigr) &= \arcsin\Bigl(-\tfrac{\sqrt{3}}{2}\Bigr) = \boxed{-\tfrac{\pi}{3}} && \text{(range } \bigl[-\tfrac{\pi}{2},\tfrac{\pi}{2}\bigr]\text{)},\\
\arctan\bigl(\tan\tfrac{4\pi}{3}\bigr) &= \arctan\Bigl(\sqrt{3}\Bigr) = \boxed{\tfrac{\pi}{3}} && \text{(range } \bigl(-\tfrac{\pi}{2},\tfrac{\pi}{2}\bigr)\text{)}.
\end{aligned}$$

The three "inverses of the same angle" disagree because each function reports the representative living in *its own* range. Always reduce the inside first, then land in the outer range.

Solving equations that contain inverse functions

An equation in arctan

Solve $\arctan(x) + \arctan(2x) = \dfrac{\pi}{4}$.
Take the tangent of both sides and use the sum formula on the left:

$$\tan\bigl(\arctan x + \arctan 2x\bigr) = \frac{x+2x}{1-(x)(2x)} = \frac{3x}{1-2x^2}, \qquad \tan\tfrac{\pi}{4} = 1.$$

So $\dfrac{3x}{1-2x^2} = 1 \Rightarrow 3x = 1-2x^2 \Rightarrow 2x^2+3x-1=0$, giving

$$x = \frac{-3\pm\sqrt{9+8}}{4} = \frac{-3\pm\sqrt{17}}{4}.$$

Check for extraneous roots. The two arctangents must add to a *positive* $\frac{\pi}{4}$, so we need $x>0$. Only $x = \dfrac{-3+\sqrt{17}}{4} \approx 0.28$ works; the negative root ≈ -1.78 makes the left side negative (and is the spurious solution introduced by taking tan). Answer: $\boxed{x = \dfrac{-3+\sqrt{17}}{4}}$.

Taking tan/sin/cos **can add fake solutions.** Those operations are not one-to-one, so *every* algebraic root must be tested back in the original equation — confirm both the value and the correct sign/quadrant of each inverse term.

Domain of a composite inverse expression

Chaining domain requirements

For a composition to be defined, the inner output must satisfy the outer function's domain *and* the inner function's own domain. For arcsin and arccos the input must lie in $[-1, 1]$; arctan accepts anything.

Find the domain of $f(x) = \arccos(2x - 1)$

The outer arccos demands its argument stay in $[-1, 1]$:

$$-1 \leq 2x - 1 \leq 1 \implies 0 \leq 2x \leq 2 \implies \boxed{0 \leq x \leq 1}.$$

So the domain is $[0, 1]$. As a bonus, the range is still all of $[0, \pi]$ because $2x - 1$ sweeps the full interval $[-1, 1]$ as x runs over $[0, 1]$.

Two-sided inputs need two inequalities. A restriction like $|u| \leq 1$ is shorthand for $-1 \leq u \leq 1$; solve *both* halves. For a nested radical such as $\arcsin(\sqrt{x})$ you must also enforce the inner domain $(x \geq 0)$ *and* the outer one $(\sqrt{x} \leq 1)$, giving $0 \leq x \leq 1$.

12 Trigonometric Identities

Fundamental identities, simplifying, and verifying

12.1 The Fundamental Identities

These are the building blocks. Memorize them: every simplification and every proof in this topic comes from combining a few of these.

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad\qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta \qquad\qquad 1 + \cot^2\theta = \csc^2\theta$$

Each has useful rearrangements, e.g. $\sin^2\theta = 1 - \cos^2\theta$ and $\tan^2\theta = \sec^2\theta - 1$.

Cofunction Identities (angles complementary)

$$\sin(90^\circ - \theta) = \cos\theta \qquad \tan(90^\circ - \theta) = \cot\theta \qquad \sec(90^\circ - \theta) = \csc\theta$$
$$\cos(90^\circ - \theta) = \sin\theta \qquad \cot(90^\circ - \theta) = \tan\theta \qquad \csc(90^\circ - \theta) = \sec\theta$$

In radians replace 90° with $\dfrac{\pi}{2}$.

Even/Odd (Negative-Angle) Identities

Even (unchanged by a sign flip):

$$\cos(-\theta) = \cos\theta \qquad \sec(-\theta) = \sec\theta$$

Odd (pick up a negative sign):

$$\sin(-\theta) = -\sin\theta \quad \tan(-\theta) = -\tan\theta \quad \csc(-\theta) = -\csc\theta \quad \cot(-\theta) = -\cot\theta$$

Applying one identity

Write $\sec\theta\cot\theta$ as a single function.

$$\begin{aligned} \sec\theta\cot\theta &= \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} && \text{reciprocal and quotient} \\ &= \frac{1}{\sin\theta} = \csc\theta. \end{aligned}$$

Tip. When you are stuck, rewrite everything in terms of $\sin\theta$ and $\cos\theta$. Almost every identity untangles once only sines and cosines remain.

12.2 Simplifying Trigonometric Expressions

To *simplify* means to reduce an expression to its shortest equivalent form. Strategy: convert to sines and cosines, combine fractions, then use a Pythagorean identity to collapse the result.

A full simplification

Simplify $\dfrac{1-\sin^2\theta}{\cos\theta}$.

$$\begin{aligned}\frac{1-\sin^2\theta}{\cos\theta} &= \frac{\cos^2\theta}{\cos\theta} && \text{since } 1-\sin^2\theta=\cos^2\theta\\ &= \cos\theta.\end{aligned}$$

Combine, then use $\sin^2+\cos^2=1$

Simplify $\cos\theta\,(\tan\theta+\cot\theta)$.

$$\begin{aligned}\cos\theta\,(\tan\theta+\cot\theta) &= \cos\theta\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)\\ &= \sin\theta+\frac{\cos^2\theta}{\sin\theta}=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta}\\ &= \frac{1}{\sin\theta}=\csc\theta.\end{aligned}$$

Tip. Watch for the Pythagorean patterns hiding in an expression: $1-\sin^2\theta$, $\sec^2\theta-1$, and $\csc^2\theta-1$ each collapse to a single squared function.

12.3 Rewriting in Terms of Sine and Cosine (or as One Function)

Any of the six functions can be written using only $\sin\theta$ and $\cos\theta$. This is the single most useful move for both simplifying and verifying.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

The universal rewrite

$$\tan\theta=\frac{\sin\theta}{\cos\theta}\qquad \cot\theta=\frac{\cos\theta}{\sin\theta}\qquad \sec\theta=\frac{1}{\cos\theta}\qquad \csc\theta=\frac{1}{\sin\theta}$$

Rewrite, then combine into one function

Express $\dfrac{\tan\theta}{\sec\theta}$ as a single function.

$$\frac{\tan\theta}{\sec\theta} = \frac{\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos\theta}} = \frac{\sin\theta}{\cos\theta}\cdot\frac{\cos\theta}{1}$$
$$= \sin\theta.$$

Tip. A complex fraction is divided by *multiplying by the reciprocal.* Rewriting in sin/cos usually turns a messy quotient into a clean cancellation.

12.4 Verifying Trigonometric Identities

To *verify* (prove) an identity, transform one side until it matches the other. You may not move terms across the = sign as in solving an equation—each side is worked independently.

Strategies

- Start with the **more complicated side**; there is more to simplify.
- Convert everything to **sine and cosine**.
- Get a **common denominator** to combine fractions.
- Multiply by a **conjugate** to create a Pythagorean form (e.g. multiply by $1+\sin\theta$ to get $1-\sin^2\theta$).
- **Factor** and look for a Pythagorean identity to substitute.
- Work **both sides** to a common third expression if neither side alone reaches the other.

Verification by converting to sin/cos

Verify $\sec\theta - \sin\theta\tan\theta = \cos\theta$.

$$\begin{aligned}\sec\theta - \sin\theta\tan\theta &= \frac{1}{\cos\theta} - \sin\theta\cdot\frac{\sin\theta}{\cos\theta}\\ &= \frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta} = \frac{1-\sin^2\theta}{\cos\theta}\\ &= \frac{\cos^2\theta}{\cos\theta} = \cos\theta. \qquad \checkmark\end{aligned}$$

Verification using a conjugate

Verify $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$.

$$\begin{aligned}
\frac{\cos\theta}{1-\sin\theta} &= \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} && \text{multiply by conjugate}\\
&= \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}\\
&= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}\\
&= \frac{1+\sin\theta}{\cos\theta}. \qquad \checkmark
\end{aligned}$$

Tip. Never divide by an expression that could be zero, and never "cross-multiply." A verification is a chain of equalities down one side, ending exactly at the other side.

12.5 Finding the Other Five Values from One

Given one function value and the quadrant of θ, you can find all six. Use a Pythagorean identity to get a partner value, then the reciprocal and quotient identities for the rest. The quadrant fixes each sign.

Signs by quadrant (ASTC)

"**A**ll **S**tudents **T**ake **C**alculus": the function that is positive in QI, QII, QIII, QIV.

Full worked value problem

Given $\sin\theta = \dfrac{3}{5}$ with θ in Quadrant II, find the other five.

$$\begin{aligned}
\cos^2\theta &= 1-\sin^2\theta = 1-\tfrac{9}{25} = \tfrac{16}{25}\\
\cos\theta &= -\tfrac{4}{5} && \text{(QII: cosine negative)}\\
\tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{3/5}{-4/5} = -\tfrac{3}{4}\\
\csc\theta &= \frac{1}{\sin\theta} = \tfrac{5}{3}, \quad \sec\theta = \frac{1}{\cos\theta} = -\tfrac{5}{4}\\
\cot\theta &= \frac{1}{\tan\theta} = -\tfrac{4}{3}.
\end{aligned}$$

Tip. Solve for the *partner* value (cos if you were given sin) with a Pythagorean identity first, choosing its sign from the quadrant. Everything else follows from reciprocal and quotient identities—no square roots needed.

12.6 Going Deeper: Advanced Identities

The identities above are enough to simplify and verify. The next level combines them into *multi-step* proofs and into *conditional* problems, where a single fact about $\sin x$ and $\cos x$ unlocks a whole family of expressions. The recurring trick: treat $s = \sin x$ and $c = \cos x$ as two unknowns tied by $s^2 + c^2 = 1$, and rewrite every target as a **symmetric function** of them.

Symmetric-Function Toolkit

Write $s = \sin x$, $c = \cos x$. Every symmetric expression in s and c is built from the sum $s + c$ and the product sc. The Pythagorean identity locks them together:

$$s^2 + c^2 = 1, \qquad (s + c)^2 = 1 + 2sc, \qquad (s - c)^2 = 1 - 2sc.$$

So knowing *either* $s + c$ *or* sc immediately gives the other. Higher powers reduce the same way:

$$s^3 + c^3 = (s + c)(1 - sc)\,, \qquad s^3 - c^3 = (s - c)(1 + sc)\,.$$

Power-Sum Reductions (memorize the pattern)

Since $s^2 + c^2 = 1$, every even power sum collapses to a polynomial in $p = sc$:

$$\sin^4 x + \cos^4 x = 1 - 2p^2,$$

$$\sin^6 x + \cos^6 x = 1 - 3p^2,$$

$$\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x, \quad \sin^6 x + \cos^6 x = 1 - 3\sin^2 x\cos^2 x.$$

These come from $(s^2 + c^2)^2 = s^4 + c^4 + 2s^2c^2$ and $(s^2 + c^2)^3 = s^6 + c^6 + 3s^2c^2(s^2 + c^2)$.

Conditional identity: given $\sin x + \cos x$

Given $\sin x + \cos x = \frac{1}{2}$, find $\sin x\cos x$ and $\sin^3 x + \cos^3 x$.

$$\begin{aligned} (\sin x + \cos x)^2 &= \sin^2 x + \cos^2 x + 2\sin x\cos x \\ \tfrac{1}{4} &= 1 + 2\sin x\cos x && \text{square the given} \\ \sin x\cos x &= \frac{\frac{1}{4} - 1}{2} = -\frac{3}{8}. \end{aligned}$$

Now use the sum-of-cubes factoring with $s^2 + c^2 = 1$:

$$\begin{aligned}\sin^3 x + \cos^3 x &= (\sin x + \cos x)\big(\sin^2 x - \sin x \cos x + \cos^2 x\big)\\ &= (\sin x + \cos x)(1 - \sin x \cos x)\\ &= \frac{1}{2}\left(1 - \left(-\frac{3}{8}\right)\right) = \frac{1}{2}\cdot\frac{11}{8} = \frac{11}{16}.\end{aligned}$$

"If $\sin^4 x + \cos^4 x = k$, find $\sin x \cos x$"

Reduce the power sum, then solve for the product $p = \sin x \cos x$.

$$\begin{aligned}\sin^4 x + \cos^4 x &= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x\\ &= 1 - 2p^2 && \text{since } \sin^2 x + \cos^2 x = 1\\ k &= 1 - 2p^2\\ p^2 &= \frac{1-k}{2}\\ \sin x \cos x &= \pm\sqrt{\frac{1-k}{2}}.\end{aligned}$$

The sign depends on the quadrant of x. As a check, $k = 1$ gives $p = 0$ (an axis angle), and $k = \frac{1}{2}$ gives $p = \pm\frac{1}{2}$, i.e. $\sin 2x = \pm 1$.

Tip. For any "given one symmetric fact, find another" problem: square the given (or use $s^2 + c^2 = 1$) to extract sc, then express the target through $s + c$ and sc only. You almost never need the individual values of $\sin x$ and $\cos x$.

Conditional identities from $\tan x$

If you are given $\tan x = t$, divide numerator and denominator by $\cos^2 x$ to turn symmetric expressions into rational functions of t:

$$\sin x \cos x = \frac{\tan x}{1 + \tan^2 x} = \frac{t}{1 + t^2},$$

$$\sin^2 x - \cos^2 x = \frac{\tan^2 x - 1}{\tan^2 x + 1} = \frac{t^2 - 1}{t^2 + 1}.$$

Both follow from dividing by $\sin^2 x + \cos^2 x = 1$ written as $\cos^2 x(\tan^2 x + 1)$.

Multi-step proof: common denominator, then a conjugate collapse

Verify $\dfrac{1+\sin x}{\cos x} + \dfrac{\cos x}{1+\sin x} = 2\sec x$.

$$
\begin{aligned}
\frac{1+\sin x}{\cos x} + \frac{\cos x}{1+\sin x} &= \frac{(1+\sin x)^2 + \cos^2 x}{\cos x\,(1+\sin x)} && \text{common denominator} \\
&= \frac{1 + 2\sin x + \sin^2 x + \cos^2 x}{\cos x\,(1+\sin x)} \\
&= \frac{2 + 2\sin x}{\cos x\,(1+\sin x)} && \text{use } \sin^2 x + \cos^2 x = 1 \\
&= \frac{2(1+\sin x)}{\cos x\,(1+\sin x)} = \frac{2}{\cos x} = 2\sec x. \quad \checkmark
\end{aligned}
$$

Multi-step proof: factor out, then substitute Pythagorean

Verify $\tan^2 x - \sin^2 x = \tan^2 x\ \sin^2 x$.

$$
\begin{aligned}
\tan^2 x - \sin^2 x &= \frac{\sin^2 x}{\cos^2 x} - \sin^2 x && \text{convert to sin/cos} \\
&= \sin^2 x\left(\frac{1}{\cos^2 x} - 1\right) && \text{factor out } \sin^2 x \\
&= \sin^2 x \cdot \frac{1 - \cos^2 x}{\cos^2 x} \\
&= \sin^2 x \cdot \frac{\sin^2 x}{\cos^2 x} = \sin^2 x\ \tan^2 x. \quad \checkmark
\end{aligned}
$$

A long expression that is secretly a constant

Show that $\sin^6 x + \cos^6 x + 3\sin^2 x \cos^2 x = 1$ for every x.

$$
\begin{aligned}
\sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x(\sin^2 x + \cos^2 x) \\
&= 1 - 3\sin^2 x \cos^2 x && \text{since } \sin^2 x + \cos^2 x = 1 \\
\sin^6 x + \cos^6 x + 3\sin^2 x \cos^2 x &= \bigl(1 - 3\sin^2 x \cos^2 x\bigr) + 3\sin^2 x \cos^2 x \\
&= 1. \quad \checkmark
\end{aligned}
$$

The $\sin^2 x \cos^2 x$ terms cancel exactly—the whole expression is constant.

Tip. A long trig expression that reduces to a *constant* almost always hides a power-sum reduction: rewrite $\sin^4/\sin^6$ (and their cosine partners) using $\sin^2 x + \cos^2 x = 1$, and watch the leftover product terms cancel. If they do not cancel, recheck a sign before doubting the identity.

13 Sum, Difference, and Multiple-Angle Formulas

Exact values, double & half-angles, power-reducing, product-to-sum

13.1 Sum and Difference Formulas

The six core formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Watch the signs: cosine *flips* the sign ($+$ becomes $-$); sine and tangent *keep* it.

Exact value: $\cos 15°$ via a difference

Write $15° = 45° - 30°$ and use $\cos(A-B)$:

$$\begin{aligned}\cos 15° &= \cos(45° - 30°) = \cos 45°\cos 30° + \sin 45°\sin 30° \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}\end{aligned}$$

Exact value: $\tan 75°$ via a sum

Write $75° = 45° + 30°$:

$$\begin{aligned}\tan 75° &= \frac{\tan 45° + \tan 30°}{1 - \tan 45°\tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \\ &= \frac{(\sqrt{3}+1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{4+2\sqrt{3}}{2} = \boxed{2+\sqrt{3}}\end{aligned}$$

Tip: A "non-special" angle is usually a sum or difference of two special angles $(30°, 45°, 60°, 90°)$. Pick the pair whose sum or difference lands on your target.

13.2 Finding Exact Values of Non-Special Angles

Common decompositions

$$15^\circ = 45^\circ - 30^\circ, \quad 75^\circ = 45^\circ + 30^\circ, \quad 105^\circ = 60^\circ + 45^\circ, \quad \tfrac{7\pi}{12} = 105^\circ = \tfrac{\pi}{3} + \tfrac{\pi}{4}$$

Then apply the matching sum or difference formula and simplify the radicals.

Exact value: $\sin \dfrac{7\pi}{12}$

Since $\dfrac{7\pi}{12} = \dfrac{\pi}{3} + \dfrac{\pi}{4}$ (that is $105^\circ = 60^\circ + 45^\circ$):

$$\begin{aligned}
\sin \frac{7\pi}{12} &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} + \cos \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}
\end{aligned}$$

For $\cos \dfrac{7\pi}{12}$ the same setup gives $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ (negative, as expected in Quadrant II).

Tip: Sanity-check the sign against the quadrant. 105° sits in Quadrant II, so its sine is positive and its cosine is negative — exactly what the radicals show.

13.3 Double-Angle Formulas

Double-angle identities

$$\sin 2A = 2 \sin A \cos A, \qquad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

Choose the cosine form that matches what you already know: use $2\cos^2 A - 1$ when you know $\cos A$, and $1 - 2\sin^2 A$ when you know $\sin A$.

Given $\sin\theta = \frac{3}{5}$ with θ in Quadrant II

In Quadrant II cosine is negative, so $\cos\theta = -\frac{4}{5}$.

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta = 2 \cdot \tfrac{3}{5} \cdot \left(-\tfrac{4}{5}\right) = -\frac{24}{25} \\
\cos 2\theta &= 1 - 2\sin^2\theta = 1 - 2 \cdot \tfrac{9}{25} = 1 - \tfrac{18}{25} = \frac{7}{25} \\
\tan 2\theta &= \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24/25}{7/25} = -\frac{24}{7}
\end{aligned}$$

Tip: 2θ can land in a different quadrant than θ. Trust the algebra: the sign comes out automatically once $\sin\theta$ and $\cos\theta$ carry their correct signs.

13.4 Half-Angle Formulas

Half-angle identities

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}, \qquad \cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$
$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

The $\pm$ is decided by the quadrant of $\frac{A}{2}$, *not* of A. (The two tangent quotient forms need no sign choice.)

Given $\cos\theta = \frac{3}{5}$ with θ in Quadrant IV

Here $270° < \theta < 360°$, so $135° < \frac{\theta}{2} < 180°$: the half-angle is in Quadrant II, where sine is positive and cosine is negative.

$$\sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{3}{5}}{2}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5}$$
$$\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{3}{5}}{2}} = -\sqrt{\frac{4}{5}} = -\frac{2\sqrt{5}}{5}$$

Tip: Always halve the interval first. If θ is in Quadrant IV, $\frac{\theta}{2}$ is in Quadrant II — a fresh sign decision every time.

13.5 Power-Reducing Formulas

Lowering even powers

$$\sin^2 A = \frac{1-\cos 2A}{2}, \qquad \cos^2 A = \frac{1+\cos 2A}{2}, \qquad \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}$$

These are just the double-angle cosine identity solved for $\sin^2 A$ and $\cos^2 A$. Apply repeatedly to reduce higher powers.

Reduce $\sin^4 x$ to first powers of cosine

$$\sin^4 x = \left(\sin^2 x\right)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1-2\cos 2x+\cos^2 2x}{4}$$
$$= \frac{1-2\cos 2x+\frac{1+\cos 4x}{2}}{4} = \boxed{\frac{3-4\cos 2x+\cos 4x}{8}}$$

Tip: Power-reducing is the key to integrating and simplifying $\sin^2$, $\cos^2$, $\sin^4, \ldots$ — keep lowering the power until every term is a first-power cosine.

13.6 Product-to-Sum and Sum-to-Product

Two families of identities

Product-to-sum:

$$\sin A\cos B = \tfrac{1}{2}\big[\sin(A+B)+\sin(A-B)\big]$$
$$\cos A\sin B = \tfrac{1}{2}\big[\sin(A+B)-\sin(A-B)\big]$$
$$\cos A\cos B = \tfrac{1}{2}\big[\cos(A-B)+\cos(A+B)\big]$$
$$\sin A\sin B = \tfrac{1}{2}\big[\cos(A-B)-\cos(A+B)\big]$$

Sum-to-product:

$$\sin A+\sin B = 2\sin\tfrac{A+B}{2}\cos\tfrac{A-B}{2}$$
$$\sin A-\sin B = 2\cos\tfrac{A+B}{2}\sin\tfrac{A-B}{2}$$
$$\cos A+\cos B = 2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2}$$
$$\cos A-\cos B = -2\sin\tfrac{A+B}{2}\sin\tfrac{A-B}{2}$$

Product-to-sum: exact value of $\cos 75^\circ \cos 15^\circ$

$$\cos 75^\circ\cos 15^\circ = \tfrac{1}{2}\big[\cos(75^\circ-15^\circ)+\cos(75^\circ+15^\circ)\big]$$
$$= \tfrac{1}{2}\big[\cos 60^\circ+\cos 90^\circ\big] = \tfrac{1}{2}\left[\tfrac{1}{2}+0\right] = \boxed{\tfrac{1}{4}}$$

Sum-to-product: exact value of $\sin 75^\circ + \sin 15^\circ$

$$\sin 75^\circ + \sin 15^\circ = 2\sin\frac{75^\circ+15^\circ}{2}\cos\frac{75^\circ-15^\circ}{2} = 2\sin 45^\circ\cos 30^\circ$$
$$= 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{6}}{2}}$$

Tip: Use *product-to-sum* when a product blocks you (integrals, exact values); use *sum-to-product* to factor a sum so it can cancel (proving identities, solving equations).

13.7 Simplifying, Verifying Identities, and Finding Values

Strategy

- To **simplify**, spot the pattern: $2\sin A\cos A = \sin 2A$, $\cos^2 A - \sin^2 A = \cos 2A$, $1 - \cos 2A = 2\sin^2 A$.
- To **verify**, transform the busier side using double/half-angle or power-reducing until it matches the other.
- To **find values given quadrants**, first recover sin and cos of each angle (with correct signs), then substitute into a sum/difference or double-angle formula.

Verify $\dfrac{\sin 2x}{1+\cos 2x} = \tan x$

$$\frac{\sin 2x}{1+\cos 2x} = \frac{2\sin x\cos x}{1+(2\cos^2 x - 1)} = \frac{2\sin x\cos x}{2\cos^2 x} = \frac{\sin x}{\cos x} = \tan x \checkmark$$

Find $\sin(\alpha+\beta)$ **given quadrants**

Let $\sin\alpha = \frac{3}{5}$ with α in Quadrant II, and $\cos\beta = \frac{5}{13}$ with β in Quadrant IV.

$$\alpha \text{ in QII} : \cos\alpha = -\tfrac{4}{5}; \qquad \beta \text{ in QIV} : \sin\beta = -\tfrac{12}{13}$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \tfrac{3}{5}\cdot\tfrac{5}{13} + \left(-\tfrac{4}{5}\right)\left(-\tfrac{12}{13}\right)$$
$$= \frac{15}{65} + \frac{48}{65} = \boxed{\frac{63}{65}}$$

Tip: Before plugging in, draw each angle's reference triangle and label signs by quadrant. A single wrong sign on $\cos\alpha$ or $\sin\beta$ ruins the whole answer.

13.8 Going Deeper: Advanced Formula Techniques

Telescoping products of cosines

Multiplying a chain of angle-doubling cosines by $2\sin\theta$ collapses it, because each step obeys $2\sin\phi\cos\phi = \sin 2\phi$. The result is the beautiful identity

$$\prod_{k=0}^{n-1} \cos\left(2^k\theta\right) = \cos\theta\,\cos 2\theta\,\cos 4\theta \cdots \cos\left(2^{n-1}\theta\right) = \frac{\sin(2^n\theta)}{2^n\sin\theta}.$$

The whole product depends only on the *first* and *last* angles — everything in between telescopes away.

A striking exact value: $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$

Take $\theta = \dfrac{\pi}{7}$ and $n = 3$, so the angles are $\theta, 2\theta, 4\theta$. Apply the telescoping identity:

$$\begin{aligned}\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} &= \frac{\sin\left(2^3\cdot\frac{\pi}{7}\right)}{2^3\sin\frac{\pi}{7}} = \frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}} \\ &= \frac{\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}} = \frac{-\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = \boxed{-\frac{1}{8}}\end{aligned}$$

The key step is $\sin\left(\pi+\frac{\pi}{7}\right) = -\sin\frac{\pi}{7}$, which cancels the leftover sine exactly.

Triple-angle formulas

Chaining a double-angle with a sum formula ($3A = 2A + A$) gives the triple-angle identities:

$$\sin 3A = 3\sin A - 4\sin^3 A, \qquad \cos 3A = 4\cos^3 A - 3\cos A,$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

Read as cubics in $\sin A$ or $\cos A$, they turn angle problems into polynomial equations — the doorway to exact values like $\cos 36°$.

$\sin 18°$ and $\cos 36°$ from a cubic

Let $\theta = 18°$, so $5\theta = 90°$ and hence $3\theta = 90° - 2\theta$. Taking sines,

$$\sin 3\theta = \sin(90° - 2\theta) = \cos 2\theta.$$

Now substitute the triple-angle and double-angle forms with $s = \sin 18°$:

$$\begin{aligned} 3s - 4s^3 &= 1 - 2s^2 \\ 4s^3 - 2s^2 - 3s + 1 &= 0 \\ (s-1)\left(4s^2 + 2s - 1\right) &= 0. \end{aligned}$$

Since $s = \sin 18° \neq 1$, solve $4s^2 + 2s - 1 = 0$ and keep the positive root:

$$s = \frac{-2 + \sqrt{4+16}}{8} = \frac{-2 + 2\sqrt{5}}{8} = \boxed{\sin 18° = \frac{\sqrt{5}-1}{4}}.$$

Then $\cos 36° = 1 - 2\sin^2 18° = 1 - 2 \cdot \dfrac{3-\sqrt{5}}{8} = \boxed{\cos 36° = \dfrac{1+\sqrt{5}}{4}}$, the golden ratio in disguise.

Summing equally spaced sines and cosines

For an arithmetic progression of angles a, $a+d$, $a+2d, \ldots$, multiply by $2\sin\frac{d}{2}$ and telescope with product-to-sum. The closed forms are

$$\sum_{k=0}^{n-1} \cos(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \cos\left(a + \tfrac{(n-1)d}{2}\right),$$

$$\sum_{k=0}^{n-1} \sin(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \sin\left(a + \tfrac{(n-1)d}{2}\right).$$

When the angles wrap evenly around the circle (so nd is a multiple of 2π), the factor $\sin\frac{nd}{2} = 0$ and both sums vanish — e.g. $\displaystyle\sum_{k=0}^{n-1} \cos\frac{2\pi k}{n} = 0$ for $n \geq 2$.

Nested half-angles for unusual exact values

Applying the half-angle cosine formula repeatedly to $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ builds a tower of nested radicals:

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} = \frac{1}{2}\sqrt{2+\sqrt{2}}, \qquad \cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}},$$

and in general $\cos\dfrac{\pi}{2^n} = \dfrac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n-1 \text{ radicals}}$. Each extra half-angle nests one more square root; the $+$ sign is used throughout because every $\frac{\pi}{2^n}$ lands in Quadrant I.

Given quadrants, find $\sin(\alpha-\beta)$ and $\tan 2\alpha$

Let $\sin\alpha = \frac{4}{5}$ with α in Quadrant II, and $\cos\beta = \frac{12}{13}$ with β in Quadrant IV. First recover the partners with correct signs:

$$\alpha \text{ in QII}: \ \cos\alpha = -\tfrac{3}{5}; \qquad \beta \text{ in QIV}: \ \sin\beta = -\tfrac{5}{13}.$$

Difference formula:

$$\begin{aligned}\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta &= \tfrac{4}{5}\cdot\tfrac{12}{13} - \left(-\tfrac{3}{5}\right)\left(-\tfrac{5}{13}\right)\\ &= \frac{48}{65} - \frac{15}{65} = \boxed{\frac{33}{65}}.\end{aligned}$$

Double-angle in tangent: with $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{4/5}{-3/5} = -\dfrac{4}{3}$,

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(-\frac{4}{3}\right)}{1-\frac{16}{9}} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{-8}{3}\cdot\frac{9}{-7} = \boxed{\frac{24}{7}}.$$

Tip: Advanced exact values almost always come from one of three moves — *telescoping* a product with $2\sin\theta$, turning a triple-angle relation into a *cubic*, or *nesting* half-angles. Spot which structure your target angle fits, and the radicals fall out.

14 Trigonometric Equations

Basic equations, factoring, quadratic form, identities, multiple angles, and inverses

14.1 Solving Basic Trigonometric Equations

A **trigonometric equation** is true only for certain values of the variable. Because sine, cosine, and tangent are periodic, most equations have *infinitely many* solutions. You will usually be asked for one of two things:

- **All solutions** — write a general formula using $+2\pi n$ (for sin/cos) or $+\pi n$ (for tan), where n is any integer.
- **Solutions on an interval**, usually $[0, 2\pi)$ — list only the values inside that interval.

The idea

To solve $\sin x = k$ (or $\cos x = k$, $\tan x = k$):

1. Find the **reference angle** from the special-angle table (or arcsin, arccos, arctan).
2. Use the sign of k to decide **which quadrants** the solutions live in.

3. Add the period to get *all* solutions: $+2\pi n$ for sine/cosine, $+\pi n$ for tangent.

Useful periods: $\sin x = 0 \Rightarrow x = \pi n$; $\cos x = 0 \Rightarrow x = \frac{\pi}{2} + \pi n$; $\tan x = 0 \Rightarrow x = \pi n$.

Worked example: all solutions and solutions on $[0, 2\pi)$

Solve $2\cos x + 1 = 0$.

$$\begin{aligned} 2\cos x + 1 &= 0 \\ \cos x &= -\tfrac{1}{2} \end{aligned}$$

Reference angle: $\dfrac{\pi}{3}$. Cosine is negative in **Quadrants II and III**, so on $[0, 2\pi)$

$$x = \frac{2\pi}{3}, \qquad x = \frac{4\pi}{3}.$$

All solutions: $x = \dfrac{2\pi}{3} + 2\pi n$ or $x = \dfrac{4\pi}{3} + 2\pi n$, $n \in \mathbb{Z}$.

The picture below shows why $\sin x = \frac{1}{2}$ has *two* solutions on $[0, 2\pi)$: the curve $y = \sin x$ crosses the line $y = \frac{1}{2}$ twice.

Don't forget the period. A common mistake is giving only one solution. For sine/cosine always attach $+2\pi n$; for tangent attach $+\pi n$. To restrict to $[0, 2\pi)$, keep adding the period until you leave the interval.

14.2 Equations Solved by Factoring

The idea

Move every term to one side so the equation equals 0, **factor**, then set each factor equal to 0 (the Zero-Product Property). Solve each simple equation separately and combine the answers.

Worked example: a full factoring solve

Solve $2\sin x \cos x = \cos x$ on $[0, 2\pi)$.

$$\begin{aligned} 2\sin x \cos x - \cos x &= 0 && \text{(get one side} = 0\text{)} \\ \cos x\,(2\sin x - 1) &= 0 && \text{(factor out } \cos x\text{)} \end{aligned}$$

Set each factor to zero:

$$\cos x = 0 \;\Rightarrow\; x = \frac{\pi}{2}, \frac{3\pi}{2}; \qquad 2\sin x - 1 = 0 \;\Rightarrow\; \sin x = \tfrac{1}{2} \;\Rightarrow\; x = \frac{\pi}{6}, \frac{5\pi}{6}.$$

Solutions: $x = \dfrac{\pi}{6},\ \dfrac{\pi}{2},\ \dfrac{5\pi}{6},\ \dfrac{3\pi}{2}.$

Never divide both sides by a trig factor. Dividing $2\sin x \cos x = \cos x$ by $\cos x$ throws away the solutions where $\cos x = 0$. Always *factor* instead.

14.3 Quadratic-Form Equations (Substitution)

The idea

If the equation looks like a quadratic in one trig function — for example $2\sin^2 x - \sin x - 1 = 0$ — substitute u for that function, solve the quadratic (factor or quadratic formula), then back-substitute. Discard any value with $|u| > 1$ for sine or cosine (impossible).

Worked example: substitution

Solve $2\sin^2 x - \sin x - 1 = 0$ on $[0, 2\pi)$. Let $u = \sin x$:

$$\begin{aligned} 2u^2 - u - 1 &= 0 \\ (2u+1)(u-1) &= 0 \\ u = -\tfrac{1}{2} \quad &\text{or} \quad u = 1. \end{aligned}$$

Back-substitute:

$$\sin x = 1 \ \Rightarrow\ x = \frac{\pi}{2}; \qquad \sin x = -\tfrac{1}{2} \ \Rightarrow\ x = \frac{7\pi}{6},\ \frac{11\pi}{6}.$$

Solutions: $x = \dfrac{\pi}{2},\ \dfrac{7\pi}{6},\ \dfrac{11\pi}{6}.$

Check the range. A factor like $\cos x = 2$ has *no* solution, since cosine never exceeds 1. Simply drop that factor and keep the others.

14.4 Equations Requiring an Identity First

The idea

When an equation mixes different functions or has a squared term, rewrite it in terms of a **single** trig function using an identity, then factor. Most common:

$$\sin^2 x = 1 - \cos^2 x, \qquad \cos^2 x = 1 - \sin^2 x, \qquad \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

Worked example: Pythagorean identity first

Solve $2\cos^2 x + \sin x - 1 = 0$ on $[0, 2\pi)$. Replace $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned} 2(1-\sin^2 x) + \sin x - 1 &= 0 \\ -2\sin^2 x + \sin x + 1 &= 0 \\ 2\sin^2 x - \sin x - 1 &= 0 \\ (2\sin x + 1)(\sin x - 1) &= 0. \end{aligned}$$

So $\sin x = -\frac{1}{2} \Rightarrow x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ and $\sin x = 1 \Rightarrow x = \dfrac{\pi}{2}$. **Solutions:** $x = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

One function at a time. You cannot factor an equation that still contains both sin and cos squared terms. Convert everything to a single function *before* you try to factor.

14.5 Multiple-Angle Equations

The idea

For an equation in $2x$, $3x$, or $\frac{x}{2}$, first solve for the **whole angle**, then divide. If $x \in [0, 2\pi)$ then:

$$2x \in [0, 4\pi), \qquad 3x \in [0, 6\pi), \qquad \tfrac{x}{2} \in [0, \pi).$$

List *all* angle solutions across the expanded interval, then divide each by the multiple.

Worked example: list all multiple-angle solutions

Solve $2\cos(2x) = 1$ on $[0, 2\pi)$.

$$\cos(2x) = \tfrac{1}{2}.$$

Let $\theta = 2x$. Since $x \in [0, 2\pi)$, we need $\theta \in [0, 4\pi)$. Cosine equals $\frac{1}{2}$ at reference angle $\frac{\pi}{3}$ in Quadrants I and IV, over two full turns:

$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}.$$

Now $x = \dfrac{\theta}{2}$:

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}.$$

General solution: $x = \dfrac{\pi}{6} + \pi n$ or $x = \dfrac{5\pi}{6} + \pi n$.

Expand the interval first. Because $2x$ sweeps around *twice* as fast, you get twice as many solutions. Solve for the angle over $[0, 4\pi)$ (or $[0, 6\pi)$ for $3x$) *before* dividing — otherwise you will lose half the answers.

14.6 Inverse Functions and Extraneous Solutions

The idea

Inverse functions: when the value is not a special angle, use arcsin, arccos, or arctan on a calculator. It returns only *one* angle; use quadrant symmetry and the period to find the rest.

Extraneous solutions: *squaring* both sides or *multiplying* by a trig factor can create false solutions. Always substitute each candidate back into the **original** equation and discard any that fail (or make a function undefined).

Worked example: inverse function

Solve $\tan x = 3$ on $[0, 2\pi)$ (round to three decimals). The reference is $\arctan 3 \approx 1.249$. Tangent is positive in Quadrants I and III:

$$x \approx 1.249, \qquad x \approx 1.249 + \pi \approx 4.391.$$

Worked example: checking for extraneous solutions

Solve $\sin x - \cos x = 1$ on $[0, 2\pi)$. Square both sides:

$$\begin{aligned}(\sin x - \cos x)^2 &= 1^2\\ \sin^2 x - 2\sin x\cos x + \cos^2 x &= 1\\ 1 - 2\sin x\cos x &= 1\\ \sin x\cos x &= 0.\end{aligned}$$

Candidates: $\sin x = 0 \Rightarrow x = 0, \pi$; $\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$. **Check each in the original:**

$$\begin{array}{llll} x = 0: & 0 - 1 = -1 \neq 1 & \text{(reject)} & \\ x = \frac{\pi}{2}: & 1 - 0 = 1\checkmark & x = \pi: & 0 - (-1) = 1\checkmark \\ x = \frac{3\pi}{2}: & -1 - 0 = -1 \neq 1 & \text{(reject)} & \end{array}$$

Solutions: $x = \dfrac{\pi}{2},\ \pi.$

Always verify after squaring or multiplying. These steps are not reversible — substitute every candidate into the *original* equation, and throw out any that do not check or that make tan, sec, etc. undefined.

14.7 Going Deeper: Advanced Trig Equations

The equations below combine every earlier technique. The recurring theme is *convert first, solve second*: reshape the equation with an identity or an R-form until it becomes one of the basic types you already know.

Sum-to-product and double-angle as an opening move

Some equations cannot be factored until you first **merge or split** terms. Two toolkits do this:

- **Sum-to-product** turns a *sum* of two sines/cosines into a *product*, which then factors:

$$\sin A + \sin B = 2\sin\tfrac{A+B}{2}\cos\tfrac{A-B}{2}, \qquad \cos A + \cos B = 2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2}.$$

- **Double-angle** rewrites $\sin 2x$ or $\cos 2x$ so a mixed equation collapses to a single function:

$$\sin 2x = 2\sin x\cos x, \qquad \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

Rule of thumb: a *sum* of trig terms $\to$ sum-to-product; a lone $\sin 2x/\cos 2x$ mixed with $\sin x/\cos x$ $\to$ double-angle.

Worked example: sum-to-product first

Solve $\cos 3x + \cos x = 0$ on $[0, 2\pi)$. With $A = 3x$, $B = x$, so $\frac{A+B}{2} = 2x$ and $\frac{A-B}{2} = x$:

$$\begin{aligned} \cos 3x + \cos x &= 0 \\ 2\cos 2x\,\cos x &= 0 && \text{(sum-to-product)} \\ \cos 2x = 0 \quad &\text{or} \quad \cos x = 0. \end{aligned}$$

For $\cos x = 0$: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$. For $\cos 2x = 0$, let $\theta = 2x \in [0, 4\pi)$, so $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$ and hence $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$.

$$\textbf{Solutions: } x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

Counting solutions of a multiple-angle equation

For $\sin(kx) = c$ or $\cos(kx) = c$ with k a positive integer and $-1 < c < 1$, the whole angle kx ranges over $[0, 2\pi k)$ when $x \in [0, 2\pi)$. Each full turn contributes **two** solutions, so there are exactly $2k$ solutions on $[0, 2\pi)$. (For $\tan(kx) = c$ there are k solutions, since tan repeats every π.) This lets you predict the count *before* solving, then check you found them all.

Worked example: find every solution and count them

Solve $\sin(3x) = \frac{\sqrt{3}}{2}$ on $[0, 2\pi)$. Here $k = 3$, so expect $2k = 6$ solutions. Let $\theta = 3x \in [0, 6\pi)$. Sine equals $\frac{\sqrt{3}}{2}$ at reference angle $\frac{\pi}{3}$ in Quadrants I and II, i.e. $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$, repeated every 2π across three turns:

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}.$$

Divide by 3 (i.e. $x = \theta/3$):

$$\textbf{Solutions: } x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}.$$

That is exactly 6 solutions, as predicted.

The R-form: solving $a\sin x + b\cos x = c$

A sum of a sine and a cosine *with the same angle* can be rewritten as a single sinusoid:

$$a\sin x + b\cos x = R\sin(x + \varphi), \qquad R = \sqrt{a^2 + b^2}, \qquad \tan\varphi = \frac{b}{a}.$$

Choose φ in the quadrant matching the signs of a (as the "cosine" part) and b (as the "sine" part). The equation becomes $R\sin(x+\varphi) = c$, i.e. $\sin(x+\varphi) = \frac{c}{R}$, a basic equation. **Existence check:** real solutions exist only when $|c| \le R = \sqrt{a^2 + b^2}$.

Worked example: the R-form

Solve $\sin x + \sqrt{3}\cos x = 1$ on $[0, 2\pi)$. Here $a = 1$, $b = \sqrt{3}$, so

$$R = \sqrt{1+3} = 2, \qquad \tan\varphi = \frac{\sqrt{3}}{1} \Rightarrow \varphi = \frac{\pi}{3}.$$

Both $a > 0$ and $b > 0$, so $\varphi = \frac{\pi}{3}$ (Quadrant I). The equation becomes

$$2\sin\left(x + \frac{\pi}{3}\right) = 1 \;\Rightarrow\; \sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}.$$

Let $u = x + \frac{\pi}{3}$. Since $x \in [0, 2\pi)$, $u \in \left[\frac{\pi}{3}, 2\pi + \frac{\pi}{3}\right)$. Sine is $\frac{1}{2}$ at $u = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}$; keep those inside the u-interval: $u = \frac{5\pi}{6}$ and $u = \frac{13\pi}{6}$. Then $x = u - \frac{\pi}{3}$:

$$\textbf{Solutions: } x = \frac{\pi}{2}, \frac{11\pi}{6}.$$

Worked example: quadratic in $\sin$ with careful casework

Solve $2\sin^2 x - 3\sin x + 1 = 0$ on $[0, 2\pi)$. Let $u = \sin x$:

$$\begin{aligned} 2u^2 - 3u + 1 &= 0 \\ (2u-1)(u-1) &= 0 \\ u = \tfrac{1}{2} \quad &\text{or} \quad u = 1. \end{aligned}$$

Now handle each case separately and note how many solutions each yields:

$$\begin{aligned} \sin x = \tfrac{1}{2}: &\quad \text{two solutions} & x &= \frac{\pi}{6}, \frac{5\pi}{6}, \\ \sin x = 1: &\quad \text{one solution (a peak)} & x &= \frac{\pi}{2}. \end{aligned}$$

A boundary value like $\sin x = 1$ gives a *single* solution because the curve just touches the line, not crosses it.

$$\textbf{Solutions:}\ x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \quad \text{(three total)}.$$

Boundary cases count once. When a quadratic in $\sin x$ or $\cos x$ produces $u = \pm 1$, that value gives *one* solution on $[0, 2\pi)$, not two, because $\sin x = 1$, $\sin x = -1$, $\cos x = 1$, $\cos x = -1$ each occur at a single point. Values with $|u| < 1$ give two. Miscounting here is the most common error in "how many solutions" problems.

Parameter conditions: how many solutions does $\sin x = k$ have?

Think of solving $f(x) = k$ as intersecting the graph $y = f(x)$ with the horizontal line $y = k$. For $\sin x = k$ on $[0, 2\pi)$:

$$\begin{aligned} |k| > 1: &\quad \text{no solution (line misses the curve),} \\ k = 1 \text{ or } k = -1: &\quad \text{exactly one solution (line is tangent at a peak/valley),} \\ -1 < k < 1,\ k \neq 0: &\quad \text{two solutions,} \\ k = 0: &\quad \text{two solutions on } [0, 2\pi)\ (x = 0, \pi). \end{aligned}$$

The same graph-versus-line reasoning fixes the count for $\cos x = k$ and, after the R-form, for $a\sin x + b\cos x = c$: that equation has **no** solution when $|c| > \sqrt{a^2+b^2}$, **one** (tangent) when $|c| = \sqrt{a^2+b^2}$, and **two** per period when $|c| < \sqrt{a^2+b^2}$.

Worked example: parameter condition

For which values of k does $\sqrt{3}\sin x + \cos x = k$ have *exactly one* solution on each period? Rewrite in R-form: $R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$, so the equation is $2\sin(x+\varphi) = k$, i.e. $\sin(x+\varphi) = \frac{k}{2}$. A basic sine equation has exactly one solution per period only at the tangent values $\frac{k}{2} = \pm 1$, giving

$$\boxed{k = 2 \quad \text{or} \quad k = -2.}$$

For $|k| < 2$ there are two solutions per period; for $|k| > 2$ there are none.

Systems of trig equations

To solve two trig equations simultaneously, find each variable's candidate set, then keep only the pairs that satisfy **both**. A frequent case is a system that hides an identity — for instance, given $\sin x + \sin y$ and $\cos x + \cos y$, apply sum-to-product to both and divide to eliminate a variable. Always finish by *checking* candidates in both original equations, since combining equations (adding, dividing, squaring) can introduce extraneous pairs.

Worked example: a small system

Solve the system on $[0, 2\pi)$:

$$\sin x + \sin y = 1, \qquad x + y = \frac{\pi}{2}.$$

From the second equation $y = \frac{\pi}{2} - x$, so $\sin y = \sin\left(\frac{\pi}{2} - x\right) = \cos x$. Substitute:

$$\sin x + \cos x = 1.$$

Use the R-form with $a = b = 1$: $R = \sqrt{2}$, $\varphi = \frac{\pi}{4}$, giving $\sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right) = 1$, so $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Then $x + \frac{\pi}{4} = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, i.e. $x = 0$ or $x = \frac{\pi}{2}$. The matching $y = \frac{\pi}{2} - x$ gives

$$\textbf{Solutions:}\ (x, y) = \left(0,\ \frac{\pi}{2}\right) \ \text{ and } \ \left(\frac{\pi}{2},\ 0\right).$$

Both check in the original system.

Convert before you count. Advanced equations rarely give up their solutions in raw form. First reshape — sum-to-product, double-angle, or the R-form — until you reach $\sin(\text{angle}) = c$ or a factored product. Only then apply the interval and period rules, and always re-check candidates from squaring, dividing, or combining equations against the *original*.

15 Course Review

Every definition, rule, and formula from all twelve topics in one place

15.1 Angles and Their Measure

Degrees, radians, and conversion

An angle in **standard position** has its vertex at the origin and its initial side on the positive

x-axis. One full revolution is $360° = 2\pi$ radians, so a straight angle is $180° = \pi$ radians.

$$\text{radians} = \text{degrees} \cdot \frac{\pi}{180°}, \qquad \text{degrees} = \text{radians} \cdot \frac{180°}{\pi}.$$

Coterminal angles share a terminal side; add or subtract full turns: $\theta + 360°n$ (degrees) or $\theta + 2\pi n$ (radians), n an integer. **Complementary** angles sum to $90°$; **supplementary** angles sum to $180°$.

Arc length, sector area, and speed

For a central angle θ measured in *radians* in a circle of radius r:

$$s = r\theta \quad \text{(arc length)}, \qquad A = \tfrac{1}{2}r^2\theta \quad \text{(sector area)}.$$

If a point moves along the circle, its **linear speed** and **angular speed** ω are

$$v = \frac{s}{t} = r\omega, \qquad \omega = \frac{\theta}{t}.$$

Worked example: arc length and area

A sector has radius $r = 6$ cm and central angle $\theta = 60° = \dfrac{\pi}{3}$.

$$s = r\theta = 6 \cdot \tfrac{\pi}{3} = 2\pi \approx 6.28 \text{ cm}, \quad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(36)\tfrac{\pi}{3} = 6\pi \approx 18.85 \text{ cm}^2.$$

Tip. The formulas $s = r\theta$, $A = \frac{1}{2}r^2\theta$, and $v = r\omega$ require θ in *radians*. Convert degrees first, or the answer will be wrong by a factor of $\pi/180$.

15.2 Right Triangle Trigonometry

SOH-CAH-TOA and reciprocals

For an acute angle θ in a right triangle,

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

The reciprocals are

$$\csc\theta = \frac{\text{hyp}}{\text{opp}}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}.$$

Special angles and cofunctions

Cofunctions of complementary angles are equal:

$$\sin\theta = \cos(90^\circ - \theta), \quad \tan\theta = \cot(90^\circ - \theta), \quad \sec\theta = \csc(90^\circ - \theta).$$

Worked example: angle of elevation

From a point 50 ft from the base of a tree, the angle of elevation to the top is 32°. The height h satisfies

$$\tan 32^\circ = \frac{h}{50} \;\Rightarrow\; h = 50\tan 32^\circ \approx 31.2 \text{ ft}.$$

Tip. An **angle of elevation** is measured *up* from the horizontal; an **angle of depression** is measured *down* from the horizontal. They are equal (alternate interior angles), so a depression angle can be moved to the object's location.

15.3 The Unit Circle and Circular Functions

Definition from the unit circle

On the unit circle $x^2 + y^2 = 1$, if the terminal side of θ meets the circle at (x, y) then

$$\cos\theta = x, \qquad \sin\theta = y, \qquad \tan\theta = \frac{y}{x},$$

with $\sec\theta = \frac{1}{x}$, $\csc\theta = \frac{1}{y}$, $\cot\theta = \frac{x}{y}$. For a point (x, y) *not* on the unit circle, let $r = \sqrt{x^2 + y^2}$; then $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$.

Reference angles, signs, period, and symmetry

The **reference angle** θ' is the acute angle to the x-axis:

$$\text{Q II: } 180^\circ - \theta, \quad \text{Q III: } \theta - 180^\circ, \quad \text{Q IV: } 360^\circ - \theta.$$

Signs follow **ASTC** ("**A**ll **S**tudents **T**ake **C**alculus"): in QI all are $+$; QII only $\sin, \csc$; QIII only $\tan, \cot$; QIV only $\cos, \sec$. Periods: $\sin, \cos, \sec, \csc$ repeat every 2π; $\tan, \cot$ every π. **Even:** $\cos, \sec$; **odd:** $\sin, \csc, \tan, \cot$.

Worked example: reference angle then sign

Evaluate $\sin 210^\circ$. It lies in QIII with reference angle $210^\circ - 180^\circ = 30^\circ$, and sine is negative in QIII, so

$$\sin 210^\circ = -\sin 30^\circ = -\tfrac{1}{2}.$$

Tip. Cosine is the x-coordinate, sine is the y-coordinate. At the quadrantal angles a coordinate is 0, making $\tan, \sec$ (when $x = 0$) or $\cot, \csc$ (when $y = 0$) undefined.

15.4 Graphs of Sine and Cosine

The general sinusoid

For $y = a\sin\big(b(x-c)\big) + d$ or $y = a\cos\big(b(x-c)\big) + d$ with $b > 0$:

$$\text{amplitude} = |a|, \qquad \text{period} = \frac{2\pi}{b},$$

$$\text{phase shift} = c \text{ (right if } c > 0), \qquad \text{vertical shift} = d.$$

The midline is $y = d$; the graph oscillates between $y = d - |a|$ and $y = d + |a|$. If $a < 0$ the basic shape is reflected across the midline.

Worked example: reading a graph's equation

For $y = 3\cos\big(2x - \pi\big) + 1 = 3\cos\big(2(x - \frac{\pi}{2})\big) + 1$:

$$|a| = 3, \quad \text{period} = \frac{2\pi}{2} = \pi, \quad \text{phase shift} = \frac{\pi}{2} \text{ right}, \quad d = 1.$$

Range: $[1-3,\ 1+3] = [-2, 4]$.

Tip. Always factor out b before reading the phase shift: in $\sin(bx - c)$ the shift is c/b, *not* c. Divide the period into quarters to plot the five key points (max, zero, min, zero).

15.5 Graphs of the Other Trig Functions

Tangent and cotangent

$$y = a\tan\big(b(x-c)\big): \quad \text{period} = \frac{\pi}{b}, \quad \text{asymptotes where } bx = \tfrac{\pi}{2} + \pi n.$$

$$y = a\cot\big(b(x-c)\big): \quad \text{period} = \frac{\pi}{b}, \quad \text{asymptotes where } bx = \pi n.$$

Both have range all real numbers and no amplitude. Tangent increases on each branch; cotangent decreases.

Secant and cosecant

$$y = a\sec\big(b(x-c)\big) \text{ and } y = a\csc\big(b(x-c)\big): \quad \text{period} = \frac{2\pi}{b}.$$

Graph the guide sine/cosine first: sec has vertical asymptotes where its guide cos $= 0$; csc where its guide sin $= 0$. Range is $(-\infty, -|a|] \cup [|a|, \infty)$; the curve never crosses the midline.

Worked example: period and asymptotes of a tangent

For $y = \tan\big(\frac{1}{2}x\big)$: period $= \dfrac{\pi}{1/2} = 2\pi$, and asymptotes occur where $\frac{1}{2}x = \frac{\pi}{2} + \pi n$, i.e. $x = \pi + 2\pi n$.

Tip. Tangent/cotangent use π/b for the period; secant/cosecant use $2\pi/b$. Sketch the reciprocal sine or cosine lightly first — its zeros become the asymptotes, and its peaks become the U-shaped turning points.

15.6 Inverse Trigonometric Functions

Domains and ranges of the inverses

$\sin^{-1} x = y$ means $\sin y = x$ with y in the range above. (Also written arcsin, arccos, arctan.)

Composition rules

$$\sin(\sin^{-1} x) = x \;\text{ for } x \in [-1, 1], \qquad \sin^{-1}(\sin\theta) = \theta \;\text{ only if } \theta \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

For $\cos(\sin^{-1} x)$ and similar mixed compositions, draw a reference triangle: let the inner inverse define an angle, label two sides, find the third by $a^2 + b^2 = c^2$.

Worked example: a range-restriction trap

Evaluate $\sin^{-1}\big(\sin\frac{3\pi}{4}\big)$. Since $\frac{3\pi}{4} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$, the answer is *not* $\frac{3\pi}{4}$. Compute $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$, then take the angle in range: $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$.

Tip. An inverse trig function returns exactly one angle in its restricted range. $\cos^{-1}$ never returns a negative angle; $\sin^{-1}$ and $\tan^{-1}$ never return an angle outside $\pm\frac{\pi}{2}$.

15.7 Trigonometric Identities

The fundamental identities

Reciprocal:

$$\csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta}$$

Quotient:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Cofunction: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$, $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$ (and the three partners).
Even/odd: $\cos(-\theta) = \cos\theta$, $\sec(-\theta) = \sec\theta$; $\sin(-\theta) = -\sin\theta$, $\csc(-\theta) = -\csc\theta$, $\tan(-\theta) = -\tan\theta$, $\cot(-\theta) = -\cot\theta$.

Worked example: verifying an identity

Verify $\sec\theta - \sin\theta\tan\theta = \cos\theta$.

$$\sec\theta - \sin\theta\tan\theta = \frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta} = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta. \quad \checkmark$$

Tip. When stuck, rewrite everything in $\sin\theta$ and $\cos\theta$, combine over a common denominator, and hunt for a Pythagorean pattern such as $1 - \sin^2\theta = \cos^2\theta$.

15.8 Sum, Difference, and Multiple-Angle Formulas

Sum and difference

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Cosine *flips* the sign; sine and tangent *keep* it.

Double-angle and half-angle

Double:

$$\sin 2A = 2\sin A\cos A$$

$$\begin{aligned}\cos 2A &= \cos^2 A - \sin^2 A\\ &= 2\cos^2 A - 1 = 1 - 2\sin^2 A\end{aligned}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Half:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A}$$

The $\pm$ on a half-angle is chosen from the quadrant of $\frac{A}{2}$.

Power-reducing, product-to-sum, sum-to-product

Power-reducing:

$$\sin^2 A = \frac{1-\cos 2A}{2}, \quad \cos^2 A = \frac{1+\cos 2A}{2}, \quad \tan^2 A = \frac{1-\cos 2A}{1+\cos 2A}.$$

Product-to-sum:

$$\begin{aligned}\sin A\cos B &= \tfrac{1}{2}\big[\sin(A+B)+\sin(A-B)\big]\\ \cos A\sin B &= \tfrac{1}{2}\big[\sin(A+B)-\sin(A-B)\big]\\ \cos A\cos B &= \tfrac{1}{2}\big[\cos(A-B)+\cos(A+B)\big]\\ \sin A\sin B &= \tfrac{1}{2}\big[\cos(A-B)-\cos(A+B)\big]\end{aligned}$$

Sum-to-product:

$$\begin{aligned}\sin A+\sin B &= 2\sin\tfrac{A+B}{2}\cos\tfrac{A-B}{2}\\ \sin A-\sin B &= 2\cos\tfrac{A+B}{2}\sin\tfrac{A-B}{2}\\ \cos A+\cos B &= 2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2}\\ \cos A-\cos B &= -2\sin\tfrac{A+B}{2}\sin\tfrac{A-B}{2}\end{aligned}$$

Worked example: exact value of $\cos 15^\circ$

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ-30^\circ) = \cos 45^\circ\cos 30^\circ + \sin 45^\circ\sin 30^\circ\\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

Tip. To find sin or cos of a non-special angle, write it as a sum or difference of $30^\circ, 45^\circ, 60^\circ$. Use $2\cos^2 A - 1$ when you know $\cos A$ and $1-2\sin^2 A$ when you know $\sin A$.

15.9 Trigonometric Equations

General solutions

Solve for the reference angle, place it in every quadrant the sign allows, then add the period:

$$\sin, \cos, \sec, \csc: \text{ add } +2\pi n, \qquad \tan, \cot: \text{ add } +\pi n,$$

where n is any integer. Example patterns:

$$\sin\theta = \tfrac{1}{2} \;\Rightarrow\; \theta = \tfrac{\pi}{6} + 2\pi n \;\text{ or }\; \theta = \tfrac{5\pi}{6} + 2\pi n.$$

$$\tan\theta = 1 \;\Rightarrow\; \theta = \tfrac{\pi}{4} + \pi n.$$

Factoring, quadratic form, and multiple angles

- **Factor** and set each factor to zero, e.g. $2\sin^2\theta - \sin\theta = 0 \Rightarrow \sin\theta(2\sin\theta - 1) = 0$.
- **Quadratic form:** let $u = \sin\theta$, solve $au^2 + bu + c = 0$, then back-substitute.
- **Multiple angle:** for $\sin 2\theta = k$, solve for 2θ over a doubled interval $[0, 4\pi)$, then divide each solution by 2.

Worked example: quadratic form on $[0, 2\pi)$

Solve $2\cos^2\theta - \cos\theta - 1 = 0$. Factor: $(2\cos\theta + 1)(\cos\theta - 1) = 0$, so $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$. Thus

$$\theta = \tfrac{2\pi}{3},\ \tfrac{4\pi}{3},\ 0.$$

Tip. If a solving step multiplies or squares, check for extraneous roots at the end. For multiple-angle equations, expand the interval *before* solving so you keep every solution in the requested range.

15.10 Law of Sines and Law of Cosines

The two laws

For any triangle with sides a, b, c opposite angles A, B, C:

$$\textbf{Law of Sines:} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\textbf{Law of Cosines:} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

(and cyclically for a^2 and b^2). Solve for an angle as $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.

Which law, plus area

- **Law of Sines:** use for AAS, ASA, or SSA.
- **Law of Cosines:** use for SAS or SSS.

Area of a triangle:

$$\text{Area} = \tfrac{1}{2}ab\sin C, \qquad \text{Heron: Area} = \sqrt{s(s-a)(s-b)(s-c)}, \quad s = \tfrac{a+b+c}{2}.$$

Worked example: Law of Cosines (SAS)

Given $a = 5$, $b = 7$, $C = 40^\circ$:

$$c^2 = 5^2 + 7^2 - 2(5)(7)\cos 40^\circ = 74 - 70\cos 40^\circ \approx 20.37,$$

so $c \approx 4.51$.

Tip: the ambiguous case (SSA). With two sides and a non-included angle there may be 0, 1, or 2 triangles. After finding one angle from the Law of Sines, test its supplement: if that supplement plus the given angle is still under 180°, a second triangle exists.

15.11 Vectors and the Dot Product

Components, magnitude, and direction

A vector $\mathbf{v} = \langle a, b\rangle$ from initial point (x_1, y_1) to terminal point (x_2, y_2) has components $\langle x_2 - x_1,\ y_2 - y_1\rangle$. Its

$$\text{magnitude} = \|\mathbf{v}\| = \sqrt{a^2 + b^2}, \qquad \text{direction } \theta = \tan^{-1}\frac{b}{a}.$$

In component form $\mathbf{v} = \|\mathbf{v}\|\langle\cos\theta, \sin\theta\rangle$. A unit vector is $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$. Standard basis: $\mathbf{i} = \langle 1, 0\rangle$, $\mathbf{j} = \langle 0, 1\rangle$.

Dot product, angle, projection, work

For $\mathbf{u} = \langle u_1, u_2\rangle$ and $\mathbf{v} = \langle v_1, v_2\rangle$:

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}, \qquad \mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\,\mathbf{v}.$$

Vectors are **orthogonal** iff $\mathbf{u}\cdot\mathbf{v} = 0$. **Work** done by force $\mathbf{F}$ along displacement $\mathbf{d}$ is $W = \mathbf{F}\cdot\mathbf{d}$.

Worked example: angle between vectors

Let $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle -1, 2\rangle$. Then $\mathbf{u}\cdot\mathbf{v} = 3(-1) + 4(2) = 5$, $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = \sqrt{5}$, so

$$\cos\theta = \frac{5}{5\sqrt{5}} = \frac{1}{\sqrt{5}} \;\Rightarrow\; \theta \approx 63.4^\circ.$$

Tip. The dot product is a *scalar*, not a vector. A positive value means the vectors point in a generally similar direction ($\theta < 90^\circ$); zero means perpendicular; negative means they oppose ($\theta > 90^\circ$).

15.12 Polar Coordinates and Complex Numbers

Polar and rectangular conversion

A point (r, θ) in polar coordinates relates to rectangular (x, y) by

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x} \text{ (choose } \theta \text{ by quadrant).}$$

Trig form, DeMoivre, and nth roots

A complex number $z = a + bi$ has **trig (polar) form**

$$z = r(\cos\theta + i\sin\theta), \quad r = \sqrt{a^2 + b^2}, \quad \tan\theta = \frac{b}{a}.$$

Product/quotient: multiply/divide moduli, add/subtract arguments. **DeMoivre's Theorem:**

$$z^n = r^n\big(\cos n\theta + i\sin n\theta\big).$$

The n **nth roots** of z are, for $k = 0, 1, \ldots, n-1$,

$$z_k = r^{1/n}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right).$$

Worked example: DeMoivre's Theorem

Compute $\big(1+i\big)^8$. Here $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$, so

$$(1+i)^8 = (\sqrt{2})^8\big(\cos 2\pi + i\sin 2\pi\big) = 16(1 + 0i) = 16.$$

Common polar curves

- **Circles:** $r = a$, $r = 2a\cos\theta$, $r = 2a\sin\theta$.
- **Cardioids/limacons:** $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$ (cardioid when $a = b$).
- **Roses:** $r = a\cos n\theta$ or $r = a\sin n\theta$ (n petals if n odd, $2n$ if n even).
- **Lemniscate:** $r^2 = a^2\cos 2\theta$.

Tip. A single point has infinitely many polar names: (r, θ), $(r, \theta + 2\pi n)$, and $(-r, \theta + \pi)$ all coincide. When converting to polar, always confirm θ lies in the correct quadrant for the signs of x and y.