

MathCastle

Proof Writing & Problem Solving

Complete course booklet • 8 topics

Contents

1	What a Proof Is	3
1.1	Statements and Definitions	3
1.2	Reading a Proof	3
2	Direct Proof	4
2.1	The Shape of a Direct Proof	4
2.2	Cases	5
3	Contrapositive and Contradiction	6
3.1	Contrapositive	6
3.2	Contradiction	6
4	Induction	7
4.1	Ordinary Induction	8
4.2	Strong Induction	8
5	Pigeonhole and Extremal Arguments	9
5.1	The Pigeonhole Principle	9
5.2	Extremal Arguments	10
6	Existence, Uniqueness and Counterexamples	11
6.1	Quantifiers	11
6.2	Uniqueness and "If and Only If"	12
7	Problem-Solving Strategies	13
7.1	Plya's Four Steps	13
7.2	Invariants and Parity	13
8	Writing It Up	14
8.1	Style Rules	14

8.2	Checking Your Own Proof	15
-----	-----------------------------------	----

1 What a Proof Is

Statements, definitions, and the difference between checking and proving

1.1 Statements and Definitions

A proof is a chain of implications

A **proof** is a sequence of statements, each of which is an assumption, a definition, a previously proved fact, or a logical consequence of earlier lines, ending at the claim. Nothing else is allowed — not examples, not diagrams that "look right", not "it is obvious". The reader must be able to check every step without trusting you.

The strange thing about proofs, for someone meeting them for the first time, is how little they resemble the way we are convinced of things in daily life. Checking twenty cases convinces; it does not prove, because the twenty-first might fail. What a proof provides is a reason that covers every case at once. The reason usually comes from a definition: to prove something about even numbers, you will write $n = 2k$, because that is what "even" means, and everything follows from that sentence.

Know the definitions cold

Even: $n = 2k$. Odd: $n = 2k + 1$. $a \mid b$: $b = ak$. Rational: p/q with integers p, q , $q \neq 0$. Prime: $p > 1$ with no divisors other than 1 and p . Every proof about these objects begins by unpacking the definition.

Checking versus proving

Claim: the sum of two odd numbers is even. Checking: $3 + 5 = 8$, $7 + 11 = 18$. ✓ — but not a proof. Proof: let the numbers be $2a + 1$ and $2b + 1$. Their sum is $2a + 2b + 2 = 2(a + b + 1)$, which is 2 times an integer, hence even. ■

Where it goes wrong

Using the same letter for two different unknowns. "Let the odd numbers be $2k + 1$ and $2k + 1$ " proves only that an odd number added to ITSELF is even.

Try it: write it out

Prove that the product of two odd numbers is odd.

Answer. Let the numbers be $2a + 1$ and $2b + 1$. Product: $4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$, which is 2 times an integer plus 1: odd. ■

1.2 Reading a Proof

How to read

Read a proof three times. First, find the claim and the assumptions. Second, at each line, ask "why is this true?" and do not move on until you can say. Third, ask "where was each assumption used?" — if some assumption was never used, either the theorem is stronger than stated or the proof is wrong.

Mathematicians read slowly. A page of proof may take an hour, most of it spent with a pencil checking a line the author called "clearly". That checking is the reading; a proof skimmed is a proof not read. The third pass — where were the hypotheses used? — is the most instructive one, because it shows what the proof is really about and what would break without each assumption.

Find where the hypothesis is used

Theorem: if n^2 is even then n is even. Proof by contrapositive: suppose n is odd, $n = 2k + 1$. Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd. So if n^2 is even, n cannot be odd. Where is " n^2 even" used? Only at the end, as the thing being contradicted. The real work is the algebra showing odd squares are odd.

Try it: spot the gap

What is wrong with this proof that all horses are the same color? "One horse is the same color as itself. If any n horses share a color, then in a group of $n + 1$, the first n share a color and the last n share a color, and the overlap forces all $n + 1$ to share it."

Answer. The step from $n = 1$ to $n = 2$ fails: with two horses, the "first 1" and the "last 1" do not overlap, so nothing forces them to match. The induction step needs $n \geq 2$, and the base case $n = 2$ was never proved.

2 Direct Proof

From hypothesis to conclusion, one justified step at a time

2.1 The Shape of a Direct Proof

Assume, unpack, compute, conclude

To prove "if P then Q " directly: assume P ; rewrite P using definitions; manipulate until Q appears in the form of its definition; state Q . Every "if-then" statement is proved this way unless there is a reason to do otherwise.

The hardest part of a direct proof is usually the middle — the algebra or the counting — and the most common failure is at the ends: forgetting to state what is assumed, or stopping one line before the conclusion has actually been reached in the required form. Ending a proof with " $= 2(a + b + 1)$ " is not the same as ending with "which is even".

Divisibility

Prove: if $a \mid b$ and $b \mid c$ then $a \mid c$. Proof. Since $a \mid b$, $b = ak$ for some integer k ; since $b \mid c$, $c = bm$ for some integer m . Then $c = (ak)m = a(km)$, and km is an integer, so $a \mid c$. ■

An inequality

Prove: for real x, y , $x^2 + y^2 \geq 2xy$. Proof. $(x - y)^2 \geq 0$ because squares are non-negative. Expanding, $x^2 - 2xy + y^2 \geq 0$, so $x^2 + y^2 \geq 2xy$. ■ (Written forwards. It was found backwards: start from the goal, subtract, recognize a square.)

Find it backwards, write it forwards

Scratch work may start from the conclusion and work toward something known. The written proof must run the other way — from known to conclusion — or it proves nothing.

Where it goes wrong

Starting the write-up with the statement to be proved and manipulating it. " $x^2 + y^2 \geq 2xy$, so $(x - y)^2 \geq 0$, which is true" assumes what it wants to prove; the steps happen to be reversible here, but the proof must SAY so, or be written in the right direction.

Try it: direct

Prove that the sum of any three consecutive integers is divisible by 3.

Answer. Let them be $n, n + 1, n + 2$. Sum $= 3n + 3 = 3(n + 1)$, which is 3 times an integer. ■

2.2 Cases

Proof by cases

When the hypothesis splits naturally (even/odd, positive/negative/zero, $n \bmod 3$), prove the claim in each case. The cases must be **exhaustive** — every possibility covered — and the conclusion must be the SAME in every case.

$n^2 + n$ is always even

Case 1: $n = 2k$. Then $n^2 + n = 4k^2 + 2k = 2(2k^2 + k)$, even. Case 2: $n = 2k + 1$. Then $n^2 + n = n(n + 1) = (2k + 1)(2k + 2) = 2(2k + 1)(k + 1)$, even. Both cases give even; every integer is in one of them. ■

Where it goes wrong

Cases that miss something ("n positive or negative" forgets zero), or proving a different conclusion in each case and calling it done.

Try it: cases

Prove that for every integer n , $n^3 - n$ is divisible by 3.

Answer. Write $n^3 - n = (n - 1)n(n + 1)$. Among three consecutive integers one is a multiple of 3 (cases $n \equiv 0, 1, 2 \pmod{3}$: the multiple is $n, n - 1, n + 1$ respectively). So the product is divisible by 3. ■

3 Contrapositive and Contradiction

Proving by assuming the opposite

3.1 Contrapositive

"If P then Q " equals "if not Q then not P "

The **contrapositive** of an implication is logically equivalent to it, and often easier to prove directly — especially when the conclusion Q is a "not" statement or when P gives you nothing to compute with.

The contrapositive is not the converse. "If it rains, the ground is wet" has contrapositive "if the ground is dry, it is not raining" (true) and converse "if the ground is wet, it is raining" (false — sprinklers). Confusing the two is the single most common logical error, in and out of mathematics.

When the hypothesis is useless

Prove: if n^2 is odd then n is odd. Directly, " n^2 odd" gives $n^2 = 2k + 1$ — and then what? Contrapositive: if n is even, $n = 2m$, then $n^2 = 4m^2 = 2(2m^2)$ is even. Done. ■

Try it: contrapositive

Prove: if $3n + 2$ is even, then n is even.

Answer. Contrapositive: if n is odd, $n = 2k + 1$, then $3n + 2 = 6k + 5 = 2(3k + 2) + 1$ is odd. ■

3.2 Contradiction

Assume the claim is false, derive nonsense

To prove P by **contradiction**: assume $\neg P$ and deduce something impossible ($1 = 0$, a number that is both even and odd, a smaller element of a set that had a smallest element). Then $\neg P$ cannot hold, so P does. It is the right tool for "there is no", "is irrational", "infinitely many".

Contradiction is more powerful than the contrapositive because you get to assume the WHOLE negation, including whatever structure " P is false" carries — a smallest counterexample, an explicit fraction p/q , a largest prime. Its cost is that a proof by contradiction, once the contradiction appears, tells you nothing about why P is true. Prefer a direct proof when one exists; reach for contradiction when the negation gives you something concrete to work with.

$\sqrt{2}$ is irrational

Suppose $\sqrt{2} = p/q$ in lowest terms. Then $p^2 = 2q^2$, so p^2 is even, so p is even (previous section), $p = 2r$. Then $4r^2 = 2q^2$, $q^2 = 2r^2$, so q is even too. Both even contradicts lowest terms. ■

Infinitely many primes

Suppose there are finitely many: $p_1, \dots, p_k$. Let $N = p_1 \cdots p_k + 1$. $N > 1$ has a prime factor p ; p is some p_i ; but $N \equiv 1 \pmod{p_i}$, so $p_i \nmid N$. Contradiction. ■

Minimal counterexample

To prove every n has property P : suppose not, and let n_0 be the SMALLEST failure (well-ordering). Use the fact that everything below n_0 works to show n_0 works too — contradiction.

Where it goes wrong

Assuming the negation and then proving P directly without ever using the assumption. That is a direct proof wearing a costume; the "contradiction" is fake. If you never used $\neg P$, delete it.

Try it: contradiction

Prove that there is no largest even integer.

Answer. Suppose N is the largest even integer. Then $N + 2$ is even (as $N = 2k$ gives $N + 2 = 2(k + 1)$) and larger than N . Contradiction. ■

4 Induction

Proving a statement for every natural number

4.1 Ordinary Induction

Base case, inductive step

To prove $P(n)$ for all $n \geq n_0$: prove $P(n_0)$ (**base case**); then prove that for every $k \geq n_0$, $P(k)$ implies $P(k+1)$ (**inductive step**). Together these knock over every domino.

Induction is the proof method built into the natural numbers: they are exactly the set you reach from 0 by adding 1 repeatedly, so a property that starts true and survives "+1" is true everywhere. The inductive step is not "assume what you are proving" — it is "assume it for one specific k and prove it for the next one". Writing the inductive hypothesis out explicitly, with a specific k , is what keeps that straight.

Sum of the first n odd numbers

Claim: $1 + 3 + \cdots + (2n - 1) = n^2$. Base: $n = 1$: $1 = 1^2$. Step: assume it for k . Then $1 + \cdots + (2k - 1) + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$. ■

A divisibility

Claim: $7 \mid 8^n - 1$ for all $n \geq 0$. Base: $8^0 - 1 = 0$. Step: $8^{k+1} - 1 = 8 \cdot 8^k - 1 = 8(8^k - 1) + 7$; by hypothesis $7 \mid 8^k - 1$, so 7 divides the whole. ■

The shape of the inductive step

Write the $k + 1$ case; find the k case inside it (peel off the last term, factor out one power); substitute the inductive hypothesis; finish with algebra. If you cannot find the k case inside, you may need strong induction.

Where it goes wrong

Forgetting the base case — " $P(k) \Rightarrow P(k+1)$ " is true of " $n = n + 1$ " and proves nothing without a start. And proving $P(k+1)$ from scratch without using $P(k)$, which usually means induction was not needed.

Try it: induction

Prove that $2^n > n$ for all $n \geq 1$.

Answer. Base: $2^1 = 2 > 1$. Step: assume $2^k > k$. Then $2^{k+1} = 2 \cdot 2^k > 2k \geq k + 1$ (since $k \geq 1$). ■

4.2 Strong Induction

Assume everything below

In **strong induction** the inductive hypothesis is " $P(j)$ holds for ALL j with $n_0 \leq j \leq k$ ", and you prove $P(k+1)$. It is needed when the case $k+1$ depends on cases further back than k — factorizations, Fibonacci-type recurrences, games. It is logically equivalent to ordinary induction.

Every $n \geq 2$ is a product of primes

Assume every j with $2 \leq j \leq k$ is a product of primes. If $k+1$ is prime, done. Otherwise $k+1 = ab$ with $2 \leq a, b \leq k$; each is a product of primes by hypothesis; so is their product. ■ (Ordinary induction fails: $k+1$'s factors are not k .)

Postage

Every $n \geq 12$ is a sum of 4s and 5s. Bases: $12 = 4 + 4 + 4$, $13 = 4 + 4 + 5$, $14 = 4 + 5 + 5$, $15 = 5 + 5 + 5$. Step: for $k+1 \geq 16$, $(k+1) - 4 \geq 12$ is a sum of 4s and 5s by hypothesis; add a 4. ■ (Four base cases because the step reaches back 4.)

Where it goes wrong

Too few base cases. If the step uses $P(k-3)$, the first four values must be checked by hand or the chain has a gap.

Try it: strong induction

Define $a_1 = 1$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$. Prove $a_n = 2^n - (-1)^n$.

Answer. Bases: $a_1 = 2 + 1 = 3$? No: $2^1 - (-1)^1 = 2 + 1 = 3 \neq 1$. Recheck: the formula must be $a_n = \frac{2^{n+1} + (-1)^n}{3}$: $a_1 = (4 - 1)/3 = 1$ ✓, $a_2 = (8 + 1)/3 = 3$ ✓. Step: $a_{k+1} = a_k + 2a_{k-1} = \frac{2^{k+1} + (-1)^k}{3} + 2 \frac{2^k + (-1)^{k-1}}{3} = \frac{2^{k+2} - (-1)^k}{3} = \frac{2^{k+2} + (-1)^{k+1}}{3}$. ■ (The lesson: check the base cases FIRST — they caught a wrong formula.)

5 Pigeonhole and Extremal Arguments

Proving that something must exist without finding it

5.1 The Pigeonhole Principle

More pigeons than holes

If $n + 1$ objects are placed in n boxes, some box holds at least two. In general, N objects in n boxes put at least $\lceil N/n \rceil$ in some box. The whole art is choosing the boxes.

Pigeonhole proofs are existence proofs: they show two objects must collide without saying which. The boxes are usually residues mod n , pairs that sum to something, or intervals. A good sign you need pigeonhole: the problem says "among any $n + 1$ " or "some two of them".

Two that differ by n

Among any $n + 1$ integers from $\{1, \dots, 2n\}$, two are consecutive. Boxes: $\{1, 2\}, \{3, 4\}, \dots, \{2n - 1, 2n\}$ — n boxes, $n + 1$ numbers; two share a box, and they are consecutive. ■

Divisible difference

Among any n integers, some non-empty subset has sum divisible by n . Consider the partial sums $s_1, \dots, s_n$ mod n . If one is 0, done. Otherwise n sums in $n - 1$ nonzero residues: two agree, $s_i \equiv s_j$, and the terms between them sum to a multiple of n . ■

Where it goes wrong

Boxes that are not disjoint or do not cover all the objects. Say explicitly what the objects are, what the boxes are, and why there are more objects than boxes.

Try it: pigeonhole

Show that among any 13 people, two were born in the same month.

Answer. 13 people (pigeons), 12 months (holes): some month has at least $\lceil 13/12 \rceil = 2$ people. ■

5.2 Extremal Arguments

Look at the biggest or smallest

Choose the object that is extreme — the largest number, the shortest path, the point farthest left — and argue that its extremity forces a property, or that a "more extreme" object would exist otherwise (contradiction). Finite sets always have a maximum; well-ordering gives a minimum for sets of natural numbers.

A largest number is a sum of neighbors?

Numbers are placed at the vertices of a polygon so that each equals the average of its two

neighbors. Prove they are all equal. Take the largest value M . Its two neighbors average to M and are at most M , so both equal M . Repeat around the polygon: all equal M . ■

No infinite descent

Find all integer solutions of $x^2 = 2y^2$. If $(x, y) \neq (0, 0)$ is a solution, take the one with smallest $|x| > 0$. Then x is even, $x = 2x_1$, giving $2x_1^2 = y^2$, so $y = 2y_1$ and $x_1^2 = 2y_1^2$ — a smaller solution. Contradiction; only $(0, 0)$. ■

Try it: extremal

In a finite set of points in the plane, no three collinear, prove some three of them form a triangle containing no other point of the set.

Answer. Take any two points A, B and the point C (among the rest) minimizing the distance from C to line AB . Triangle ABC contains no other point: such a point would be closer to AB than C is. ■

6 Existence, Uniqueness and Counterexamples

The four kinds of claims and what each needs

6.1 Quantifiers

"For all" and "there exists"

A **universal** claim ("for all x , $P(x)$ ") needs a proof that works for an arbitrary x and is refuted by ONE counterexample. An **existential** claim ("there exists x with $P(x)$ ") is proved by ONE example (or a non-constructive argument) and refuted only by a proof covering every x . Negating swaps them: not-for-all is there-exists-not.

Most errors in reading a problem are quantifier errors. "Every even number greater than 2 is a sum of two primes" is refuted by a single even number that is not; it is proved only by an argument for all of them — which is why it is still open. "There is a prime between n and $2n$ " is proved by showing one exists for each n ; it is not refuted by " $n = 5$: 7 is prime, but so what". Underline the quantifiers before you start.

One counterexample kills a universal claim

Claim: $n^2 + n + 41$ is prime for every natural number n . It is prime for $n = 0, 1, \dots, 39$. At $n = 40$: $40^2 + 40 + 41 = 41^2$. False. One case suffices; forty successes did not.

Existence by construction

Claim: for every n there are n consecutive composite numbers. Proof: $(n+1)! + 2, (n+1)! + 3, \dots, (n+1)! + (n+1)$ are divisible by $2, 3, \dots, n+1$ respectively. ■

Try it: negate

Write the negation of: "for every $\varepsilon > 0$ there exists N such that for all $n > N$, $|a_n| < \varepsilon$."

Answer. "There exists $\varepsilon > 0$ such that for every N there is some $n > N$ with $|a_n| \geq \varepsilon$." (Each quantifier flips; the inner inequality negates.)

6.2 Uniqueness and "If and Only If"

Two halves each

To prove **uniqueness**: suppose x and y both have the property, and show $x = y$. To prove " **P if and only if Q** ": prove $P \Rightarrow Q$ and $Q \Rightarrow P$ separately (or write a chain of reversible steps and SAY they are reversible).

Unique identity

In any group, the identity is unique: if e and e' are both identities, then $e = ee' = e'$ (the first equality because e' is an identity, the second because e is). ■

Iff

An integer is divisible by 4 iff its last two digits form a number divisible by 4. Write $n = 100a + b$ with $0 \leq b < 100$. Since $4 \mid 100a$, $4 \mid n$ iff $4 \mid b$ — one line, both directions, because " $4 \mid 100a + b \iff 4 \mid b$ " is a genuine equivalence. ■

Where it goes wrong

Proving only one direction of an iff, or proving existence and calling it uniqueness. "There is a solution" and "there is exactly one solution" are different theorems.

Try it: iff

Prove: an integer n is odd if and only if n^2 is odd.

Answer. $(\Rightarrow)$ $n = 2k + 1$ gives $n^2 = 2(2k^2 + 2k) + 1$, odd. $(\Leftarrow)$ Contrapositive: if n is even, $n = 2k$, then $n^2 = 2(2k^2)$ is even. ■

7 Problem-Solving Strategies

How to find a proof when you do not know where to start

7.1 Plya's Four Steps

Understand, plan, execute, look back

Understand: restate the problem in your own words; identify the unknown, the data, the condition; draw a figure; try small cases. **Plan:** connect to a known problem; specialize, generalize, or work backwards; introduce notation. **Execute:** carry it out, checking each step. **Look back:** check the result, find another route, ask what the method can also solve.

The first step is where beginners spend too little time and experts spend the most. Computing the first five cases of a problem about all n is not a waste — it is how the pattern, and usually the proof, is discovered. Working backwards from the goal ("what would I need to be true one line before the end?") is the second most useful habit; it turns a blank page into a specific sub-goal.

Small cases reveal the pattern

How many regions do n lines in general position cut the plane into? $n = 0, 1, 2, 3$: 1, 2, 4, 7. Differences 1, 2, 3: each new line crosses all previous ones and adds (crossings + 1) regions. So $R(n) = 1 + (1 + 2 + \cdots + n) = 1 + \frac{n(n+1)}{2}$, proved by induction with exactly that step.

Wishful thinking

Ask: "what would make this easy?" If the numbers were smaller, if the shape were symmetric, if one variable were zero — then solve THAT and see whether the solution survives being un-wished.

Try it: work backwards

You must reach exactly 100 starting from 1 using only "add 1" and "double". What is the fewest moves? (Hint: go backwards from 100.)

Answer. Backwards, halve when even, subtract 1 when odd: $100 \rightarrow 50 \rightarrow 25 \rightarrow 24 \rightarrow 12 \rightarrow 6 \rightarrow 3 \rightarrow 2 \rightarrow 1$: 8 moves. Halving whenever possible is optimal because doubling is the only way to gain more than 1.

7.2 Invariants and Parity

Find what does not change

When a process or game is described by moves, look for an **invariant**: a quantity (often a parity, a sum mod k , or a coloring count) that every move preserves. If the start and the goal

have different values of the invariant, the goal is unreachable — a proof of impossibility with no case analysis at all.

The chessboard

Remove two opposite corners of an 8×8 board. Can the rest be tiled by dominoes? Each domino covers one black and one white square; the opposite corners are the same color, so 32 of one color and 30 of the other remain. No. ■

A cup game

Seven cups, all face up; a move flips exactly two. Can all end face down? Each move changes the number of face-up cups by -2 , 0 , or $+2$: the parity of face-up cups never changes. It starts odd (7) and the goal is 0, even. Impossible. ■

Where it goes wrong

Finding a quantity that changes "usually" but not always, or proving a state is unreachable when the problem asked whether it is reachable (an invariant proves impossibility only; possibility needs a construction).

Try it: invariant

The numbers $1, 2, \dots, 10$ are on a board. A move erases two numbers a, b and writes $a + b$. What single number remains at the end?

Answer. The SUM is invariant: $1 + 2 + \dots + 10 = 55$. (Harder variant: erase a, b , write $|a - b|$ — then the parity of the sum is invariant, so the last number is odd.)

8 Writing It Up

How a proof should read on the page

8.1 Style Rules

The reader's rules

Write in full sentences. State what you are proving and what you assume. Name every variable before using it ("let k be an integer such that"). Give the method up front ("we proceed by induction on n "). Justify each step with a reason a skeptical reader accepts. End with what has been shown, and mark it: ■ or "which completes the proof".

A proof is written for someone who is trying to find a mistake. Assume that reader is intelligent, careful, and unwilling to guess what you meant. Symbols save space but hide gaps: a page of equal

signs with no words is usually a page of unjustified steps. The right proportion is about half words, half mathematics, and the words carry the logic — "so", "because", "suppose", "therefore", "for all".

Words that do work

"Suppose", "let", "since", "therefore", "it follows that", "in particular", "conversely", "without loss of generality" (say why), "by the inductive hypothesis". Avoid "obviously", "clearly", and "it is easy to see" — if it is, show it in a line; if it is not, the word is hiding the gap.

Before and after

Draft: " n odd $\rightarrow n = 2k + 1 \rightarrow n^2 = 4k^2 + 4k + 1 \rightarrow$ odd." Final: "Let n be odd, so $n = 2k + 1$ for some integer k . Then $n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd since $2k^2 + 2k$ is an integer. ■" Same mathematics; the second can be checked."

Where it goes wrong

Arrows ($\Rightarrow$) used as sentence connectors, unquantified variables (" $x^2 \geq 0$ " — for which x ?), and a proof that ends in mid-computation without stating the conclusion.

Try it: rewrite

Rewrite as a proof: " $a \mid b$, $b = ak$, $a \mid bc$ since $bc = a(kc)$."

Answer. "Suppose $a \mid b$. Then $b = ak$ for some integer k . For any integer c , $bc = (ak)c = a(kc)$, and kc is an integer, so $a \mid bc$. ■"

8.2 Checking Your Own Proof

The audit

Before calling a proof finished: (1) Did you use every hypothesis? (2) Did you handle every case, including degenerate ones ($n = 0$, empty set, equal points)? (3) Does each step follow from the ones before, not from the conclusion? (4) Would the argument also "prove" something false? — if so, it is wrong. (5) Does the last line say exactly what was claimed?

Test 4 in action

"Proof" that $1 = 2$: let $a = b$; $a^2 = ab$; $a^2 - b^2 = ab - b^2$; $(a - b)(a + b) = b(a - b)$; $a + b = b$; $2b = b$; $2 = 1$. The argument proves something false, so a step is wrong: dividing by $a - b = 0$. Every proof that divides must say why the divisor is nonzero.

Try it: audit

Audit: "Every positive integer is interesting. Suppose not; let n be the smallest uninteresting integer. But being the smallest uninteresting integer is interesting — contradiction." What is wrong?

Answer. "Interesting" is not a mathematical property with a definition; the proof uses the word in two different senses. A proof about a set needs the set to be well-defined. (The pattern — smallest counterexample, then contradiction — is a valid one; the object is the problem.)