MathCastle

Precalculus

Complete course booklet • 14 topics

Contents

1 Core Ideas in Plain Terms

The bridge to calculus, in everyday language

1.1 Functions and Transformations

A function is a machine

A function takes an input, does something to it, and gives one output. Writing $f(x)$ just means "the output when the input is x." Changing the formula slightly *moves* the graph: $f(x)+k$ slides it up k, $f(x-h)$ slides it right h, and a minus sign in front flips it over.

Shifting a parabola

$y=(x-2)^2+3$ is the basic $y=x^2$ moved 2 to the right and 3 up, so its lowest point is at $(2,3)$.

(diagram in the online version)

1.2 Exponentials and Logarithms

Logarithms undo exponents

The equation $\log_b x = y$ is just another way of writing $b^y = x$ — it asks "what power turns b into x?" So logs are the reverse of exponentials, and that is why they turn multiplication into addition ($\log(xy) = \log x + \log y$) and pull an exponent down front.

Solving $2^x = 8$

$\log_2 8 = 3$ because $2^3 = 8$. The log simply reports the exponent.

1.3 Sequences, Series, and Limits

A limit is where a value is heading

A **limit** asks: as the input gets closer and closer to some number, what does the output approach? You don't have to reach it — just see where it is heading. This idea is the whole foundation of calculus.

Where 1/x heads as x grows

As x gets huge, $\frac{1}{x}$ gets tiny — it heads toward 0. We write $\lim_{x\to\infty} \frac{1}{x} = 0$.

(diagram in the online version)

1.4 Going Deeper: Log Rules and Infinite Series

Log rules are exponent rules in reverse

Multiplying powers adds exponents, so logs turn multiplication into addition — the **product rule**, $\log(ab) = \log a + \log b$ — and pull exponents down front — the **power rule**, $\log(a^n) = n \log a$. That second rule is why logs solve for unknowns stuck in an exponent — the n comes down where you can reach it.

How an infinite series can add to a finite number

In a geometric series each term is the same fraction r of the one before. When $|r| < 1$ the terms shrink fast enough that the total closes in on the **infinite geometric series formula** $S = \dfrac{a}{1-r}$. So $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \dfrac{1}{1-\frac{1}{2}} = 2$: you never pass 2, but you get as close as you like.

Putting the log rules to work

$\log_2 8 + \log_2 4 = 3 + 2 = 5$, and indeed $\log_2(8 \cdot 4) = \log_2 32 = 5$. Adding the logs multiplied the numbers.

A function and its inverse (multi-step)

Let $f(x) = 2x - 3$. Step 1 — evaluate: $f(5) = 7$. Step 2 — find the inverse by swapping and solving: $x = 2y - 3$ gives $f^{-1}(x) = \dfrac{x+3}{2}$. Step 3 — check with a composition: $f^{-1}(11) = \dfrac{14}{2} = 7$ and $f(7) = 11$ — the functions undo each other.

1.5 Problem-Solving Playbook

Rewrite before you grind

Most precalc problems get easy after ONE rewrite: exponential equations → **same base**; log equations → **exponent form** ($\log_b x = y \iff b^y = x$); rational expressions → **factor first**. Only start computing after the rewrite.

Worked: same-base rewrite

Solve $4^x = 8^{x-1}$. Both are powers of 2: $2^{2x} = 2^{3(x-1)}$. Equal bases force equal exponents: $2x = 3x - 3$, so $x = 3$. (Check: $4^3 = 64$ and $8^2 = 64$.)

2 Sequences, Series, and the Binomial Theorem

Explicit & recursive formulas, factorials, sigma notation, arithmetic & geometric series, induction, the Binomial Theorem, counting, and probability

2.1 Sequences: Explicit & Recursive Formulas, Factorials

Sequences and factorial notation

A **sequence** is a function whose domain is the positive integers. Its outputs $a_1, a_2, a_3, \dots$ are the **terms**, and a_n is the n**th (general) term**.

- **Explicit formula:** a_n is given directly as a formula in n. To list terms, substitute $n = 1, 2, 3, \dots$
- **Recursive formula:** a starting term (or terms) plus a rule that builds each term from earlier ones, e.g. $a_1 = 3,\ a_n = a_{n-1} + 4$.
- **Factorial:** $n! = n(n-1)(n-2)\cdots 2 \cdot 1$, and by definition $0! = 1$. So $5! = 120$.
- **Simplifying factorials:** cancel! $\dfrac{8!}{6!} = \dfrac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56$.

Listing terms from a formula

Write the first four terms of $a_n = \dfrac{(-1)^n\, n}{n+1}$.

$a_1 = \dfrac{-1}{2}, \quad a_2 = \dfrac{2}{3}, \quad a_3 = \dfrac{-3}{4}, \quad a_4 = \dfrac{4}{5}.$

The factor $(-1)^n$ makes the signs alternate, starting negative.

From recursive to terms

Let $a_1 = 2$ and $a_n = 3a_{n-1} - 1$. Find a_2, a_3, a_4.

$a_2 = 3(2) - 1 = 5, \quad a_3 = 3(5) - 1 = 14, \quad a_4 = 3(14) - 1 = 41.$

Tip: A factorial in a denominator is your friend—always cancel the largest common factorial before multiplying. Never multiply out 8! if a 6! will cancel.

2.2 Summation (Sigma) Notation & Its Properties

Sigma notation

The sum of terms a_k from $k = 1$ to n is written

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

Here k is the **index**, 1 the lower limit, n the upper limit. Useful properties:

$$\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k, \qquad \sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k, \qquad \sum_{k=1}^{n} c = cn.$$

Handy closed forms:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Evaluating a sum with properties

Evaluate $\displaystyle\sum_{k=1}^{5} (2k+3)$.

$$\sum_{k=1}^{5} (2k+3) = 2\sum_{k=1}^{5} k + \sum_{k=1}^{5} 3 = 2 \cdot \frac{5 \cdot 6}{2} + 3 \cdot 5 = 30 + 15 = 45.$$

Check by hand: $5 + 7 + 9 + 11 + 13 = 45$. ✓

Tip: $\sum_{k=1}^{n} c = cn$, *not* c. A constant is added once for every value of the index.

2.3 Arithmetic Sequences and Partial Sums

Arithmetic sequences

An **arithmetic** sequence adds a fixed **common difference** d each step, so $d = a_n - a_{n-1}$.

$$\textbf{nth term:} \quad a_n = a_1 + (n-1)d.$$

The sum of the first n terms (an **arithmetic series**) is

$$S_n = \frac{n}{2}\big(a_1 + a_n\big) = \frac{n}{2}\big[\,2a_1 + (n-1)d\,\big].$$

Finding a term and a sum

For $3, 7, 11, 15, \ldots$ find a_{20} and S_{20}.

Here $a_1 = 3$, $d = 4$. $\quad a_{20} = 3 + (20-1)(4) = 3 + 76 = 79.$

$S_{20} = \dfrac{20}{2}(a_1 + a_{20}) = 10(3 + 79) = 10(82) = 820.$

Tip: It is $(n-1)d$, not nd. The first term already "used up" zero steps, so the nth term has taken only $n-1$ steps of size d.

2.4 Geometric Sequences & Series (incl. Infinite)

Geometric sequences and series

A **geometric** sequence multiplies by a fixed **common ratio** r each step, so $r = \dfrac{a_n}{a_{n-1}}$.

$$n\textbf{th term:} \quad a_n = a_1\, r^{\,n-1}.$$

Finite sum of the first n terms:

$$S_n = a_1 \cdot \frac{1 - r^{\,n}}{1 - r} \qquad (r \neq 1).$$

Infinite geometric series: if $|r| < 1$ the partial sums converge and

$$S = \sum_{k=1}^{\infty} a_1 r^{\,k-1} = \frac{a_1}{1 - r}.$$

If $|r| \geq 1$ the series **diverges** (no finite sum).

Finite geometric sum

Sum the first 6 terms of $2, 6, 18, 54, \ldots$

Here $a_1 = 2$, $r = 3$. $\quad S_6 = 2 \cdot \dfrac{1 - 3^6}{1 - 3} = 2 \cdot \dfrac{1 - 729}{-2} = 2 \cdot \dfrac{-728}{-2} = 728.$

Infinite geometric series

Evaluate $\displaystyle\sum_{k=1}^{\infty} 5\left(\tfrac{2}{3}\right)^{k-1}$.

Here $a_1 = 5$ and $r = \frac{2}{3}$, and $|r| < 1$, so it converges:

$$S = \frac{a_1}{1-r} = \frac{5}{1-\frac{2}{3}} = \frac{5}{\frac{1}{3}} = 15.$$

Tip: Only check convergence with $|r| < 1$. A repeating decimal such as $0.\overline{7} = \frac{7}{10} + \frac{7}{100} + \cdots$ is an infinite geometric series with $a_1 = \frac{7}{10}$, $r = \frac{1}{10}$, giving $\frac{7/10}{9/10} = \frac{7}{9}$.

2.5 Mathematical Induction

The principle of induction

To prove a statement $P(n)$ holds for every positive integer n:

1. **Base case:** verify $P(1)$ is true.
2. **Inductive step:** *assume* $P(k)$ is true (the **inductive hypothesis**) and use it to prove $P(k+1)$.

Then $P(n)$ is true for all $n \geq 1$.

A full induction proof of a summation formula

Prove that $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all integers $n \geq 1$.

Base case ($n = 1$): the left side is 1; the right side is $\dfrac{1(1+1)}{2} = 1$. They agree. ✓

Inductive step: assume $\displaystyle\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ for some integer $m \geq 1$. We must show the formula holds for $m+1$:

$$\sum_{k=1}^{m+1} k = \left(\sum_{k=1}^{m} k\right) + (m+1) = \frac{m(m+1)}{2} + (m+1).$$

Factor out $(m+1)$:

$$= (m+1)\left(\frac{m}{2} + 1\right) = (m+1) \cdot \frac{m+2}{2} = \frac{(m+1)\big((m+1)+1\big)}{2}.$$

This is exactly the formula with $n = m+1$. By induction the statement holds for all $n \geq 1$. ■

Tip: In the inductive step, always start from $\sum_{k=1}^{m+1} = \left(\sum_{k=1}^{m}\right) + a_{m+1}$, replace the bracket using the hypothesis, then algebra should rebuild the formula with $n = m + 1$.

2.6 The Binomial Theorem & Pascal's Triangle

Binomial coefficients and the theorem

The **binomial coefficient** is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}, \qquad \binom{n}{0} = \binom{n}{n} = 1.$$

The **Binomial Theorem** expands a power of a binomial:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k} = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n} y^n.$$

The $(k+1)$**st term** of the expansion is $\binom{n}{k} x^{n-k} y^{k}$.

Pascal's triangle gives the coefficients; each entry is the sum of the two above it:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

A full binomial expansion

Expand $(x+2)^4$.

Row of coefficients: $1, 4, 6, 4, 1$. With $y = 2$:

$$(x+2)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3(2) + \binom{4}{2} x^2(2^2) + \binom{4}{3} x(2^3) + \binom{4}{4}(2^4).$$

$$= x^4 + 4x^3(2) + 6x^2(4) + 4x(8) + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16.$$

Finding a specific term

Find the term containing x^4 in $(2x-1)^6$.

General term: $\binom{6}{k}(2x)^{6-k}(-1)^k$. We need power $6-k=4$, so $k=2$:

$$\binom{6}{2}(2x)^4(-1)^2 = 15 \cdot 16x^4 \cdot 1 = 240x^4.$$

The coefficient of x^4 is 240.

Tip: The $(k+1)$st term uses index k, and the exponents on x and y always add to n. When a term has a coefficient like $2x$ or a sign like -1, raise the *whole* thing to its power.

2.7 Counting Principles: FCP, Permutations, Combinations

Three counting tools

- **Fundamental Counting Principle (FCP):** if a task is done in stages with $m_1, m_2, \ldots$ choices, the total number of outcomes is $m_1 \cdot m_2 \cdots$.
- **Permutations** (order *matters*): number of ways to arrange r of n objects is

$$_nP_r = \frac{n!}{(n-r)!}.$$

- **Combinations** (order does *not* matter): number of ways to choose r of n objects is

$$_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Permutation vs. combination

From 10 students, (a) how many ways to pick a president then a vice-president? (b) how many ways to pick a 2-person committee?

(a) Order matters: $_{10}P_2 = \frac{10!}{8!} = 10 \cdot 9 = 90$.

(b) Order does not matter: $_{10}C_2 = \binom{10}{2} = \frac{10 \cdot 9}{2 \cdot 1} = 45$.

Tip: Ask "does swapping two chosen items give a *different* outcome?" If yes, use a **permutation**; if no, use a **combination**. Note $_nP_r = r! \cdot {_nC_r}$.

2.8 Probability Using Counting

Basic probability

For equally likely outcomes,

$$P(E) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{n(E)}{n(S)}.$$

Always $0 \le P(E) \le 1$. The **complement** satisfies $P(\text{not } E) = 1 - P(E)$. Use combinations to count favorable and total outcomes when order does not matter.

Probability with combinations

A bag has 5 red and 3 blue marbles. Two are drawn at random. Find $P(\text{both red})$.

Total ways to choose 2 of 8: $\binom{8}{2} = 28$. Favorable (both red): $\binom{5}{2} = 10$.

$$P(\text{both red}) = \frac{10}{28} = \frac{5}{14}.$$

Tip: When "at least one" appears, the complement is usually faster: $P(\text{at least one}) = 1 - P(\text{none})$.

2.9 Going Deeper: Advanced Sequences, Series & Binomial

Telescoping sums via partial fractions

A sum **telescopes** when each term splits into a difference so that interior terms cancel in pairs. The key tool is a **partial-fraction** split. For a product of consecutive linear factors,

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \qquad \frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right).$$

Once written as $t_k = b_k - b_{k+1}$, the partial sum collapses:

$$\sum_{k=1}^{n} \left(b_k - b_{k+1}\right) = b_1 - b_{n+1}.$$

Only the *first* surviving piece and the *last* surviving piece remain.

A telescoping series (worked)

Evaluate $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ and then take $n \to \infty$.

Split each term: $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right) = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{n+1} = \frac{n}{n+1}.$$

As $n \to \infty$, $\dfrac{1}{n+1} \to 0$, so the infinite sum is $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$.

Weighted geometric sums: $\sum k\, r^{k}$

The plain geometric sum has a closed form; multiplying each term by k needs one extra idea. For $|r| < 1$ the infinite weighted sum converges to

$$\sum_{k=1}^{\infty} k\, r^{k} = \frac{r}{(1-r)^2} \qquad (|r| < 1).$$

Why (difference trick): let $S = \sum_{k=1}^{n} k\, r^{k}$. Form $S - rS$; each column telescopes so that

$$(1-r)S = \left(r + r^2 + \cdots + r^n\right) - n\, r^{n+1} = \frac{r(1-r^n)}{1-r} - n\, r^{n+1}.$$

When $|r| < 1$ the terms r^n and $nr^{n+1} \to 0$, leaving $S = \dfrac{r}{(1-r)^2}$. Compare with the pure power sum $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$, which grows without bound (it has no infinite value).

Computing $\sum k\, r^{k}$ and $\sum k^2$ (worked)

(a) Evaluate $\displaystyle\sum_{k=1}^{\infty} k\left(\tfrac{1}{2}\right)^{k}$.

Here $r = \frac{1}{2}$ and $|r| < 1$, so

$$\sum_{k=1}^{\infty} k\left(\tfrac{1}{2}\right)^{k} = \frac{r}{(1-r)^2} = \frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2.$$

(b) Evaluate $\displaystyle\sum_{k=1}^{10} k^2$ with the closed form:

$$\sum_{k=1}^{10} k^2 = \frac{10 \cdot 11 \cdot 21}{6} = \frac{2310}{6} = 385.$$

Solving a linear recurrence for a closed form

A first-order recurrence $a_n = r\,a_{n-1} + c$ (with $r \neq 1$) can be turned into an explicit formula. Find the fixed point $L = \dfrac{c}{1-r}$ (solve $L = rL + c$); then the shifted sequence $a_n - L$ is purely geometric:

$$a_n - L = r\,(a_{n-1} - L) \implies a_n = L + (a_1 - L)\,r^{\,n-1}.$$

This converts a step-by-step rule into a formula you can evaluate directly at any n.

Recurrence to closed form (worked)

Solve $a_1 = 2,\ a_n = 3a_{n-1} - 1$ for an explicit formula, and check against the earlier hand-computed terms $2, 5, 14, 41$.

Fixed point: $L = 3L - 1 \Rightarrow L = \frac{1}{2}$. So $a_n - \frac{1}{2} = 3\big(a_{n-1} - \frac{1}{2}\big)$, a geometric sequence with ratio 3 and first term $a_1 - \frac{1}{2} = \frac{3}{2}$:

$$a_n = \frac{1}{2} + \frac{3}{2}\cdot 3^{\,n-1} = \frac{1 + 3^{\,n}}{2}.$$

Check: $a_1 = \frac{1+3}{2} = 2$, $a_2 = \frac{1+9}{2} = 5$, $a_3 = \frac{1+27}{2} = 14$, $a_4 = \frac{1+81}{2} = 41$. ✓

Binomial-coefficient identities

These follow from the Binomial Theorem and from Pascal's rule $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$:

- **Row sum:** setting $x = y = 1$ in $(x+y)^n$ gives $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^{\,n}$.
- **Alternating sum:** setting $x = 1,\ y = -1$ gives $\displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ (for $n \geq 1$).
- **Weighted sum:** $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n\,2^{\,n-1}$, using $k\binom{n}{k} = n\binom{n-1}{k-1}$.
- **Hockey-stick:** $\displaystyle\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$ — a diagonal of Pascal's triangle sums to the entry just below its end.

Using the identities (worked)

(a) A pizza shop offers 6 toppings. How many topping-subsets are possible (including plain)?

Choose any subset: $\sum_{k=0}^{6} \binom{6}{k} = 2^6 = 64$.

(b) Verify the hockey-stick identity $\sum_{i=2}^{5} \binom{i}{2} = \binom{6}{3}$.

Left: $\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 1 + 3 + 6 + 10 = 20$. Right: $\binom{6}{3} = 20$. ✓

Induction proof of the row-sum identity

Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for all integers $n \geq 0$.

Base case ($n = 0$): the left side is $\binom{0}{0} = 1$ and the right side is $2^0 = 1$. They agree. ✓

Inductive step: assume $\sum_{k=0}^{m} \binom{m}{k} = 2^m$. Using Pascal's rule $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$ (with $\binom{m}{-1} = \binom{m}{m+1} = 0$),

$$\sum_{k=0}^{m+1} \binom{m+1}{k} = \sum_{k=0}^{m+1} \binom{m}{k-1} + \sum_{k=0}^{m+1} \binom{m}{k} = \underbrace{\sum_{k=0}^{m} \binom{m}{k}}_{2^m} + \underbrace{\sum_{k=0}^{m} \binom{m}{k}}_{2^m} = 2^m + 2^m = 2^{m+1}.$$

This is the formula with $n = m + 1$. By induction it holds for all $n \geq 0$. ■

Generating-function-flavored counting

A product of binomials is a bookkeeping device: the coefficient of a power records *how many ways* to reach it. Since $(1 + x)^n = \sum_k \binom{n}{k} x^k$, the coefficient of x^k counts the ways to choose k items from n. Multiplying expansions combines independent choices, and Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

comes from matching the x^r coefficient on both sides of $(1 + x)^m (1 + x)^n = (1 + x)^{m+n}$.

Extracting a specific term from a product

Find the coefficient of x^2 in $(1 + x)^3 (1 + x)^4$.

The product is $(1 + x)^7$, so the coefficient of x^2 is $\binom{7}{2} = 21$. As a Vandermonde check with

$m = 3, n = 4, r = 2$:

$$\sum_{k=0}^{2} \binom{3}{k}\binom{4}{2-k} = \binom{3}{0}\binom{4}{2} + \binom{3}{1}\binom{4}{1} + \binom{3}{2}\binom{4}{0} = 6 + 12 + 3 = 21. \checkmark$$

A constant-term extraction (worked)

Find the constant term in $\left(2x + \frac{1}{x}\right)^6$.

General term: $\binom{6}{k}(2x)^{6-k}\left(\frac{1}{x}\right)^k = \binom{6}{k}2^{6-k}x^{6-2k}$. The constant term needs $6 - 2k = 0$, so $k = 3$:

$$\binom{6}{3}2^3x^0 = 20 \cdot 8 = 160.$$

The constant term is 160.

Tips for the advanced toolkit: (1) Before any *infinite* sum, confirm convergence—geometric-type sums need $|r| < 1$. (2) To spot a telescoping sum, try a partial-fraction split and look for a $b_k - b_{k+1}$ pattern. (3) For "specific term" problems, write the general term first, set the exponent equal to the target, solve for k, then substitute. (4) Many binomial identities are just the Binomial Theorem with a clever choice of x and y.

3 Introduction to Limits

The idea of a limit, evaluating limits, one-sided & infinite limits, limits at infinity, continuity, and a first look at the derivative

3.1 The Idea of a Limit; Estimating Numerically and Graphically

What a limit means

The **limit** of $f(x)$ as x approaches a is the single value L that the outputs $f(x)$ get arbitrarily close to as x gets close to a (from both sides), *without* necessarily equaling a. We write

$$\lim_{x \to a} f(x) = L.$$

The value $f(a)$ itself may be different from L, or may not even exist—the limit only cares about the behavior *near* a.

Estimating a limit from a table

Estimate $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$. The function is undefined at $x=2$ (it gives $\frac{0}{0}$), so we build a table of nearby inputs:

From both sides the outputs approach 4, so $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = 4$, even though $f(2)$ is undefined.

Tip: A hole in a graph does *not* stop a limit from existing. Read a limit by tracing the curve toward $x=a$ with your finger—the height you approach is L, whether or not the point is filled in.

3.2 Evaluating Limits Algebraically

The limit laws

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then limits distribute over the arithmetic:

$$\lim_{x\to a} \big[f \pm g\big] = \lim f \pm \lim g, \qquad \lim_{x\to a} \big[f \cdot g\big] = \lim f \cdot \lim g,$$

$$\lim_{x\to a} \frac{f}{g} = \frac{\lim f}{\lim g} \quad (\lim g \neq 0), \qquad \lim_{x\to a} \big[f(x)\big]^n = \Big[\lim_{x\to a} f(x)\Big]^n.$$

Direct substitution: for any polynomial, and for any rational/radical function *where the denominator is nonzero and the radicand is valid*, $\lim_{x\to a} f(x) = f(a)$.

Factoring away a $\frac{0}{0}$ form

Evaluate $\lim_{x\to 3} \dfrac{x^2-9}{x-3}$.

Direct substitution gives $\dfrac{9-9}{3-3} = \dfrac{0}{0}$ (indeterminate), so we factor and cancel:

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3 \quad (x \neq 3).$$

Now substitute: $\lim_{x\to 3}(x+3) = 3+3 = 6$.

Rationalizing a $\frac{0}{0}$ form

Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$. Substitution gives $\frac{0}{0}$. Multiply by the conjugate:

$$\frac{\sqrt{x+4}-2}{x} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)} = \frac{x}{x\left(\sqrt{x+4}+2\right)} = \frac{1}{\sqrt{x+4}+2}.$$

Then $\lim_{x\to 0} \dfrac{1}{\sqrt{x+4}+2} = \dfrac{1}{2+2} = \dfrac{1}{4}$.

Tip: Always *try direct substitution first*. Only if it produces the indeterminate form $\frac{0}{0}$ do you switch tools: **factor and cancel** (for polynomial ratios) or **multiply by the conjugate** (when a square root is involved).

3.3 One-Sided Limits; When a Limit Fails; Infinite Limits

One-sided limits and existence

The **right-hand limit** $\lim_{x\to a^+} f(x)$ uses inputs slightly larger than a; the **left-hand limit** $\lim_{x\to a^-} f(x)$ uses inputs slightly smaller. The two-sided limit exists *only when they agree*:

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

A limit **fails to exist (DNE)** when: the one-sided limits differ (a *jump*), the function grows without bound (*infinite*), or the function oscillates wildly near a.

A limit that fails to exist

For the graph above, $\lim_{x\to 2^-} f(x) = 2$ but $\lim_{x\to 2^+} f(x) = 4$. Since the one-sided limits disagree, $\lim_{x\to 2} f(x)$ **does not exist**. (Note $f(2) = 4$ from the closed dot—but that has no bearing on the two-sided limit.)

Infinite limits and vertical asymptotes

If $f(x)$ increases or decreases without bound as $x \to a$, we write $\lim_{x\to a} f(x) = \infty$ (or $-\infty$). This is a way of *describing* a limit that fails to exist; the line $x = a$ is a **vertical asymptote**. For $f(x) = \dfrac{1}{x}$:

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty, \qquad \lim_{x\to 0^+} \frac{1}{x} = +\infty.$$

Tip: To find a vertical asymptote of a rational function, look for inputs that make the *denominator zero but the numerator nonzero* after all common factors are cancelled. A factor that cancels gives a *hole*, not an asymptote.

3.4 Limits at Infinity and End Behavior

What happens as $x \to \pm\infty$

$\lim_{x\to\infty} f(x) = L$ means the outputs level off at the height $y = L$ as x grows large; the line $y = L$ is a **horizontal asymptote**. Two building blocks:

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = 0 \ (n > 0), \qquad \lim_{x\to\infty} x^n = \infty \ (n > 0).$$

Rational functions $\dfrac{p(x)}{q(x)}$—compare degrees:

- degree top < degree bottom ⇒ limit 0 (H.A. $y = 0$);
- degree top = degree bottom ⇒ limit = ratio of leading coefficients;
- degree top > degree bottom ⇒ limit $\pm\infty$ (no H.A.).

A limit at infinity

Evaluate $\lim_{x\to\infty} \dfrac{2x^2 - x}{x^2 + 1}$. Divide numerator and denominator by the highest power, x^2:

$$\frac{2x^2 - x}{x^2 + 1} = \frac{2 - \frac{1}{x}}{1 + \frac{1}{x^2}} \longrightarrow \frac{2 - 0}{1 + 0} = 2.$$

The degrees match, so the limit is the ratio of leading coefficients, $\frac{2}{1} = 2$. The line $y = 2$ is a horizontal asymptote.

Tip: For polynomials, end behavior is governed entirely by the leading term: $\lim_{x\to\pm\infty}(3x^3 - \cdots)$ behaves like $3x^3$. Lower-degree terms are negligible for large $|x|$.

3.5 Continuity at a Point; Types of Discontinuity

The three-part continuity test

A function f is **continuous at** $x = a$ if all three hold:

$$(1)\ f(a) \text{ is defined}, \quad (2)\ \lim_{x\to a} f(x) \text{ exists}, \quad (3)\ \lim_{x\to a} f(x) = f(a).$$

Informally: you can draw the graph through $x = a$ without lifting your pencil. If any part fails, f has a **discontinuity** there.

Classifying discontinuities

- **Removable** (a hole): $\lim_{x\to a} f$ exists but $\neq f(a)$ (or $f(a)$ is undefined). "Removable" because redefining one point would fix it.
- **Jump:** the left- and right-hand limits both exist but disagree.
- **Infinite:** the function blows up to $\pm\infty$ (a vertical asymptote).

Testing continuity

Is $f(x) = \dfrac{x^2-9}{x-3}$ continuous at $x = 3$? Here $f(3) = \frac{0}{0}$ is undefined, so part (1) fails—f is **discontinuous** at 3. Because $\lim_{x\to 3} f(x) = 6$ exists, the discontinuity is **removable**: defining $f(3) = 6$ would patch the hole.

Tip: To classify a discontinuity from an equation, factor the rational expression. A factor that *cancels* signals a removable hole; a leftover zero in the denominator signals an infinite discontinuity (vertical asymptote).

3.6 The Tangent-Line Problem, the Derivative, and the Area Problem

From secant slopes to the tangent slope

The slope of the **secant** line through $\big(x, f(x)\big)$ and $\big(x+h, f(x+h)\big)$ is the difference quotient $\dfrac{f(x+h)-f(x)}{h}$. Letting $h \to 0$ slides the second point into the first; the secant slopes approach the slope of the **tangent** line. That limiting slope is the **derivative**:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Derivative from the limit definition

Find $f'(x)$ for $f(x) = x^2$.

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x + h.$$

Then

$$f'(x) = \lim_{h\to 0}(2x+h) = 2x.$$

So the tangent slope at $x = 3$ is $f'(3) = 6$.

The area problem

The other great problem of calculus is finding the **area under a curve** $y = f(x)$ over $[a, b]$. Approximate it with n thin rectangles of width $\Delta x = \frac{b-a}{n}$, sum their areas, and take a limit:

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x.$$

As $n \to \infty$ the rectangles shrink and the approximation becomes exact. This limit is the **definite integral**.

Tip: Both headline ideas of calculus are *limits*: the derivative is a limit of secant slopes ($h \to 0$), and the definite integral is a limit of rectangle-sum areas ($n \to \infty$). Mastering limits now is what makes calculus possible later.

3.7 Going Deeper: Advanced Limit Ideas

A bigger toolbox for the $\frac{0}{0}$ form

When direct substitution gives $\frac{0}{0}$, the expression hides a common factor you must expose and cancel. Choose the tool by what you see:

- **Polynomial ratio** $\Rightarrow$ factor top and bottom, cancel the shared factor.
- **A square root** $\Rightarrow$ multiply by the conjugate to turn $\sqrt{\ }$ differences into plain differences.
- **A complex fraction** (fractions stacked inside a fraction) $\Rightarrow$ combine the small fractions over a common denominator first, then simplify the big fraction.
- **A trig ratio** $\Rightarrow$ rewrite in terms of sin and cos and steer toward the small-angle facts $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

A complex-fraction limit

Evaluate $\lim_{x\to 0} \dfrac{\dfrac{1}{x+2}-\dfrac{1}{2}}{x}$. Substitution gives $\frac{0}{0}$. First combine the top over the common denominator $2(x+2)$:

$$\frac{1}{x+2}-\frac{1}{2}=\frac{2-(x+2)}{2(x+2)}=\frac{-x}{2(x+2)}.$$

Dividing by x means multiplying by $\frac{1}{x}$, and the x's cancel:

$$\frac{1}{x}\cdot\frac{-x}{2(x+2)}=\frac{-1}{2(x+2)} \quad\longrightarrow\quad \frac{-1}{2(0+2)}=-\frac{1}{4}.$$

A three-factor trig limit: $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$

Substitution gives $\frac{0}{0}$. Rewrite $\tan x=\dfrac{\sin x}{\cos x}$ and factor $\sin x$ out of the numerator:

$$\tan x-\sin x=\frac{\sin x}{\cos x}-\sin x=\sin x\left(\frac{1-\cos x}{\cos x}\right).$$

Now split the x^3 in the denominator as $x\cdot x^2$ and group into three familiar pieces:

$$\frac{\tan x-\sin x}{x^3}=\frac{\sin x\,(1-\cos x)}{x^3\cos x}=\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{1-\cos x}{x^2}}_{\to\frac{1}{2}}\cdot\underbrace{\frac{1}{\cos x}}_{\to 1}.$$

Each factor has a known limit as $x\to 0$, so the product is $1\cdot\frac{1}{2}\cdot 1=\boxed{\dfrac{1}{2}}$.

Limits of the floor function $\lfloor x\rfloor$

The **floor** $\lfloor x\rfloor$ is the greatest integer $\le x$; its graph is a staircase of flat steps with jumps at every integer. So at an integer n the one-sided limits disagree:

$$\lim_{x\to n^-}\lfloor x\rfloor=n-1,\qquad \lim_{x\to n^+}\lfloor x\rfloor=n \quad\Rightarrow\quad \lim_{x\to n}\lfloor x\rfloor \text{ DNE.}$$

Between integers the floor is constant, so for any non-integer c, $\lim_{x\to c}\lfloor x\rfloor=\lfloor c\rfloor$ (continuous there).

Piecewise functions: solving for TWO parameters

Suppose a piecewise f has two unknown constants a,b and you want it **smooth** (continuous *and* differentiable) at the split point $x = c$. You get two equations from two conditions:

$$\text{(continuity)} \quad \lim_{x\to c^-} f = \lim_{x\to c^+} f, \qquad \text{(differentiability)} \quad \lim_{x\to c^-} f' = \lim_{x\to c^+} f'.$$

Match the *heights* for continuity and the *slopes* for differentiability, then solve the resulting 2×2 system for a and b.

Making a piecewise function differentiable

Find a, b so that

$$f(x) = \begin{cases} x^2, & x \le 1, \\ ax + b, & x > 1, \end{cases}$$

is differentiable at $x = 1$.

Slopes (differentiability). For $x < 1$, $f'(x) = 2x \to 2$; for $x > 1$, $f'(x) = a$. Matching slopes gives $a = 2$.

Heights (continuity). The left value is $1^2 = 1$; the right value is $a(1) + b = 2 + b$. Matching gives $2 + b = 1$, so $b = -1$.

Thus $a = 2,\ b = -1$: the parabola hands off to the tangent line $y = 2x - 1$ with no corner.

The derivative of a harder function from the definition

Find $f'(x)$ for $f(x) = \dfrac{1}{x}$. Form the difference quotient and combine the small fractions:

$$\frac{f(x+h) - f(x)}{h} = \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \frac{\dfrac{x - (x+h)}{x(x+h)}}{h} = \frac{-h}{h\,x(x+h)} = \frac{-1}{x(x+h)}.$$

The h cancels, so

$$f'(x) = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x \cdot x} = -\frac{1}{x^2}.$$

A limit at infinity with a radical

Evaluate $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2 + 3x} - x\right)$. This is an $\infty - \infty$ form, so multiply by the conjugate:

$$\left(\sqrt{x^2 + 3x} - x\right) \cdot \frac{\sqrt{x^2 + 3x} + x}{\sqrt{x^2 + 3x} + x} = \frac{(x^2 + 3x) - x^2}{\sqrt{x^2 + 3x} + x} = \frac{3x}{\sqrt{x^2 + 3x} + x}.$$

Divide top and bottom by x (and, since $x > 0$, note $\sqrt{x^2+3x} = x\sqrt{1+3/x}$):

$$\frac{3x}{\sqrt{x^2+3x}+x} = \frac{3}{\sqrt{1+\frac{3}{x}}+1} \longrightarrow \frac{3}{\sqrt{1}+1} = \frac{3}{2}.$$

Preview—area as a limit of Riemann sums. To find the exact area under $y = x^2$ on $[0,1]$, cut it into n rectangles of width $\Delta x = \frac{1}{n}$ using right endpoints $x_k = \frac{k}{n}$:

$$S_n = \sum_{k=1}^{n} \left(\tfrac{k}{n}\right)^2 \cdot \tfrac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}.$$

As $n \to \infty$ this tends to $\dfrac{2}{6} = \dfrac{1}{3}$, so the area is $\frac{1}{3}$. Every definite integral is exactly this kind of limit.

4 Functions and Their Graphs

Notation, domain & range, graph analysis, parent functions, transformations, combinations, inverses, and piecewise functions

4.1 Function Notation, Domain & Range, Evaluating

What is a function?

A **function** f assigns to each input x (from the **domain**) exactly one output $y = f(x)$. The set of all outputs is the **range**.

- **Evaluating:** substitute the input for every x. In $f(x) = x^2 - 3x$, we get $f(a+1) = (a+1)^2 - 3(a+1)$.
- **Domain of a polynomial:** all reals, $(-\infty, \infty)$.
- **Radical** $\sqrt{g(x)}$: require $g(x) \geq 0$.
- **Rational** $\dfrac{p(x)}{q(x)}$: require $q(x) \neq 0$.

Finding a domain

Find the domain of $f(x) = \dfrac{\sqrt{x+2}}{x-3}$.

Need $x + 2 \geq 0 \Rightarrow x \geq -2$, and $x - 3 \neq 0 \Rightarrow x \neq 3$.

Domain: $[-2, 3) \cup (3, \infty)$.

Evaluating

For $f(x) = x^2 - 3x$, compute $f(-2)$ and $f(a+1)$.

$f(-2) = (-2)^2 - 3(-2) = 4 + 6 = 10$.

$f(a+1) = (a+1)^2 - 3(a+1) = a^2 + 2a + 1 - 3a - 3 = a^2 - a - 2$.

Tip: A domain restriction from a radical uses $\geq$; a restriction from a denominator uses $\neq$. Always combine *both* kinds of restriction when they appear together.

4.2 The Difference Quotient

Difference quotient

The **difference quotient** of f is

$$\frac{f(x+h) - f(x)}{h}, \qquad h \neq 0.$$

It measures the average rate of change of f over $[x, x+h]$ and is the foundation of the derivative. Simplify until the h in the denominator cancels.

Difference quotient of a quadratic

Let $f(x) = x^2 + 1$. Then

$$f(x+h) = (x+h)^2 + 1 = x^2 + 2xh + h^2 + 1,$$

$$\frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 + 1) - (x^2 + 1)}{h} = \frac{2xh + h^2}{h} = 2x + h.$$

Tip: The $f(x+h) - f(x)$ numerator should always lose its constant term and its lone x^2 term, leaving only factors of h. If no h cancels, recheck the algebra.

4.3 Analyzing Graphs

Reading a graph

- **x-intercepts (zeros):** where $f(x) = 0$; **y-intercept:** the value $f(0)$.
- **Even** (symmetric about y-axis): $f(-x) = f(x)$. **Odd** (symmetric about origin): $f(-x) = -f(x)$.
- **Increasing / decreasing / constant** over an interval as x moves left to right.
- **Relative maximum / minimum:** a peak or valley of the graph.

Slope is rise over run.

(diagram in the online version)

Analyzing $f(x) = x^3 - 3x$

- **Zeros:** $x^3 - 3x = x(x^2 - 3) = 0 \Rightarrow x = 0,\ \pm\sqrt{3}$.
- **Symmetry:** $f(-x) = -x^3 + 3x = -f(x)$, so f is **odd**.
- **Relative max** $(-1, 2)$; **relative min** $(1, -2)$.
- **Increasing** on $(-\infty, -1)$ and $(1, \infty)$; **decreasing** on $(-1, 1)$.

Tip: To test symmetry algebraically, substitute $-x$. If you get the *same* expression it is even; if you get the *opposite* of the original it is odd; otherwise neither.

4.4 The Library of Parent Functions

Eight parents to memorize

Constant $f(x) = c$; Identity $f(x) = x$; Absolute value $f(x) = |x|$; Square root $f(x) = \sqrt{x}$; Quadratic $f(x) = x^2$; Cubic $f(x) = x^3$; Reciprocal $f(x) = \dfrac{1}{x}$; Greatest integer (step) $f(x) = \lfloor x \rfloor$.

Tip: $\sqrt{x}$ has domain $[0, \infty)$ and $\frac{1}{x}$ excludes $x = 0$ (a vertical asymptote). All other parents above have domain $(-\infty, \infty)$.

4.5 Transformations of Functions

Reading $y = a\,f(bx - c) + d$

Starting from $y = f(x)$:

- $+d$: shift **up** d; $-d$: shift down.
- $f(x - c)$: shift **right** c; $f(x + c)$: shift left. (Horizontal moves are opposite the sign.)
- $a < 0$: reflect over the x-axis; $|a| > 1$: vertical **stretch**; $0 < |a| < 1$: vertical shrink.
- $b < 0$: reflect over the y-axis; $|b| > 1$: horizontal **shrink**; $0 < |b| < 1$: horizontal stretch.

A multi-step transformation

Graph $g(x) = -|x - 1| + 3$ from the parent $f(x) = |x|$: shift right 1, reflect over the x-axis, then shift up 3. The vertex moves from $(0, 0)$ to $(1, 3)$ and the graph opens downward. Below, f is dashed (pre-image) and g is solid (image).

Tip: Apply transformations in order: horizontal shift/stretch (inside the function) first, then vertical stretch/reflection, then vertical shift. When $b \neq 1$, factor first: $f(bx - c) = f\big(b(x - \frac{c}{b})\big)$ so the horizontal shift is c/b, not c.

4.6 Combinations and Composition of Functions

Arithmetic and composition

For functions f and g:

$$(f \pm g)(x) = f(x) \pm g(x), \quad (fg)(x) = f(x)\,g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0.$$

The **composition** $(f \circ g)(x) = f\big(g(x)\big)$ feeds the output of g into f.

Composition and its domain

Let $f(x) = \sqrt{x}$ and $g(x) = x - 4$. Then

$$(f \circ g)(x) = f(x-4) = \sqrt{x-4}.$$

The domain needs $x - 4 \geq 0$, so **domain** $= [4, \infty)$. Also $(g \circ f)(x) = \sqrt{x} - 4$, with domain $[0, \infty)$.

Tip: The domain of $f \circ g$ is the set of x in the domain of g whose outputs $g(x)$ lie in the domain of f. Composition is *not* commutative: usually $f \circ g \neq g \circ f$.

4.7 Inverse Functions

One-to-one and inverses

f is **one-to-one** if each output comes from exactly one input; equivalently it passes the **horizontal line test**. Only one-to-one functions have an inverse f^{-1}, which satisfies

$$f\big(f^{-1}(x)\big) = x \quad \text{and} \quad f^{-1}\big(f(x)\big) = x.$$

The graph of f^{-1} is the reflection of the graph of f over the line $y = x$.

Finding and verifying an inverse

Find the inverse of $f(x) = 2x + 6$. Write $y = 2x + 6$, swap: $x = 2y + 6$, solve: $y = \dfrac{x-6}{2}$.

So $f^{-1}(x) = \dfrac{x-6}{2}$. Verify: $f\big(f^{-1}(x)\big) = 2 \cdot \dfrac{x-6}{2} + 6 = x$. (verified)

Tip: If a function fails the horizontal line test (like $f(x) = x^2$), *restrict its domain* (e.g. $x \geq 0$) to make it one-to-one before inverting.

4.8 Piecewise Functions and Average Rate of Change

Piecewise functions

A **piecewise** function uses different formulas on different parts of the domain. A **closed dot** $\bullet$ includes an endpoint; an **open dot** $\circ$ excludes it. Evaluate by choosing the rule whose condition the input satisfies.

$$f(x) = \begin{cases} x+1, & x<1 \\ 4-x, & x \geq 1 \end{cases}$$

Here $f(0) = 1$ and $f(1) = 3$.

Average rate of change

The average rate of change of f from $x = a$ to $x = b$ is the slope of the secant line:

$$\frac{f(b) - f(a)}{b - a}.$$

Average rate of change

For $f(x) = x^2$ from $a = 1$ to $b = 3$:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Tip: Average rate of change is the difference quotient with $x = a$ and $h = b - a$; it is a *secant* slope, not the value of the function.

4.9 Going Deeper: Advanced Function Ideas

Functional equations

A **functional equation** constrains an unknown function f by an identity that must hold for *all* allowed x. The key tactic is **substitution**: replace x with a related expression (such as $\frac{1}{x}$, $-x$, or $1-x$) to generate a *second* equation, then solve the resulting system for $f(x)$ as if $f(x)$ and the substituted value were two separate unknowns.

- Common self-inverse swaps: $x \mapsto \frac{1}{x}$, $x \mapsto -x$, $x \mapsto 1 - x$.
- After substituting, treat $f(x)$ and $f(\text{new input})$ as variables and eliminate.

Solving a functional equation

Find f if $f(x) + 2f\left(\frac{1}{x}\right) = 3x$ for all $x \neq 0$.

Replace x with $\frac{1}{x}$ to get a companion equation, giving the system

$$\begin{aligned} f(x) + 2f\left(\tfrac{1}{x}\right) &= 3x, \\ 2f(x) + f\left(\tfrac{1}{x}\right) &= \tfrac{3}{x}. \end{aligned}$$

Multiply the second by 2 and subtract the first: $3f(x) = \dfrac{6}{x} - 3x$, so

$$f(x) = \frac{2}{x} - x.$$

Check: $f(x) + 2f\left(\frac{1}{x}\right) = \left(\frac{2}{x} - x\right) + 2\left(2x - \frac{1}{x}\right) = 3x.$ (verified)

Iterated compositions and involutions

Composing f with itself n times is written

$$f^{[n]} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies}}, \qquad f^{[1]} = f, \quad f^{[n+1]} = f \circ f^{[n]}.$$

(The bracket distinguishes this from a power $f(x)^n$ or the inverse f^{-1}.) A function is an **involution** if $f \circ f = \mathrm{id}$, i.e. $f\big(f(x)\big) = x$; then $f^{-1} = f$ and the iterates *cycle*: $f^{[\text{even}]} = \mathrm{id}$ and $f^{[\text{odd}]} = f$. Some maps have longer periods (e.g. $f(x) = 1 - \frac{1}{x}$ satisfies $f^{[3]} = \mathrm{id}$).

An involution and its iterates

Let $f(x) = \dfrac{x}{x-1}$ with $x \neq 1$. Compute $f \circ f$:

$$f\big(f(x)\big) = \frac{\frac{x}{x-1}}{\frac{x}{x-1} - 1} = \frac{\frac{x}{x-1}}{\frac{x-(x-1)}{x-1}} = \frac{\frac{x}{x-1}}{\frac{1}{x-1}} = x.$$

So f is an **involution**: $f^{[2]} = \mathrm{id}$ and $f^{-1} = f$. The iterates therefore alternate, and for instance $f^{[100]} = \mathrm{id}$ while $f^{[101]} = f$.

Monotonicity, symmetry, and composed transformations

Three high-leverage ideas:

- **Monotonicity $\Rightarrow$ invertibility.** If f is *strictly* increasing (or strictly decreasing) on its domain, it automatically passes the horizontal line test, so f^{-1} exists there — no need to solve algebraically to know it is one-to-one.
- **Exploiting even/odd symmetry.** An even function is fully determined by its values on $[0, \infty)$; an odd function must have $f(0) = 0$ (when 0 is in the domain). Products/quotients obey: even·even = even, odd·odd = even, even·odd = odd.

- **Composing transformations into one mapping.** Applying transformation T_1 then T_2 means the combined rule is $T_2 \circ T_1$. Order matters: a shift-then-stretch gives a different single formula $y = a\,f(bx - c) + d$ than stretch-then-shift.

Domain of a nested composition

Find the domain of $F(x) = \sqrt{4 - \sqrt{x}}$, a composition $F = f \circ g$ with $f(u) = \sqrt{u}$ and $g(x) = 4 - \sqrt{x}$.

Work from the *inside out*, collecting every restriction:

- Inner radical: $x \geq 0$.
- Outer radical: $4 - \sqrt{x} \geq 0 \Rightarrow \sqrt{x} \leq 4 \Rightarrow x \leq 16$.

Both must hold, so **domain** $= [0, 16]$.

A parameter that makes a piecewise function continuous

For which c is

$$f(x) = \begin{cases} cx + 1, & x \leq 2 \\ x^2 - c, & x > 2 \end{cases}$$

continuous at the split point $x = 2$?

The two branches must agree in the limit at $x = 2$:

$$\underbrace{c(2) + 1}_{\text{left value}} = \underbrace{(2)^2 - c}_{\text{right limit}} \implies 2c + 1 = 4 - c \implies 3c = 3 \implies c = 1.$$

With $c = 1$ both pieces give 3 at $x = 2$, so the graph joins without a jump.

Difference quotient of a harder function

Compute the difference quotient of $f(x) = \sqrt{x}$ using the **conjugate**:

$$\frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \frac{(x+h) - x}{h\big(\sqrt{x+h} + \sqrt{x}\big)} = \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

The h cancels once the radicals are rationalized. (For $f(x) = \frac{1}{x}$ the same idea gives $\dfrac{f(x+h) - f(x)}{h} = \dfrac{-1}{x(x+h)}$.)

Tip: For nested radicals and rational compositions, never simplify before finding the domain — restrictions from an inner piece survive even if they cancel algebraically. And for iterated maps, always confirm a suspected involution or period by *computing* $f \circ f$ (and $f \circ f \circ f$), not by eyeballing the formula.

5 Polynomial and Rational Functions

Quadratics, end behavior, zeros, division, complex numbers, and rational graphs

5.1 Quadratic Functions

Two useful forms

Standard (vertex) form: $f(x) = a(x-h)^2+k$ has vertex (h, k) and axis of symmetry $x = h$.
General form: $f(x) = ax^2+bx+c$ has vertex at $x = -\dfrac{b}{2a}$, so the vertex is $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.
If $a > 0$ the parabola opens up (vertex is a *minimum*); if $a < 0$ it opens down (vertex is a *maximum*).

Completing the square to find the vertex

Write $f(x) = 2x^2 - 8x + 5$ in vertex form.

$$f(x) = 2\left(x^2 - 4x\right) + 5 = 2\left(x^2 - 4x + 4 - 4\right) + 5 = 2\left(x-2\right)^2 - 8 + 5 = 2(x-2)^2 - 3.$$

Vertex $(2, -3)$, axis $x = 2$, minimum value -3. Check: $-\dfrac{b}{2a} = \dfrac{8}{4} = 2$ and $f(2) = 8-16+5 = -3$. ✓

Modeling: maximum height

A ball's height is $h(t) = -16t^2 + 64t + 5$ (feet, t in seconds). Since $a < 0$, the vertex is the max.

$$t = -\frac{b}{2a} = -\frac{64}{-32} = 2 \text{ s}, \qquad h(2) = -16(4) + 64(2) + 5 = 69 \text{ ft}.$$

The ball reaches a maximum height of 69 ft after 2 seconds.

Tip: The axis of symmetry always passes through the vertex. Read the sign of a first: it tells you the shape (up/down) and whether the vertex is a min or max.

5.2 Polynomial Functions: End Behavior, Zeros, and Sketching

Leading Coefficient Test and multiplicity

For $f(x) = a_n x^n + \cdots + a_0$, the ends are controlled by the degree n and sign of a_n:

	$a_n > 0$	$a_n < 0$
n even	up / up	down / down
n odd	down / up	up / down

Real zeros & multiplicity: if $(x-c)^k$ is a factor, c is a zero of multiplicity k. If k is *odd* the graph *crosses* at c; if k is *even* it *touches* (bounces). A degree-n polynomial has at most n real zeros and at most $n-1$ turning points.

Reading a factored polynomial

Analyze $f(x) = (x+2)^2(x-1)(x-3)$. Degree $= 2+1+1 = 4$ (even), leading coefficient > 0, so **end behavior: up / up**. Zeros: $x = -2$ (multiplicity 2, *touches*), $x = 1$ (mult. 1, *crosses*), $x = 3$ (mult. 1, *crosses*). At most 3 turning points.

Tip: Higher multiplicity flattens the graph near the zero. Multiplicity $3, 5, \ldots$ still crosses but with a flat "wiggle"; multiplicity $2, 4, \ldots$ bounces off the axis.

5.3 Polynomial Long Division; Remainder & Factor Theorems

Division algorithm and two theorems

For polynomials, $f(x) = d(x)\,q(x) + r(x)$ with $\deg r < \deg d$. **Synthetic division** is a shortcut when the divisor is $x - k$.

Remainder Theorem: dividing $f(x)$ by $x - k$ leaves remainder $r = f(k)$.
Factor Theorem: $(x-k)$ is a factor of $f(x)$ if and only if $f(k) = 0$.

Long division

Divide $x^3 + 2x^2 - 4$ by $x - 1$.

$$
x^2+3x+3 \;\Big|\; \begin{array}{rrrrrrr}
 & x^3 & + & 2x^2 & + & 0x & -4 \\
-(& x^3 & - & x^2 & & &) \\
\hline
 & & & 3x^2 & + & 0x & \\
 & & -(& 3x^2 & - & 3x &) \\
\hline
 & & & & & 3x & -4 \\
 & & & & -(& 3x & -3) \\
\hline
 & & & & & & -1
\end{array}
$$

Quotient $x^2 + 3x + 3$, remainder -1. By the Remainder Theorem $f(1) = 1 + 2 - 4 = -1$. ✓

Synthetic division

Divide $2x^3 - 5x^2 + 3x - 7$ by $x - 3$ (so $k = 3$).

$$\begin{array}{c|cccc} 3 & 2 & -5 & 3 & -7 \\ & & 6 & 3 & 18 \\ \hline & 2 & 1 & 6 & 11 \end{array}$$

Quotient $2x^2 + x + 6$, remainder 11. Check: $f(3) = 54 - 45 + 9 - 7 = 11$. ✓ Since $r \neq 0$, $(x - 3)$ is *not* a factor.

Tip: Use synthetic division *only* when the divisor has the form $x - k$. Always insert 0 coefficients for any missing powers before you start.

5.4 Finding Real Zeros: Rational Zero Test, Descartes, Bounds

Three tools

Rational Zero Test: every rational zero of f has the form $\dfrac{p}{q}$, where $p \mid$ (constant term) and $q \mid$ (leading coefficient).

Descartes' Rule of Signs: the number of *positive* real zeros equals the number of sign changes in $f(x)$, or less by an even number. The number of *negative* real zeros uses the sign changes in $f(-x)$.

Upper/Lower Bound Rule (via synthetic division by $x - c$): if $c > 0$ and the bottom row is all ≥ 0, then c is an *upper* bound for the real zeros. If $c < 0$ and the bottom row *alternates* in sign, then c is a *lower* bound.

Full zero-finding

Find all real zeros of $f(x) = 2x^3 - 3x^2 - 11x + 6$.
Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}$. Try $x = 3$:

$$\begin{array}{c|cccc} 3 & 2 & -3 & -11 & 6 \\ & & 6 & 9 & -6 \\ \hline & 2 & 3 & -2 & 0 \end{array}$$

Remainder 0, so 3 is a zero and $f(x) = (x - 3)(2x^2 + 3x - 2) = (x - 3)(2x - 1)(x + 2)$.
Zeros: $x = 3,\ \frac{1}{2},\ -2$. Check $f(\frac{1}{2}) = 2(\frac{1}{8}) - 3(\frac{1}{4}) - \frac{11}{2} + 6 = \frac{1}{4} - \frac{3}{4} - \frac{11}{2} + 6 = 0$. ✓

Descartes' Rule of Signs

For $f(x) = 2x^3 - 3x^2 - 11x + 6$ the signs are $+, -, -, +$: two sign changes $\Rightarrow$ 2 or 0 positive real zeros.
$f(-x) = -2x^3 - 3x^2 + 11x + 6$ has signs $-, -, +, +$: one sign change $\Rightarrow$ exactly 1 negative real zero. This matches the zeros 3, $\frac{1}{2}$ (positive) and -2 (negative).

Tip: Bounds shrink your search. Once synthetic division by a positive c gives an all-nonnegative bottom row, no zero can be larger than c, so stop testing bigger candidates.

5.5 Complex Numbers and the Fundamental Theorem of Algebra

Complex zeros come in pairs

With $i^2 = -1$, a complex number is $a + bi$ and its **conjugate** is $a - bi$; note $(a + bi)(a - bi) = a^2 + b^2$.
Fundamental Theorem of Algebra: every polynomial of degree $n \geq 1$ has exactly n complex zeros (counting multiplicity).
Conjugate Pairs Theorem: if f has *real* coefficients and $a + bi$ is a zero, then $a - bi$ is also a zero.

Finding all zeros

Find all zeros of $f(x) = x^3 - x^2 + 9x - 9$. Factor by grouping:

$$f(x) = x^2(x - 1) + 9(x - 1) = (x - 1)(x^2 + 9).$$

So $x - 1 = 0$ or $x^2 = -9$. **Zeros:** $x = 1,\ 3i,\ -3i$ (the imaginary zeros are a conjugate pair).

Writing a polynomial from its zeros

Build a degree-3 polynomial with real coefficients having zeros 2 and $3i$. By conjugate pairs, $-3i$ is also a zero.

$$f(x) = (x - 2)(x - 3i)(x + 3i) = (x - 2)(x^2 + 9) = x^3 - 2x^2 + 9x - 18.$$

Tip: Multiply conjugate factors first: $(x - (a + bi))(x - (a - bi)) = x^2 - 2ax + (a^2 + b^2)$ is always a *real* quadratic, which keeps the arithmetic clean.

5.6 Rational Functions: Domain, Asymptotes, Holes, and Sketching

Asymptote rules

For $f(x) = \dfrac{N(x)}{D(x)}$ in lowest terms:
Domain: all reals except the zeros of $D(x)$.
Vertical asymptotes: at zeros of $D(x)$ that remain after cancelling.
Holes: at any zero common to N and D (a cancelled factor).
Horizontal / slant asymptotes (let $n = \deg N,\ m = \deg D$):

$n < m:\ y = 0;$ $\quad n = m:\ y = \frac{\text{lead } N}{\text{lead } D};$ $\quad n = m{+}1:$ slant (divide to find it); $\quad n > m{+}1:$ none.

Full analysis with a hole

Analyze $f(x) = \dfrac{x^2 - x - 6}{x^2 - 9} = \dfrac{(x-3)(x+2)}{(x-3)(x+3)} = \dfrac{x+2}{x+3}$.
Hole: at $x = 3$ (cancelled factor); its y-value is $\dfrac{3+2}{3+3} = \dfrac{5}{6}$, so the hole is $\left(3, \frac{5}{6}\right)$.
Vertical asymptote: $x = -3$. **Horizontal:** degrees equal $\Rightarrow y = \dfrac{1}{1} = 1$.
Intercepts: x-int at $x = -2$; y-int $f(0) = \dfrac{0+2}{0+3} = \dfrac{2}{3}$.

Slant asymptote

Analyze $f(x) = \dfrac{x^2 - 4}{x - 1}$. Divide: $x^2 - 4 = (x-1)(x+1) - 3$, so

$$f(x) = x + 1 - \frac{3}{x-1}.$$

Slant asymptote: $y = x + 1$. **Vertical asymptote:** $x = 1$. **x-intercepts:** $x = \pm 2$; **y-int:** $f(0) = \dfrac{-4}{-1} = 4$.

Tip: Simplify the fraction *first*. A factor that cancels gives a *hole*, not a vertical asymptote. The remaining denominator zeros are the true vertical asymptotes.

5.7 Going Deeper: Advanced Polynomial & Rational Ideas

Vieta's formulas: coefficients from roots

If $x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ (monic) has roots $r_1, \ldots, r_n$, then the elementary symmetric sums of the roots are the coefficients (up to sign):

$$\sum r_i = -c_{n-1}, \qquad \sum_{i<j} r_i r_j = c_{n-2}, \qquad \dots, \qquad r_1 r_2 \cdots r_n = (-1)^n c_0.$$

Cubic $x^3 + bx^2 + cx + d$: $r_1 + r_2 + r_3 = -b,\; r_1r_2 + r_1r_3 + r_2r_3 = c,\; r_1r_2r_3 = -d.$
For a non-monic $a_n x^n + \cdots + a_0$, divide by a_n first: e.g. $\sum r_i = -\dfrac{a_{n-1}}{a_n}$ and $\prod r_i = (-1)^n \dfrac{a_0}{a_n}$.

Power sums and Newton's identities

Let $p_k = r_1^k + r_2^k + \cdots + r_n^k$ be the k-th *power sum* of the roots and let $e_1, e_2, \dots$ be the elementary symmetric sums ($e_1 = \sum r_i,\; e_2 = \sum_{i<j} r_i r_j,\; \dots$). **Newton's identities** link them:

$$\begin{aligned} p_1 &= e_1, \\ p_2 &= e_1 p_1 - 2e_2, \\ p_3 &= e_1 p_2 - e_2 p_1 + 3e_3, \\ p_4 &= e_1 p_3 - e_2 p_2 + e_3 p_1 - 4e_4. \end{aligned}$$

These let you compute $\sum r_i^k$ *without ever finding the roots* — just read $e_1, e_2, \dots$ off the coefficients via Vieta.

Power sums via Vieta & Newton (no roots needed)

The roots of $x^3 - 2x^2 + 3x - 4 = 0$ are r_1, r_2, r_3. Find $r_1^2 + r_2^2 + r_3^2$ and $r_1^3 + r_2^3 + r_3^3$.
By Vieta: $e_1 = 2,\; e_2 = 3,\; e_3 = 4$. Then by Newton's identities:

$$\begin{aligned} p_1 &= e_1 = 2, \\ p_2 &= e_1 p_1 - 2e_2 = (2)(2) - 2(3) = -2, \\ p_3 &= e_1 p_2 - e_2 p_1 + 3e_3 = (2)(-2) - (3)(2) + 3(4) = -4 - 6 + 12 = 2. \end{aligned}$$

So $\sum r_i^2 = -2$ and $\sum r_i^3 = 2$. (A negative sum of squares is fine: the roots are complex.) ✓

Remainder modulo $(x-a)(x-b)$

Dividing $f(x)$ by a *quadratic* $(x-a)(x-b)$ leaves a remainder of degree < 2, so $r(x) = px + q$ is *linear*:

$$f(x) = (x-a)(x-b)\,q(x) + px + q.$$

Evaluating at the two roots kills the quotient term, giving a 2×2 system:

$$f(a) = pa + q, \qquad f(b) = pb + q.$$

Solve for p, q. (The idea generalizes: the remainder mod a degree-m divisor has degree $< m$, and m evaluation points pin it down — this is polynomial interpolation.)

Finding a linear remainder

Find the remainder when $f(x) = x^{50}$ is divided by $(x-1)(x-2)$.
Write $r(x) = px + q$. Then $f(1) = 1^{50} = 1$ and $f(2) = 2^{50}$ give

$$\begin{cases} p + q = 1, \\ 2p + q = 2^{50}. \end{cases}$$

Subtracting: $p = 2^{50} - 1$, and $q = 1 - p = 2 - 2^{50}$. **Remainder:** $(2^{50} - 1)x + (2 - 2^{50})$. ✓

Constructing polynomials from constraints; roots in AP or GP

From constraints: a degree-n polynomial has $n+1$ coefficients, so $n+1$ independent conditions (values, zeros, matching derivatives/slopes, a leading coefficient) determine it. Set up one linear equation per condition and solve.
Roots in arithmetic progression (AP): write three roots symmetrically as $\alpha - \delta,\ \alpha,\ \alpha + \delta$. Then $\sum r_i = 3\alpha$, so Vieta immediately gives the middle root $\alpha = -\frac{b}{3}$ (for monic $x^3 + bx^2 + cx + d$).
Roots in geometric progression (GP): write them as $\frac{\alpha}{\rho},\ \alpha,\ \alpha\rho$. Then the product $r_1 r_2 r_3 = \alpha^3 = -d$, so the middle root is $\alpha = \sqrt[3]{-d}$.

Roots in arithmetic progression

The equation $x^3 - 6x^2 + 11x - 6 = 0$ has three real roots in AP. Find them.
Let the roots be $\alpha - \delta,\ \alpha,\ \alpha + \delta$. Sum $= 3\alpha = 6 \Rightarrow \alpha = 2$, so 2 is a root. Product $= \alpha(\alpha^2 - \delta^2) = 6 \Rightarrow 2(4 - \delta^2) = 6 \Rightarrow \delta^2 = 1 \Rightarrow \delta = 1$.
Roots: $1,\ 2,\ 3$. Check: $(x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$. ✓

Tip: Whenever roots are described as "in AP" or "in GP," pick the *symmetric* labeling ($\alpha \pm \delta$ or $\alpha/\rho, \alpha\rho$). The symmetry makes one Vieta relation collapse to a single unknown, so you get the middle root almost for free.

Partial-fraction decomposition

To split a *proper* rational function $\dfrac{N(x)}{D(x)}$ (with $\deg N < \deg D$; otherwise divide first), fully factor D and assign one term per factor:

$$\frac{A}{x-a}, \qquad \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\cdots \text{ (repeated)}, \qquad \frac{Bx+C}{x^2+px+q} \text{ (irreducible quadratic)}.$$

Clear denominators, then solve for the constants by substituting convenient x-values (the roots) and/or matching coefficients.

Partial fractions with a repeated factor

Decompose $\dfrac{3x+5}{(x-1)^2(x+2)}$.
Set $\dfrac{3x+5}{(x-1)^2(x+2)} = \dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x+2}$. Clearing denominators:

$$3x+5 = A(x-1)(x+2)+B(x+2)+C(x-1)^2.$$

Let $x=1$: $8=3B \Rightarrow B=\frac{8}{3}$. Let $x=-2$: $-1=9C \Rightarrow C=-\frac{1}{9}$.
Matching x^2 coefficients: $0=A+C \Rightarrow A=\frac{1}{9}$. So

$$\frac{3x+5}{(x-1)^2(x+2)} = \frac{1/9}{x-1}+\frac{8/3}{(x-1)^2}-\frac{1/9}{x+2}.$$

Slant asymptotes and crossing the horizontal asymptote

Slant asymptote occurs exactly when $\deg N = \deg D + 1$: polynomial-divide N by D; the quotient (a line $y = mx+b$) is the slant asymptote and the remainder term $\to 0$ as $x \to \pm\infty$.
Crossing the horizontal asymptote: an asymptote controls only *end* behavior — a graph *may* cross its horizontal (or slant) asymptote at finite x. To find where, set $f(x)$ equal to the asymptote value and solve; real solutions are the crossing points. (A graph can *never* cross a *vertical* asymptote, but horizontal/slant ones are fair game.)

Where a graph meets its horizontal asymptote

For $f(x) = \dfrac{x^2-3x+2}{x^2+1}$, degrees are equal, so the horizontal asymptote is $y = \dfrac{1}{1} = 1$. Does the graph cross it? Set $f(x)=1$:

$$\frac{x^2-3x+2}{x^2+1} = 1 \implies x^2-3x+2 = x^2+1 \implies -3x+2=1 \implies x=\tfrac{1}{3}.$$

So the graph *does* cross $y=1$, at the single point $\left(\frac{1}{3}, 1\right)$. The curve approaches $y=1$ at the far ends but dips across it near the origin. ✓

Tip: "Asymptote" describes behavior only as $x \to \pm\infty$ (horizontal/slant) or as $x \to$ a forbidden value (vertical). Crossings of a horizontal or slant asymptote are perfectly normal and often happen exactly once — don't assume a graph stays on one side of it.

6 Exponential and Logarithmic Functions

Graphs, properties, equations, and modeling — everything on one reference

6.1 Exponential Functions and Their Graphs

Definition and key features

An **exponential function** has the form

$$f(x) = a\,b^x, \qquad a \neq 0, \quad b > 0, \quad b \neq 1.$$

The number b is the **base**. For the parent function $f(x) = b^x$:

- Domain: all real numbers, $(-\infty, \infty)$. Range: $(0, \infty)$.
- **Horizontal asymptote** $y = 0$ (the x-axis).
- y-intercept $(0, 1)$, since $b^0 = 1$; no x-intercept.
- If $b > 1$ the graph *grows*; if $0 < b < 1$ it *decays*. Note $b^{-x} = (1/b)^x$.

Transformations of $f(x) = 2^x$

Describe $g(x) = 2^{x-1} + 3$ and give its asymptote.

- $x - 1$: shift the graph of 2^x **right** 1.
- $+3$: shift **up** 3, which lifts the asymptote from $y = 0$ to $\boxed{y = 3}$.

y-intercept: $g(0) = 2^{-1} + 3 = \frac{1}{2} + 3 = \frac{7}{2}$, so $\left(0, \frac{7}{2}\right)$. Range: $(3, \infty)$.

Tip. Read a transformation from the outside in: horizontal shifts/reflections act on x *inside* the exponent; vertical shifts/reflections and the asymptote come from what happens *outside*.

6.2 The Natural Base e

The number e

The **natural base** is the irrational constant

$$e = \lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n \approx 2.71828.$$

The **natural exponential function** is $f(x) = e^x$; it behaves like any base $b > 1$ (asymptote $y = 0$, intercept $(0, 1)$) and is the base used for *continuous* growth and decay. Any base can be rewritten with e: $b^x = e^{(\ln b)x}$.

Rewriting a base with e

Write $f(x) = 5 \cdot 2^x$ in the form $a\,e^{kx}$.

$$2^x = e^{(\ln 2)x} \;\Rightarrow\; f(x) = 5\,e^{(\ln 2)x} \approx 5\,e^{0.693x}.$$

Because $\ln 2 > 0$, this confirms f is a growth function.

Tip. e is just a specific base between 2 and 3, so $y = e^x$ sits between $y = 2^x$ and $y = 3^x$. Reach for e whenever growth or decay is *continuous*.

6.3 Logarithmic Functions and Their Graphs

Definition, common and natural logs

For $b > 0,\ b \neq 1$, the **logarithm** is the inverse of the exponential:

$$y = \log_b x \iff b^y = x.$$

- **Common log:** $\log x = \log_{10} x$. **Natural log:** $\ln x = \log_e x$.
- **Domain** of $\log_b x$ is $(0, \infty)$: you may only take a log of a *positive* number.
- Range: $(-\infty, \infty)$. **Vertical asymptote** $x = 0$; x-intercept $(1, 0)$.
- Inverse identities: $b^{\log_b x} = x$ and $\log_b b^x = x$.

For a shifted log $\log_b(x - h)$, the domain is $x > h$ and the asymptote moves to $x = h$.

Forms and domain

(a) Convert $\log_2 32 = 5$ to exponential form: $2^5 = 32$.
(b) Convert $3^4 = 81$ to log form: $\log_3 81 = 4$.

(c) Domain of $f(x) = \ln(2x-6)$: require $2x - 6 > 0 \Rightarrow x > 3$, so $(3, \infty)$; asymptote $x = 3$.
(d) Evaluate $\log_2 \frac{1}{8} = \log_2 2^{-3} = -3$ and $\ln e^5 = 5$.

Tip. A log graph is the reflection of the matching exponential graph across $y = x$: the exponential's asymptote $y = 0$ and intercept $(0, 1)$ become the log's asymptote $x = 0$ and intercept $(1, 0)$.

6.4 Properties of Logarithms and Change of Base

Log laws (all bases $b > 0,\ b \neq 1$; $M, N > 0$)

$$\log_b(MN) = \log_b M + \log_b N \qquad \log_b\left(\tfrac{M}{N}\right) = \log_b M - \log_b N$$
$$\log_b(M^p) = p \log_b M \qquad \log_b 1 = 0 \qquad \log_b b = 1$$

Change of base: $\log_b M = \dfrac{\log M}{\log b} = \dfrac{\ln M}{\ln b}$.

Expanding a logarithm

Expand $\log_2 \dfrac{8x^3}{\sqrt{y}}$ completely.

$$\begin{aligned} \log_2 \frac{8x^3}{\sqrt{y}} &= \log_2(8x^3) - \log_2 y^{1/2} && \text{quotient rule} \\ &= \log_2 8 + \log_2 x^3 - \tfrac{1}{2}\log_2 y && \text{product + power} \\ &= 3 + 3\log_2 x - \tfrac{1}{2}\log_2 y. && \log_2 8 = 3 \end{aligned}$$

Condensing to a single logarithm

Condense $3\ln x - \frac{1}{2}\ln y + \ln z$.

$$\begin{aligned} 3\ln x - \tfrac{1}{2}\ln y + \ln z &= \ln x^3 - \ln y^{1/2} + \ln z && \text{power rule} \\ &= \ln \frac{x^3 z}{\sqrt{y}}. && \text{combine products/quotients} \end{aligned}$$

Tip. There is *no* rule for $\log_b(M + N)$ or $\log_b M \cdot \log_b N$. The laws only convert products, quotients, and powers *inside* a single log into sums, differences, and coefficients.

6.5 Exponential and Logarithmic Equations

Solving strategy

- **Exponential:** isolate the power, then take a log of both sides (use ln, or $\log_b$ directly). $b^x = c \Rightarrow x = \log_b c = \dfrac{\ln c}{\ln b}$.
- **Logarithmic:** condense to one log, rewrite in exponential form, then solve.
- **Always check the domain:** any solution making an argument ≤ 0 inside a log is **extraneous** and must be rejected.
- Give an **exact** answer (logs) and, if asked, an **approximate** decimal.

Exponential equation (exact and approximate)

Solve $3^{x+1} = 17$.

$$\begin{aligned} \ln 3^{x+1} &= \ln 17 \\ (x+1)\ln 3 &= \ln 17 \\ x+1 &= \frac{\ln 17}{\ln 3} \\ x &= \frac{\ln 17}{\ln 3} - 1 \approx 2.579 - 1 = \mathbf{1.579}. \end{aligned}$$

Logarithmic equation with an extraneous solution

Solve $\log_3 x + \log_3(x-6) = 3$.

$$\begin{aligned} \log_3\big(x(x-6)\big) &= 3 && \text{product rule} \\ x(x-6) &= 3^3 = 27 && \text{exponential form} \\ x^2 - 6x - 27 = 0 &\;\Rightarrow\; (x-9)(x+3) = 0 \\ x = 9 \quad &\text{or} \quad x = -3. \end{aligned}$$

Domain needs $x > 6$, so $x = -3$ is **extraneous**. Solution: $\boxed{x = 9}$.

Tip. A negative answer is not automatically extraneous — test it in the *original* equation. Only reject values that make some log's argument zero or negative.

6.6 Modeling with Exponential and Logarithmic Functions

Standard models

Growth/decay: $A = A_0 e^{kt}$ ($k > 0$ grows, $k < 0$ decays) **Half-life:** $A = A_0 \left(\frac{1}{2}\right)^{t/h}$

Compound: $A = P\left(1 + \frac{r}{n}\right)^{nt}$ **Continuous:** $A = Pe^{rt}$

Logistic: $P(t) = \dfrac{c}{1 + ae^{-bt}}$ (limit c) **Newton cooling:** $T = T_s + (T_0 - T_s)e^{-kt}$

Richter: $M = \log \dfrac{I}{I_0}$ **pH:** $\text{pH} = -\log[\text{H}^+]$

Compound vs. continuous interest

Invest \$3000 at 4.5% for 6 years.

$$\text{Continuous: } A = 3000\, e^{(0.045)(6)} = 3000\, e^{0.27} \approx \$\mathbf{3929.89}.$$

$$\text{Quarterly: } A = 3000\left(1 + \tfrac{0.045}{4}\right)^{4\cdot 6} = 3000(1.01125)^{24} \approx \$3924.01.$$

Continuous compounding earns slightly more.

Half-life

An isotope has half-life 12 years; a sample starts at 50 g.

$$A = 50\left(\tfrac{1}{2}\right)^{t/12}.$$

After 30 years: $A = 50\left(\frac{1}{2}\right)^{30/12} = 50\left(\frac{1}{2}\right)^{2.5} \approx \mathbf{8.84}$ g.

Tip. The Richter and pH scales are *logarithmic*: each whole-number step is a factor of 10. Two quakes differing by 2 magnitudes differ in intensity by $10^2 = 100$ times.

6.7 Going Deeper: Advanced Exponential & Log Ideas

Change-of-base chains and telescoping products

Change of base can be strung into a **chain**. Because

$$\log_b M = \frac{\ln M}{\ln b},$$

a product of logs whose bases and arguments interlock *telescopes*:

$$\log_a b \cdot \log_b c \cdot \log_c d = \frac{\ln b}{\ln a} \cdot \frac{\ln c}{\ln b} \cdot \frac{\ln d}{\ln c} = \frac{\ln d}{\ln a} = \log_a d.$$

Two consequences worth memorizing:

- **Reciprocal rule:** $\log_a b = \dfrac{1}{\log_b a}$, since $\log_a b \cdot \log_b a = 1$.
- **Swap-the-power rule:** $a^{\log_b c} = c^{\log_b a}$ (take ln of both sides to verify).

Telescoping a product of logarithms

Evaluate $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

$$\begin{aligned}
\prod &= \frac{\ln 3}{\ln 2}\cdot\frac{\ln 4}{\ln 3}\cdot\frac{\ln 5}{\ln 4}\cdot\frac{\ln 6}{\ln 5}\cdot\frac{\ln 7}{\ln 6}\cdot\frac{\ln 8}{\ln 7} && \text{change each to ln}\\
&= \frac{\ln 8}{\ln 2} = \log_2 8 = 3. && \text{everything cancels}
\end{aligned}$$

The same collapse turns any sum $\sum_{k=2}^{n} \log_k(k+1)$-style *product* into $\log_2(n+1)$ when written multiplicatively.

Telescoping sums of logs

A **sum** of logs telescopes through the quotient rule rather than change of base:

$$\sum_{k=1}^{n} \log_b \frac{k+1}{k} = \log_b\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right) = \log_b(n+1).$$

Likewise $\displaystyle\sum_{k=2}^{n} \log_b\left(1-\tfrac{1}{k^2}\right) = \log_b\frac{n+1}{2n}$, because $1-\frac{1}{k^2} = \frac{(k-1)(k+1)}{k\cdot k}$ collapses as a product. The trick: turn each term into a *ratio*, multiply, and watch the middle die.

Expressing logs in terms of given letters

If you are told, e.g., $\log_2 3 = a$ and $\log_2 5 = b$, then *every* log whose argument factors over $\{2,3,5\}$ can be written in a, b using the log laws — and change of base for a different outer base.

- Factor the argument into powers of the "known" primes.
- Apply product/quotient/power rules to split it.
- $\log_2 2 = 1$ supplies any factor of 2 for free.

For a different base, e.g. $\log_5(\cdot)$, first change to base 2: $\log_5 x = \dfrac{\log_2 x}{\log_2 5} = \dfrac{\log_2 x}{b}$.

“$\log_2 3 = a,\ \log_2 5 = b$ — express $\log_2 7.5$ and $\log_6 20$”

(a) $7.5 = \frac{15}{2} = \frac{3\cdot 5}{2}$, so

$$\log_2 7.5 = \log_2 3 + \log_2 5 - \log_2 2 = a + b - 1.$$

(b) $\log_6 20 = \dfrac{\log_2 20}{\log_2 6}$. Now $20 = 2^2 \cdot 5$ and $6 = 2 \cdot 3$:

$$\begin{aligned} \log_2 20 &= 2\log_2 2 + \log_2 5 = 2 + b, \\ \log_2 6 &= \log_2 2 + \log_2 3 = 1 + a, \\ \Rightarrow \quad \log_6 20 &= \frac{2+b}{1+a}. \end{aligned}$$

Exponential equations that hide a quadratic

When an equation contains a base raised to x *and* to $2x$ (or $-x$), substitute $u = b^x$ (with $u > 0$) to expose a quadratic:

$$b^{2x} = (b^x)^2 = u^2, \qquad b^{-x} = \frac{1}{u}.$$

Solve for u, discard any $u \le 0$ (an exponential is never negative), then recover $x = \log_b u$. Equations mixing e^x and e^{-x} clear to a quadratic after multiplying through by e^x.

Reducing an exponential equation to a quadratic

Solve $e^{2x} - 5e^x + 6 = 0$.

$$\begin{aligned} u = e^x\ (u>0):\quad u^2 - 5u + 6 &= 0 && \text{substitute} \\ (u-2)(u-3) = 0 \ \Rightarrow\ u &= 2 \text{ or } u = 3 && \text{both positive, keep} \\ e^x = 2 \ \Rightarrow\ x = \ln 2, \qquad e^x &= 3 \ \Rightarrow\ x = \ln 3. \end{aligned}$$

Contrast $e^x - 6 + 8e^{-x} = 0$: multiply by e^x to get $u^2 - 6u + 8 = 0$, giving $u = 2, 4$, so $x = \ln 2,\ \ln 4$.

The $x^{\log x}$ type: take logs twice in spirit

Solve $x^{\log_{10} x} = 1000\,x^2$ for $x > 0$.

$$\begin{aligned}
\log\left(x^{\log x}\right) &= \log\left(1000\,x^2\right) && \text{take } \log_{10} \\
(\log x)(\log x) &= 3 + 2\log x && \log x^p = p\log x \\
t^2 - 2t - 3 = 0, &\quad t = \log x && \text{let } t = \log x \\
(t-3)(t+1) = 0 &\;\Rightarrow\; t = 3 \text{ or } t = -1 \\
\log x = 3 \;\Rightarrow\; x = 1000, &\qquad \log x = -1 \;\Rightarrow\; x = \tfrac{1}{10}.
\end{aligned}$$

Key move: an unknown appearing in *both* base and exponent means **take a log first**, then treat $\log x$ as the variable.

Systems of logarithmic equations

Treat the logs as unknowns. A system such as

$$\log_2 x + \log_2 y = 5, \qquad \log_2 x - \log_2 y = 1$$

is *linear* in $u = \log_2 x$ and $v = \log_2 y$: add/subtract to get $u = 3,\ v = 2$, hence $x = 8,\ y = 4$. When the equations mix a log-sum with an ordinary equation (e.g. $x + y = \text{const}$), condense the logs into one, convert to exponential form to get a product xy, and solve the resulting system by substitution. **Always** re-check each solution against every domain restriction.

A mixed log system

Solve $\log(x) + \log(y) = 2$ and $x - y = 15$ (common logs, so $x, y > 0$).

$$\begin{aligned}
\log(xy) = 2 \;\Rightarrow\; xy = 10^2 &= 100 && \text{condense + exponential form} \\
x &= y + 15 && \text{from the second equation} \\
(y+15)y = 100 &\;\Rightarrow\; y^2 + 15y - 100 = 0 \\
(y+20)(y-5) = 0 &\;\Rightarrow\; y = 5 \;\; (\text{reject } y = -20).
\end{aligned}$$

Then $x = 20$. Check: both positive, $\log 20 + \log 5 = \log 100 = 2$. $\boxed{x = 20,\ y = 5}$

Logarithmic and exponential inequalities

Solve the corresponding *equation* to find boundary points, then respect monotonicity and the domain:

- $\log_b$ with $b > 1$ is **increasing**: $\log_b u < \log_b v \iff 0 < u < v$. Never forget $u > 0$.
- $\log_b$ with $0 < b < 1$ is **decreasing**: the inequality *flips*.
- $b^x < c$ with $b > 1$: take $\log_b$ to get $x < \log_b c$ (flip if $0 < b < 1$).

The domain condition is part of the answer — intersect it with the inequality's solution.

Tip. For $\log_b u \le k$ rewrite $k = \log_b b^k$ so both sides are logs, then compare arguments *and* impose $u > 0$. Example: $\log_2(x-1) \le 3 \iff 0 < x-1 \le 2^3 \iff 1 < x \le 9$.

Continuous growth versus logistic (bounded) growth

Pure continuous growth $A = A_0 e^{kt}$ is unbounded, so it only models the *early* phase of real populations. The **logistic model**

$$P(t) = \frac{c}{1 + ae^{-bt}} \qquad (a, b, c > 0)$$

adds a ceiling: $P(0) = \dfrac{c}{1+a}$, and $P(t) \to c$ (the **carrying capacity**) as $t \to \infty$. Growth is nearly exponential while $P \ll c$, fastest at $P = \frac{c}{2}$ (the inflection point), then levels off. To fit or solve, isolate the exponential:

$$e^{-bt} = \frac{c-P}{aP} \;\Rightarrow\; t = -\frac{1}{b}\ln\frac{c-P}{aP}.$$

Reading a logistic model

A rumor spreads as $P(t) = \dfrac{1000}{1 + 49e^{-0.4t}}$ people after t days.

$$\begin{aligned}
P(0) &= \frac{1000}{1+49} = 20 && \text{initial spreaders}\\
\lim_{t\to\infty} P(t) &= 1000 && \text{carrying capacity } c\\
P(t) = 500:\quad 1 + 49e^{-0.4t} &= 2 \;\Rightarrow\; e^{-0.4t} = \tfrac{1}{49} &&\\
t &= \frac{\ln 49}{0.4} \approx 9.7 \text{ days} && \text{half-saturation / inflection}
\end{aligned}$$

So the rumor spreads fastest around day 10, then saturates near 1000.

Tip. Spot the model by its ceiling: data that keeps accelerating fits $A_0 e^{kt}$; data that accelerates then flattens toward a limit fits the logistic $\dfrac{c}{1+ae^{-bt}}$. In the logistic, c is the horizontal asymptote and $\frac{c}{1+a}$ is the y-intercept.

7 Trigonometric Functions

Angles, the unit circle, graphs, and inverse trig functions

7.1 Angles: Degree and Radian Measure

Measuring Angles

An angle in **standard position** has its vertex at the origin and initial side on the positive x-axis. A positive angle rotates counterclockwise; a negative angle rotates clockwise.

- **Degrees:** one full revolution $= 360°$.
- **Radians:** one full revolution $= 2\pi$. One radian is the central angle subtending an arc equal in length to the radius.

Conversions: $\text{radians} = \text{degrees} \times \dfrac{\pi}{180°}, \qquad \text{degrees} = \text{radians} \times \dfrac{180°}{\pi}.$

An angle in standard position.

(diagram in the online version)

Converting between degrees and radians

Convert 150° to radians: $150° \cdot \dfrac{\pi}{180°} = \dfrac{150\pi}{180} = \dfrac{5\pi}{6}.$

Convert $\dfrac{7\pi}{4}$ to degrees: $\dfrac{7\pi}{4} \cdot \dfrac{180°}{\pi} = \dfrac{7 \cdot 180°}{4} = 315°.$

Coterminal, Complementary, Supplementary

- **Coterminal** angles share a terminal side: add or subtract 360° (or 2π) any whole number of times.
- **Complementary** angles sum to 90° (or $\frac{\pi}{2}$).
- **Supplementary** angles sum to 180° (or π).

Coterminal and complements

A positive coterminal angle for $-\dfrac{\pi}{3}$: $-\dfrac{\pi}{3} + 2\pi = \dfrac{5\pi}{3}.$

The complement of 40° is 50°; the supplement of 40° is 140°.

Tip: Complements and supplements only make sense for angles between the required bounds, but coterminal angles always exist. To find the smallest nonnegative coterminal angle, add/subtract full turns until you land in $[0°, 360°)$ or $[0, 2\pi)$.

7.2 Arc Length, Sector Area, and Speed

Circular Arcs and Sectors

For a circle of radius r and a central angle θ **measured in radians**:

$$\text{Arc length: } s = r\theta \qquad \text{Sector area: } A = \tfrac{1}{2}r^2\theta.$$

For an object moving along the circle:

$$\text{Angular speed: } \omega = \frac{\theta}{t} \qquad \text{Linear speed: } v = \frac{s}{t} = r\omega.$$

Arc length and sector area

A circle has radius $r = 6$ cm and central angle $\theta = \dfrac{\pi}{3}$.

$$s = r\theta = 6 \cdot \tfrac{\pi}{3} = 2\pi \text{ cm}, \qquad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(36)\tfrac{\pi}{3} = 6\pi \text{ cm}^2.$$

Angular and linear speed

A wheel of radius 0.5 m turns at 120 rev/min. In radians per second:

$$\omega = 120 \cdot 2\pi \text{ rad/min} = 240\pi \text{ rad/min} = \frac{240\pi}{60} = 4\pi \text{ rad/s}.$$

Linear speed: $v = r\omega = 0.5 \cdot 4\pi = 2\pi \approx 6.28$ m/s.

Tip: $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ require *radians*. Convert first if the angle is in degrees. One revolution $= 2\pi$ radians.

7.3 The Unit Circle and the Six Trig Functions

Definition on the Unit Circle

On the unit circle ($x^2 + y^2 = 1$), the terminal side of angle θ meets the circle at $(\cos\theta, \sin\theta)$. For any angle with terminal point (x, y) at radius $r = \sqrt{x^2 + y^2}$:

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x},$$

$$\csc\theta = \frac{r}{y}, \quad \sec\theta = \frac{r}{x}, \quad \cot\theta = \frac{x}{y}.$$

An angle in standard position.

(diagram in the online version)

Exact Values at Special Angles

θ	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

θ	sin / cos / tan
$\frac{\pi}{2}$	1/ 0/ undef.

Reading the unit circle

Find the six trig functions of $\theta = \dfrac{\pi}{4}$. The terminal point is $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$, so

$$\sin\tfrac{\pi}{4} = \cos\tfrac{\pi}{4} = \tfrac{\sqrt{2}}{2}, \quad \tan\tfrac{\pi}{4} = 1,$$

$$\csc\tfrac{\pi}{4} = \sec\tfrac{\pi}{4} = \sqrt{2}, \quad \cot\tfrac{\pi}{4} = 1.$$

Tip (“$\sqrt{n}/2$” pattern): For sin at $0, 30, 45, 60, 90$ read $\dfrac{\sqrt{0}}{2}, \dfrac{\sqrt{1}}{2}, \dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{4}}{2}$. Cosine runs the same list backward.

7.4 Right-Triangle Trig and Fundamental Identities

SOH-CAH-TOA

For an acute angle θ in a right triangle with the opposite leg, adjacent leg, and hypotenuse:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

The reciprocals are $\csc\theta, \sec\theta, \cot\theta$.

Using the triangle

For the 3-4-5 triangle above, $\sin\theta = \dfrac{3}{5}$, $\cos\theta = \dfrac{4}{5}$, $\tan\theta = \dfrac{3}{4}$, and $\sec\theta = \dfrac{5}{4}$.

Fundamental Identities

Reciprocal: $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$.

Quotient: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Pythagorean: $\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$

Applying an identity

If $\sin\theta = \dfrac{3}{5}$ and θ is in Quadrant I, then $\cos\theta = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \dfrac{4}{5}$, so $\tan\theta = \dfrac{3/5}{4/5} = \dfrac{3}{4}$.

Tip: From $\sin^2\theta + \cos^2\theta = 1$ divide by $\cos^2\theta$ to get $1 + \tan^2\theta = \sec^2\theta$; divide by $\sin^2\theta$ to get $1 + \cot^2\theta = \csc^2\theta$. You only need to memorize one.

7.5 Reference Angles and Signs by Quadrant

Reference Angle

The **reference angle** θ' is the acute angle between the terminal side and the x-axis. The value of a trig function at θ equals its value at θ' up to a sign.

Quadrant of θ	Reference angle θ'
I	θ
II	$180^\circ - \theta \quad (\pi - \theta)$
III	$\theta - 180^\circ \quad (\theta - \pi)$
IV	$360^\circ - \theta \quad (2\pi - \theta)$

An angle in standard position.

(diagram in the online version)

Signs by Quadrant ("All Students Take Calculus")

- **QI:** all positive. **QII:** only sin (and csc) positive.
- **QIII:** only tan (and cot) positive. **QIV:** only cos (and sec) positive.

Evaluating any angle

Evaluate $\cos 210^\circ$. It lies in QIII with reference angle $210^\circ - 180^\circ = 30^\circ$. Cosine is negative in QIII, so

$$\cos 210^\circ = -\cos 30^\circ = -\tfrac{\sqrt{3}}{2}.$$

A radian example

Evaluate $\sin\dfrac{5\pi}{6}$. QII, reference angle $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$. Sine is positive in QII, so $\sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$.

Tip: Steps to evaluate any angle: (1) find a coterminal angle in $[0, 2\pi)$; (2) identify the quadrant; (3) find the reference angle; (4) evaluate at the reference angle; (5) attach the correct sign.

7.6 Graphs of Sine and Cosine

Sinusoidal Form

For $y = a\sin\big(b(x-c)\big) + d$ (and likewise for cosine) with $b > 0$:

- **Amplitude** $= |a|$ **Period** $= \dfrac{2\pi}{b}$
- **Phase shift** $= c$ (right if $c > 0$) **Vertical shift** $= d$ (midline $y = d$)

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

$y = 2\sin x$ (blue) and $y = 2\cos x$ (red): amplitude 2, period 2π.

Write and read a graph

For $y = 3\sin\big(2(x - \frac{\pi}{4})\big) + 1$: amplitude $= 3$, period $= \dfrac{2\pi}{2} = \pi$, phase shift $= \dfrac{\pi}{4}$ right, midline $y = 1$. The graph oscillates between $y = -2$ and $y = 4$.

Tip: Always factor out b before reading the phase shift: $y = \sin(2x - \pi) = \sin\big(2(x - \frac{\pi}{2})\big)$ shifts right $\frac{\pi}{2}$, *not* π.

7.7 Graphs of Tangent, Cotangent, Secant, Cosecant

Asymptotes and Periods

- $y = \tan x$: period π; vertical asymptotes where $\cos x = 0$, i.e. $x = \frac{\pi}{2} + n\pi$.
- $y = \cot x$: period π; asymptotes where $\sin x = 0$, i.e. $x = n\pi$.
- $y = \sec x = \dfrac{1}{\cos x}$: asymptotes where $\cos x = 0$; period 2π.

- $y = \csc x = \dfrac{1}{\sin x}$: asymptotes where $\sin x = 0$; period 2π.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

$y = \tan x$: period π, asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$.

Secant from cosine

Because $\sec x = \dfrac{1}{\cos x}$, the secant curve has a U-shaped branch opening *up* wherever $\cos x > 0$ (minimum value 1) and opening *down* wherever $\cos x < 0$ (maximum value -1), with asymptotes at each zero of cosine.

Tip: Sketch sec/csc by first lightly drawing the matching cosine/sine wave: each hill becomes a U opening away from the midline, and each zero becomes an asymptote.

7.8 Inverse Trigonometric Functions

Definitions, Domains, and Ranges

To be invertible, each trig function is restricted:

Function	Domain	Range
$\arcsin x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\arcsin x$ is the *angle* in the listed range whose sine is x (similarly for the others).

Evaluating inverse trig

$\arcsin\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\pi}{4}$ (since $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{4}$ is in range).

$\arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$ (the angle in $[0, \pi]$ with cosine $-\frac{1}{2}$).

$\arctan(-1) = -\dfrac{\pi}{4}$.

Composition $\sin(\arccos x)$

Let $\theta = \arccos x$, so $\cos\theta = x$ with $\theta \in [0, \pi]$ (where $\sin\theta \geq 0$). Then

$$\sin(\arccos x) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - x^2}.$$

For a numeric case: $\tan\big(\arcsin \frac{3}{5}\big)$: with $\sin\theta = \frac{3}{5}$ (QI), $\cos\theta = \frac{4}{5}$, so the value is $\frac{3}{4}$.

Tip: For compositions, draw a right triangle with the inner function's ratio, find the missing side by the Pythagorean theorem, then read off the outer function. Watch the range: arccos outputs QI or QII, arcsin and arctan output QI or QIV.

7.9 Going Deeper: Advanced Trig Ideas

Product-to-Sum and Sum-to-Product

These identities trade a product for a sum (easier to integrate or evaluate) and vice versa:

$$\begin{aligned}
\cos A \cos B &= \tfrac{1}{2}[\cos(A - B) + \cos(A + B)] \\
\sin A \sin B &= \tfrac{1}{2}[\cos(A - B) - \cos(A + B)] \\
\sin A \cos B &= \tfrac{1}{2}[\sin(A + B) + \sin(A - B)]
\end{aligned}$$

Running them backward gives sum-to-product, e.g. $\sin A + \sin B = 2 \sin\dfrac{A+B}{2} \cos\dfrac{A-B}{2}$.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

An unusual exact product: $\cos 20^\circ \cos 40^\circ \cos 80^\circ$

Let $P = \cos 20^\circ \cos 40^\circ \cos 80^\circ$. Multiply and divide by $\sin 20^\circ$ and use $\sin 2\theta = 2\sin\theta\cos\theta$ repeatedly:

$$\begin{aligned}
P &= \frac{\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 80^\circ}{\sin 20^\circ} = \frac{\frac{1}{2}\sin 40^\circ \cos 40^\circ \cos 80^\circ}{\sin 20^\circ}. \\
&= \frac{\frac{1}{4}\sin 80^\circ \cos 80^\circ}{\sin 20^\circ} = \frac{\frac{1}{8}\sin 160^\circ}{\sin 20^\circ} = \frac{\frac{1}{8}\sin 20^\circ}{\sin 20^\circ} = \frac{1}{8},
\end{aligned}$$

using $\sin 160^\circ = \sin(180^\circ - 20^\circ) = \sin 20^\circ$. So the exact value is $\dfrac{1}{8}$.

An unusual exact sum: $\cos\dfrac{\pi}{7} + \cos\dfrac{3\pi}{7} + \cos\dfrac{5\pi}{7}$

Call the sum S. Multiply by $2\sin\dfrac{\pi}{7}$ and apply $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ to each term; the pieces telescope:

$$2\sin\tfrac{\pi}{7}\,S = \sin\tfrac{2\pi}{7} + \left(\sin\tfrac{4\pi}{7} - \sin\tfrac{2\pi}{7}\right) + \left(\sin\tfrac{6\pi}{7} - \sin\tfrac{4\pi}{7}\right) = \sin\tfrac{6\pi}{7}.$$

Since $\sin\frac{6\pi}{7} = \sin\frac{\pi}{7}$, we get $2\sin\frac{\pi}{7}\,S = \sin\frac{\pi}{7}$, hence $S = \dfrac{1}{2}$.

Building a Sinusoid from Data

Given a periodic phenomenon with maximum value M, minimum value m, and known timing, model it as $y = a\cos\big(b(t-c)\big) + d$ (or with sin):

$$d = \frac{M+m}{2}\ \text{(midline)}, \qquad a = \frac{M-m}{2}\ \text{(amplitude)}, \qquad b = \frac{2\pi}{\text{period}}.$$

Choose cos with $c =$ (time of the maximum) so no reflection is needed; the midline crossings occur a quarter period from each extreme.

Full sinusoid from tide data

High tide of 12 ft occurs at $t = 3$ h; the next low tide of 2 ft at $t = 9$ h. Find a model $y(t)$ and the height at $t = 0$.

- Midline $d = \dfrac{12+2}{2} = 7$; amplitude $a = \dfrac{12-2}{2} = 5$.
- High-to-low is half a period, so period $= 2(9-3) = 12$ h and $b = \dfrac{2\pi}{12} = \dfrac{\pi}{6}$.
- Maximum at $t = 3 \Rightarrow c = 3$, so $y(t) = 5\cos\left(\dfrac{\pi}{6}(t-3)\right) + 7$.

At $t = 0$: $y(0) = 5\cos\left(-\frac{\pi}{2}\right) + 7 = 5(0) + 7 = 7$ ft (a midline crossing, as expected a quarter period before the peak).

Inverse Composition and Range Traps

$f^{-1}(f(x))$ returns x *only* when x already lies in the restricted range of the inverse. Otherwise you must replace the inner angle with the coterminal/reference angle that *does* lie in range but has the same sine (or cosine, tangent):

$$\arcsin(\sin x) = x \iff x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], \qquad \arccos(\cos x) = x \iff x \in [0,\pi].$$

Going the other way, $\sin(\arcsin u) = u$ and $\cos(\arccos u) = u$ hold for *all* $u \in [-1,1]$ with no trap.

Springing the range trap

Evaluate $\arcsin\left(\sin\dfrac{5\pi}{6}\right)$. Here $\frac{5\pi}{6}\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so the answer is *not* $\frac{5\pi}{6}$. Since $\sin\frac{5\pi}{6}=\frac{1}{2}$ and the in-range angle with sine $\frac{1}{2}$ is $\frac{\pi}{6}$,

$$\arcsin\left(\sin\tfrac{5\pi}{6}\right)=\frac{\pi}{6}.$$

Similarly $\arccos\left(\cos\dfrac{7\pi}{4}\right)$: cosine $=\frac{\sqrt{2}}{2}$, and the angle in $[0,\pi]$ with that cosine is $\frac{\pi}{4}$, so the value is $\dfrac{\pi}{4}$ (not $\frac{7\pi}{4}$).

Linked Gears and Angular-Speed Chains

Where two gears (or pulleys/wheels) mesh, the **contact point shares one linear speed**: $v=r_1\omega_1=r_2\omega_2$. Hence

$$\frac{\omega_1}{\omega_2}=\frac{r_2}{r_1}=\frac{N_2}{N_1},$$

where N_i is the tooth count. Wheels rigidly on the *same axle* share ω, not v. Chain these relations to pass speed through a train.

Angular-speed chain

Gear A ($r_A=8$ cm) turns at 60 rev/min and meshes with gear B ($r_B=2$ cm). Gear B is fixed on the same axle as gear C ($r_C=5$ cm). Find the linear speed at the rim of C.

- $\omega_A=60\cdot 2\pi=120\pi$ rad/min.
- Mesh A–B: $r_A\omega_A=r_B\omega_B\Rightarrow\omega_B=\dfrac{r_A}{r_B}\omega_A=\dfrac{8}{2}(120\pi)=480\pi$ rad/min.
- Same axle B–C: $\omega_C=\omega_B=480\pi$ rad/min.
- Rim of C: $v_C=r_C\omega_C=5\cdot 480\pi=2400\pi\approx 7540$ cm/min.

Counting Solutions in $[0,2\pi)$

To count solutions of a trig equation on one period, reduce it to $F(u)=k$ where $u=n\theta$:

- A single $\sin\theta=k$ or $\cos\theta=k$ has **2** solutions in $[0,2\pi)$ when $|k|<1$, **1** when $|k|=1$, **0** when $|k|>1$.
- If the argument is $n\theta$, then $\theta\in[0,2\pi)$ makes $n\theta$ sweep $[0,2n\pi)$, i.e. n full periods, so multiply the base count by n.
- Factor first: a product $=0$ contributes the (deduplicated) union of each factor's solutions.

How many solutions?

How many $\theta \in [0, 2\pi)$ satisfy $2\sin(3\theta) = 1$, i.e. $\sin(3\theta) = \frac{1}{2}$?

Let $u = 3\theta$. As θ runs over $[0, 2\pi)$, u runs over $[0, 6\pi)$ — three full periods. In each period $\sin u = \frac{1}{2}$ has 2 solutions, so there are $3 \times 2 = \boxed{6}$ solutions.

Contrast: $\cos^2\theta - \cos\theta = 0$ factors as $\cos\theta(\cos\theta - 1) = 0$. Then $\cos\theta = 0$ gives 2 solutions $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\cos\theta = 1$ gives 1 solution (0); no overlap, so 3 solutions total.

Tip (Pythagorean simplifying): When an expression mixes $\sin^2/\cos^2$ or $\sec^2/\tan^2$, look to collapse it with $\sin^2\theta + \cos^2\theta = 1$ or $\sec^2\theta - \tan^2\theta = 1$. For instance

$$\frac{1 - \cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta, \qquad \sec^4\theta - \tan^4\theta = (\sec^2\theta + \tan^2\theta)(\sec^2\theta - \tan^2\theta) = \sec^2\theta + \tan^2\theta.$$

Spotting the identity turns a messy fraction or difference of squares into a single term.

8 Analytic Trigonometry

Identities, verifications, trig equations, and the sum/difference & multiple-angle formulas

8.1 Fundamental Identities & Simplifying

The identities you build everything from

Reciprocal: $\csc\theta = \dfrac{1}{\sin\theta}, \quad \sec\theta = \dfrac{1}{\cos\theta}, \quad \cot\theta = \dfrac{1}{\tan\theta}.$

Quotient: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}, \quad \cot\theta = \dfrac{\cos\theta}{\sin\theta}.$

Pythagorean: $\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta.$

Even/Odd: $\sin(-\theta) = -\sin\theta, \quad \cos(-\theta) = \cos\theta, \quad \tan(-\theta) = -\tan\theta.$

Cofunction: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta.$

Simplify to one function: $\dfrac{\sec\theta - \cos\theta}{\tan\theta}$

$$\begin{aligned}\frac{\sec\theta - \cos\theta}{\tan\theta} &= \frac{\frac{1}{\cos\theta} - \cos\theta}{\frac{\sin\theta}{\cos\theta}} = \frac{\frac{1-\cos^2\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}} \\ &= \frac{1 - \cos^2\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta.\end{aligned}$$

Rewrite in terms of $\sin\theta$ only: $\dfrac{\cos^2\theta}{1-\sin\theta}$

$$\begin{aligned}\frac{\cos^2\theta}{1-\sin\theta} &= \frac{1-\sin^2\theta}{1-\sin\theta} = \frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta}\\ &= 1+\sin\theta.\end{aligned}$$

Tip. To "rewrite in terms of one function," turn everything into sin and cos first, combine over a common denominator, then use $\sin^2\theta+\cos^2\theta=1$ to collapse the result.

8.2 Verifying Identities

Strategy for a proof

Work **one side only** (usually the messier one) until it matches the other. Do *not* move terms across the $=$ sign as if solving an equation. Useful moves:

- Convert to sin and cos.
- Get a common denominator.
- Use a Pythagorean identity, or multiply by a conjugate such as $\dfrac{1+\sin\theta}{1+\sin\theta}$.
- Factor, or split a fraction into separate terms.

Verify $\tan\theta\sin\theta+\cos\theta=\sec\theta$

$$\begin{aligned}\tan\theta\sin\theta+\cos\theta &= \frac{\sin\theta}{\cos\theta}\cdot\sin\theta+\cos\theta\\ &= \frac{\sin^2\theta}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}\\ &= \frac{\sin^2\theta+\cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta. \quad \checkmark\end{aligned}$$

Verify $\dfrac{\cos\theta}{1-\sin\theta}=\dfrac{1+\sin\theta}{\cos\theta}$ **(conjugate trick)**

Work the left side and multiply by $\dfrac{1+\sin\theta}{1+\sin\theta}$:

$$\frac{\cos\theta}{1-\sin\theta} = \frac{\cos\theta}{1-\sin\theta}\cdot\frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$
$$= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta} = \frac{1+\sin\theta}{\cos\theta}. \quad \checkmark$$

Tip. When a proof stalls, look at the *other* side for a clue about what form you are aiming for (a single fraction? a difference of squares?), then steer toward it.

8.3 Solving Trigonometric Equations

Method

- Isolate a single trig function, or move all terms to one side and **factor**.
- **Quadratic type** ($a\,u^2 + b\,u + c = 0$ with $u = \sin x$, etc.): factor or use the quadratic formula.
- Use identities to reduce to *one* function before factoring.
- **On** $[0, 2\pi)$**:** list only the angles in that interval. **All solutions:** add the period: $+2\pi n$ for $\sin, \cos$; $+\pi n$ for tan (here n is any integer).
- If you squared or divided, **check for extraneous solutions** in the original equation.

Solve $2\cos^2 x - \cos x - 1 = 0$ (quadratic type)

Let $u = \cos x$: $2u^2 - u - 1 = (2u+1)(u-1) = 0$, so $\cos x = -\frac{1}{2}$ or $\cos x = 1$.

$$\cos x = -\tfrac{1}{2}: \quad x = \tfrac{2\pi}{3},\ \tfrac{4\pi}{3}$$
$$\cos x = 1: \quad x = 0$$

On $[0, 2\pi)$**:** $x = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$. **All solutions:** $x = 2\pi n,\ \frac{2\pi}{3} + 2\pi n,\ \frac{4\pi}{3} + 2\pi n$.

Tip. $\cos x = \frac{1}{2}$ has *two* solutions per period (Quadrants I and IV); $\sin x = \frac{1}{2}$ has two as well (I and II). Never drop a quadrant.

8.4 Sum and Difference Formulas

Formulas

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Watch the sign flip: cos and tan use the *opposite* sign of the one in $(A \pm B)$.

Exact value of a non-special angle: $\cos 15°$

Write $15° = 45° - 30°$:

$$\begin{aligned}\cos 15° &= \cos(45° - 30°)\\ &= \cos 45° \cos 30° + \sin 45° \sin 30°\\ &= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

Likewise $\sin 75° = \sin(45° + 30°) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

Tip. Any multiple-of-15° angle can be split into two special angles: $15 = 45-30$, $75 = 45+30$, $105 = 60 + 45$. For tan, rationalize the denominator to finish.

8.5 Double-Angle, Half-Angle & Power-Reducing

Formulas

Double-angle:

$$\sin 2\theta = 2 \sin\theta \cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Power-reducing: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$.

Half-angle: $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$, $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$, $\tan\dfrac{\theta}{2} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$.

The $\pm$ is chosen by the quadrant of $\theta/2$.

Half-angle exact value: $\sin\dfrac{\pi}{12}$

Since $\dfrac{\pi}{12} = \dfrac{1}{2} \cdot \dfrac{\pi}{6}$ and $\dfrac{\pi}{12}$ is in Quadrant I (take +):

$$\begin{aligned}\sin\frac{\pi}{12} &= \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \\ &= \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}.\end{aligned}$$

Tip. $\cos 2\theta$ has three forms; pick the one that matches what you already know. Given $\sin\theta$, use $1 - 2\sin^2\theta$; given $\cos\theta$, use $2\cos^2\theta - 1$.

8.6 Product-to-Sum & Sum-to-Product

Formulas

Product-to-sum:

$$\begin{aligned}\sin A\cos B &= \tfrac{1}{2}\big[\sin(A+B) + \sin(A-B)\big] \\ \cos A\cos B &= \tfrac{1}{2}\big[\cos(A-B) + \cos(A+B)\big] \\ \sin A\sin B &= \tfrac{1}{2}\big[\cos(A-B) - \cos(A+B)\big]\end{aligned}$$

Sum-to-product:

$$\begin{aligned}\sin A + \sin B &= 2\sin\tfrac{A+B}{2}\cos\tfrac{A-B}{2} \\ \cos A + \cos B &= 2\cos\tfrac{A+B}{2}\cos\tfrac{A-B}{2} \\ \cos A - \cos B &= -2\sin\tfrac{A+B}{2}\sin\tfrac{A-B}{2}\end{aligned}$$

Product-to-sum value: $\sin 75^\circ \sin 15^\circ$

$$\begin{aligned}\sin 75^\circ \sin 15^\circ &= \tfrac{1}{2}\big[\cos(75^\circ - 15^\circ) - \cos(75^\circ + 15^\circ)\big] \\ &= \tfrac{1}{2}\big[\cos 60^\circ - \cos 90^\circ\big] = \tfrac{1}{2}\big[\tfrac{1}{2} - 0\big] = \frac{1}{4}.\end{aligned}$$

Tip. Sum-to-product turns a *sum* of trig terms into a *product*, which is perfect for solving equations by factoring (set each factor to zero).

8.7 Combining Identities with Inverse Trig Functions

Approach

To evaluate something like $\sin(\arccos u)$, let $\theta = \arccos u$, so $\cos\theta = u$ with θ in the range of arccosine ($[0, \pi]$). Draw a right triangle (or use a Pythagorean identity) to find the other functions, respecting the sign forced by the range. The same idea handles $\sin(2\arctan x)$, $\cos(\arcsin a + \arccos b)$, etc.

Evaluate $\sin\left(\arccos \frac{3}{5}\right)$ and $\cos(2\arcsin x)$

Let $\theta = \arccos\frac{3}{5}$, so $\cos\theta = \frac{3}{5}$ with $\theta \in [0, \pi]$, hence $\sin\theta \geq 0$:

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \tfrac{9}{25}} = \tfrac{4}{5}.$$

For the second, let $\phi = \arcsin x$ so $\sin\phi = x$, then use the double-angle form of cosine:

$$\cos(2\arcsin x) = 1 - 2\sin^2\phi = 1 - 2x^2.$$

Tip. Always honor the *range* of the inverse function: $\arcsin, \arctan$ give values in Quadrant I or IV, and arccos in Quadrant I or II. That range fixes the sign of your answer.

8.8 Going Deeper: Advanced Analytic Trig

Telescoping products of cosines

Repeatedly using $\sin 2\alpha = 2\sin\alpha\cos\alpha$ collapses a whole product of doubling-angle cosines into a single ratio of sines. For any θ with $\sin\theta \neq 0$,

$$\prod_{k=0}^{n-1} \cos\left(2^k\theta\right) = \cos\theta\,\cos 2\theta\,\cos 4\theta \cdots \cos\left(2^{n-1}\theta\right) = \frac{\sin\left(2^n\theta\right)}{2^n\sin\theta}.$$

The trick: multiply the product by $2^n\sin\theta$ and peel off one factor at a time. This single identity powers a surprising number of "impossible-looking" exact values.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Prove the telescoping cosine product formula

Let $P = \prod_{k=0}^{n-1}\cos\left(2^k\theta\right)$. Multiply by $2^n\sin\theta$ and absorb one power of 2 at a time using

$2\sin\alpha\cos\alpha = \sin 2\alpha$:

$$\begin{aligned}
2^n \sin\theta \cdot P &= 2^{n-1}\big(2\sin\theta\cos\theta\big)\cos 2\theta \cdots \cos\big(2^{n-1}\theta\big) \\
&= 2^{n-1}\sin 2\theta\,\cos 2\theta \cdots \cos\big(2^{n-1}\theta\big) \\
&= 2^{n-2}\big(2\sin 2\theta\cos 2\theta\big)\cos 4\theta \cdots \cos\big(2^{n-1}\theta\big) \\
&= 2^{n-2}\sin 4\theta\,\cos 4\theta \cdots \cos\big(2^{n-1}\theta\big) \\
&\;\vdots \\
&= 2\sin\big(2^{n-1}\theta\big)\cos\big(2^{n-1}\theta\big) = \sin\big(2^n\theta\big).
\end{aligned}$$

Dividing by $2^n \sin\theta$ gives $P = \dfrac{\sin(2^n\theta)}{2^n \sin\theta}$. ✓

The famous value $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = -\frac{1}{8}$

The angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ are exactly $2^0\theta$, $2^1\theta$, $2^2\theta$ with $\theta = \frac{\pi}{7}$ and $n = 3$, so the telescoping formula applies directly:

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \frac{\sin\big(2^3\cdot\frac{\pi}{7}\big)}{2^3\sin\frac{\pi}{7}} = \frac{\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}.$$

Now $\dfrac{8\pi}{7} = \pi + \dfrac{\pi}{7}$, and $\sin(\pi + x) = -\sin x$, so $\sin\dfrac{8\pi}{7} = -\sin\dfrac{\pi}{7}$. Therefore

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \frac{-\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}} = -\frac{1}{8}.$$

$\cos 36^\circ$ from a cubic equation

Not every "nice" angle needs a sum/difference split — some fall out of a polynomial. Let $\theta = 36^\circ$, so $5\theta = 180^\circ$, giving $2\theta = 180^\circ - 3\theta$ and hence $\cos 2\theta = -\cos 3\theta$. With $x = \cos\theta$ and the identities $\cos 2\theta = 2x^2 - 1$, $\cos 3\theta = 4x^3 - 3x$:

$$\begin{aligned}
2x^2 - 1 &= -\big(4x^3 - 3x\big) \\
4x^3 + 2x^2 - 3x - 1 &= 0 \\
(x+1)\big(4x^2 - 2x - 1\big) &= 0.
\end{aligned}$$

Since $x = \cos 36^\circ > 0$, discard $x = -1$ and solve $4x^2 - 2x - 1 = 0$:

$$\cos 36^\circ = \frac{2 + \sqrt{4+16}}{8} = \frac{2 + 2\sqrt{5}}{8} = \frac{1+\sqrt{5}}{4}.$$

This is the golden-ratio value $\frac{\varphi}{2}$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

Summing equally spaced sines and cosines (Dirichlet kernel)

For an arithmetic progression of angles a, $a+d$, $a+2d, \ldots,$ the telescoping idea from sum-to-product gives closed forms. For $\sin\frac{d}{2} \neq 0$,

$$\sum_{k=0}^{n-1} \cos(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \cos\left(a + \frac{(n-1)d}{2}\right),$$
$$\sum_{k=0}^{n-1} \sin(a+kd) = \frac{\sin\left(\frac{nd}{2}\right)}{\sin\left(\frac{d}{2}\right)} \sin\left(a + \frac{(n-1)d}{2}\right).$$

Why it works: multiply each term by $2\sin\frac{d}{2}$ and apply $2\sin\frac{d}{2}\cos(a+kd) = \sin\left(a+kd+\frac{d}{2}\right) - \sin\left(a+kd-\frac{d}{2}\right)$. Consecutive terms cancel in pairs, leaving only the first and last.

Solving with multiple angles and factoring: $\sin 3x = \sin x$ on $[0, 2\pi)$

Move everything to one side and apply sum-to-product ($\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$):

$$\sin 3x - \sin x = 0$$
$$2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right) = 0$$
$$2\cos 2x\,\sin x = 0.$$

Set each factor to zero on $[0, 2\pi)$:

$$\sin x = 0: \quad x = 0,\ \pi$$
$$\cos 2x = 0: \quad 2x = \tfrac{\pi}{2},\ \tfrac{3\pi}{2},\ \tfrac{5\pi}{2},\ \tfrac{7\pi}{2} \ \Rightarrow\ x = \tfrac{\pi}{4},\ \tfrac{3\pi}{4},\ \tfrac{5\pi}{4},\ \tfrac{7\pi}{4}.$$

Solutions on $[0, 2\pi)$: $x = 0,\ \frac{\pi}{4},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4},\ \frac{7\pi}{4}$. Factoring beats expanding $\sin 3x$ into a cubic in $\sin x$.

Arctangent sums and Machin-style identities

The addition law, read backward, adds arctangents:

$$\arctan a + \arctan b = \arctan\frac{a+b}{1-ab} \pmod{\pi},$$

valid outright when $ab < 1$. This yields elegant collapses such as

$$\arctan\tfrac{1}{2} + \arctan\tfrac{1}{3} = \arctan\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = \arctan 1 = \frac{\pi}{4}.$$

Chaining the rule gives *Machin's formula*, historically used to compute π:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

Conditional identities in a triangle. When $A+B+C=\pi$ (the angles of a triangle), extra relations hold that are *false* for arbitrary angles. Because $C = \pi - (A+B)$:

$$\tan A + \tan B + \tan C = \tan A\,\tan B\,\tan C, \qquad \sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C.$$

The first follows from $\tan C = \tan\big(\pi - (A+B)\big) = -\dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, then clearing the denominator. Always exploit the constraint $C = \pi - A - B$ before pushing symbols around.

9 Triangle Trigonometry and Vectors

Laws of Sines and Cosines, area, vectors, dot products, and applications

9.1 The Law of Sines

Throughout, a triangle is labeled so that side a is opposite angle A, side b is opposite B, and side c is opposite C. Decimal answers are **rounded to the nearest tenth** (angles in degrees, lengths in the given units) unless stated otherwise.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Law of Sines

For any triangle ABC,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Use it when you know **AAS**, **ASA**, or **SSA** (an angle and its opposite side, plus one more piece).

AAS: solve the triangle

Given $A = 35^\circ$, $B = 65^\circ$, $a = 12$. Then $C = 180^\circ - 35^\circ - 65^\circ = 80^\circ$.

$$b = \frac{a\sin B}{\sin A} = \frac{12\sin 65^\circ}{\sin 35^\circ} = \frac{12(0.9063)}{0.5736} \approx 19.0, \qquad c = \frac{a\sin C}{\sin A} = \frac{12\sin 80^\circ}{\sin 35^\circ} \approx 20.6.$$

9.2 The ambiguous case (SSA): 0, 1, or 2 triangles

When you are given two sides and an angle opposite one of them (say a, b, and A), the height from C to the base is $h = b\sin A$. Compare a with h and b:

A **acute:** if $a < b\sin A$ **no triangle**; if $a = b\sin A$ **one** (right) triangle; if $b\sin A < a < b$ **two** triangles; if $a \geq b$ **one** triangle. A **obtuse:** one triangle if $a > b$, otherwise none.

Full ambiguous case: two triangles

Solve the triangle with $a = 8$, $b = 10$, $A = 40°$. Since A is acute and $b\sin A = 10\sin 40° \approx 6.43$, and $6.43 < 8 < 10$, there are **two** triangles.

$$\sin B = \frac{b\sin A}{a} = \frac{10\sin 40°}{8} = \frac{6.4279}{8} \approx 0.8035 \;\Rightarrow\; B \approx 53.5° \;\text{ or }\; B' \approx 180° - 53.5° = 126.5°.$$

Triangle 1: $B \approx 53.5°$, so $C = 180° - 40° - 53.5° = 86.5°$ and $c = \dfrac{a\sin C}{\sin A} = \dfrac{8\sin 86.5°}{\sin 40°} \approx 12.4$.
Triangle 2: $B' \approx 126.5°$, so $C' = 180° - 40° - 126.5° = 13.5°$ and $c' = \dfrac{8\sin 13.5°}{\sin 40°} \approx 2.9$.

9.3 The Law of Cosines

Law of Cosines

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad b^2 = a^2 + c^2 - 2ac\cos B, \qquad c^2 = a^2 + b^2 - 2ab\cos C.$$

Use it for **SAS** (find the third side) or **SSS** (find any angle). Solving for an angle: $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$.
Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

SAS then SSS

SAS: $a = 5$, $b = 7$, $C = 45°$. Then $c^2 = 5^2 + 7^2 - 2(5)(7)\cos 45° = 74 - 70(0.7071) \approx 24.50$, so $c \approx 4.9$.
SSS: $a = 7$, $b = 8$, $c = 9$. The largest angle is C: $\cos C = \dfrac{7^2 + 8^2 - 9^2}{2(7)(8)} = \dfrac{32}{112} = 0.2857$, so $C \approx 73.4°$.

Tip: The Law of Cosines has no ambiguity. When SSS or SAS is given, start there; only switch to the Law of Sines once you have a side and its opposite angle. To avoid the ambiguous case entirely, find the *largest* unknown angle with the Law of Cosines first.

9.4 Area of a Triangle

Two area formulas

SAS formula (two sides and the included angle):

$$\text{Area} = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A = \tfrac{1}{2}ac\sin B.$$

Heron's formula (three sides), with semiperimeter $s = \dfrac{a+b+c}{2}$:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Both formulas

SAS: $a = 5$, $b = 7$, $C = 45°$: Area $= \tfrac{1}{2}(5)(7)\sin 45° = 17.5(0.7071) \approx 12.4$ square units.
Heron: $a = 7$, $b = 8$, $c = 9$: $s = \frac{7+8+9}{2} = 12$, so Area $= \sqrt{12(5)(4)(3)} = \sqrt{720} \approx 26.8$ square units.

9.5 Vectors: Component Form and Operations

Vector basics

A vector in component form is $\vec{u} = \langle a, b\rangle$.

- **Magnitude:** $\|\vec{u}\| = \sqrt{a^2 + b^2}$.
- **Direction angle** θ (from the positive x-axis): $\tan\theta = \dfrac{b}{a}$; place θ in the correct quadrant.
- **Add / subtract:** $\langle a, b\rangle \pm \langle c, d\rangle = \langle a \pm c,\ b \pm d\rangle$.
- **Scalar multiple:** $k\langle a, b\rangle = \langle ka, kb\rangle$.
- **Unit vector** in the direction of $\vec{u}$: $\hat{u} = \dfrac{1}{\|\vec{u}\|}\vec{u}$.
- **Polar / trig form:** $\vec{u} = \|\vec{u}\|\,\langle\cos\theta, \sin\theta\rangle$.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Components, magnitude, direction, unit vector

Let $\vec{u} = \langle 3, 4\rangle$. Then $\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5$, direction $\theta = \tan^{-1}\frac{4}{3} \approx 53.1^\circ$ (Quadrant I), and $\hat{u} = \frac{1}{5}\langle 3, 4\rangle = \langle 0.6, 0.8\rangle$. In trig form, $\vec{u} = 5\langle \cos 53.1^\circ, \sin 53.1^\circ\rangle$.
For $\vec{v} = \langle -2, 5\rangle$: $\|\vec{v}\| = \sqrt{29} \approx 5.4$ and the reference angle is $\tan^{-1}\frac{5}{2} \approx 68.2^\circ$; since $\vec{v}$ is in Quadrant II, $\theta \approx 180^\circ - 68.2^\circ = 111.8^\circ$.

9.6 The Dot Product, Angle, Projection, and Work

Dot product and its uses

For $\vec{u} = \langle a, b\rangle$ and $\vec{v} = \langle c, d\rangle$:

$$\vec{u} \cdot \vec{v} = ac + bd = \|\vec{u}\|\,\|\vec{v}\| \cos\theta.$$

- **Angle between:** $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$.
- **Orthogonal** (perpendicular) exactly when $\vec{u} \cdot \vec{v} = 0$.
- **Projection** of $\vec{u}$ onto $\vec{v}$: $\operatorname{proj}_{\vec{v}} \vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\right) \vec{v}$.
- **Work** done by a constant force $\vec{F}$ over displacement $\vec{d}$: $W = \vec{F} \cdot \vec{d}$.

Full dot-product and projection

Let $\vec{u} = \langle 3, 4\rangle$ and $\vec{v} = \langle 5, 2\rangle$.

$$\vec{u} \cdot \vec{v} = 3(5) + 4(2) = 23, \qquad \|\vec{u}\| = 5, \quad \|\vec{v}\| = \sqrt{29} \approx 5.385.$$

Angle: $\cos\theta = \dfrac{23}{5\sqrt{29}} = \dfrac{23}{26.93} \approx 0.8541$, so $\theta \approx 31.3^\circ$ (not orthogonal, since $\vec{u} \cdot \vec{v} \neq 0$).
Projection of $\vec{u}$ onto $\vec{v}$:

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{23}{(\sqrt{29})^2}\langle 5, 2\rangle = \frac{23}{29}\langle 5, 2\rangle = \left\langle \tfrac{115}{29}, \tfrac{46}{29}\right\rangle \approx \langle 3.97, 1.59\rangle.$$

Work: A force $\vec{F} = \langle 4, 3\rangle$ N moving an object along $\vec{d} = \langle 10, 0\rangle$ m does $W = \vec{F} \cdot \vec{d} = 4(10) + 3(0) = 40$ joules.

Tip: $\vec{u} \cdot \vec{v} > 0$ means the angle is acute; $\vec{u} \cdot \vec{v} < 0$ means obtuse; $\vec{u} \cdot \vec{v} = 0$ means perpendicular. The projection is a *vector* along $\vec{v}$; the scalar $\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ is its signed length (the "component of $\vec{u}$ along $\vec{v}$").

9.7 Applications: Bearings, Forces, and Navigation

Setting up applied problems

Bearings are measured clockwise from north (e.g. N40°E is 40° east of due north). To convert a compass bearing to a standard direction angle from the positive x-axis (east), use $\theta_{\text{std}} = 90° - (\text{bearing east of north})$.
Resultant force / velocity: write each vector in components $\langle \|\vec{F}\| \cos\theta,\ \|\vec{F}\| \sin\theta \rangle$, add them, then find the magnitude and direction of the sum.

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Resultant force

Two forces act on a point: $\vec{F}_1 = 50$ N at 30° and $\vec{F}_2 = 40$ N at 120° (standard angles).

$$\vec{F}_1 = \langle 50\cos 30°, 50\sin 30° \rangle \approx \langle 43.3, 25.0 \rangle, \quad \vec{F}_2 = \langle 40\cos 120°, 40\sin 120° \rangle = \langle -20.0, 34.6 \rangle.$$

Resultant $\vec{R} = \langle 23.3, 59.6 \rangle$, so $\|\vec{R}\| = \sqrt{23.3^2 + 59.6^2} \approx 64.0$ N at $\theta = \tan^{-1}\frac{59.6}{23.3} \approx 68.6°$.

Navigation with the Law of Cosines

A plane flies 120 mi on bearing N50°E, then turns and flies 90 mi on bearing S70°E. At the turning point the angle between the reversed first leg (S50°W) and the new heading (S70°E) is $50° + 70° = 120°$. Its distance from the start is

$$d = \sqrt{120^2 + 90^2 - 2(120)(90)\cos 120°} = \sqrt{14400 + 8100 + 10800} = \sqrt{33300} \approx 182.5 \text{ mi}.$$

9.8 Going Deeper: Advanced Triangle Trig & Vectors

Stewart's Theorem: cevians, medians, and bisectors

A **cevian** is a segment from a vertex to a point on the opposite side. Let a cevian from A meet side a (i.e. BC) at a point that splits it into segments m (adjacent to B) and n (adjacent to C), with length d. Then Stewart's Theorem says

$$b^2 m + c^2 n = a\,(d^2 + mn), \qquad a = m + n.$$

A mnemonic: "**a man and his dad put a bomb in the sink**" $\Rightarrow$ $man + dad = bmb + cnc$, i.e. $m\,a\,n + d\,a\,d = b\,m\,b + c\,n\,c$. Two special cevians follow at once:

- **Median** to side a ($m = n = \frac{a}{2}$): $m_a = \dfrac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.
- **Angle bisector** from A (it divides a in the ratio $c : b$, so $m = \dfrac{ac}{b+c}$, $n = \dfrac{ab}{b+c}$):

$$t_a = \sqrt{bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]}.$$

Opposite, adjacent and hypotenuse are named from the angle.

(diagram in the online version)

Median and angle-bisector lengths

Let $a = 7$, $b = 8$, $c = 9$ (so side $a = BC$, and the cevians issue from A).
Median to a: $m_a = \frac{1}{2}\sqrt{2(8^2) + 2(9^2) - 7^2} = \frac{1}{2}\sqrt{128 + 162 - 49} = \frac{1}{2}\sqrt{241} \approx 7.8$.
Check with Stewart ($m = n = 3.5$, $d = m_a$): $8^2(3.5) + 9^2(3.5) = 7(d^2 + 3.5^2) \Rightarrow 507.5 = 7d^2 + 85.75 \Rightarrow d^2 = 60.25$, so $d \approx 7.8$. ✓

Angle bisector from A: $t_a = \sqrt{(8)(9)\left[1 - \left(\frac{7}{8+9}\right)^2\right]} = \sqrt{72\,(1 - 0.1696)} = \sqrt{59.79} \approx 7.7$.

The ambiguous case as a parameter count

Solving a triangle means finding all 6 parts from 3 given ones. "How many triangles" is really "how many solutions does the given data admit," and it is governed by *how many pieces of data are fixed versus free*:

- **SSS, SAS, ASA, AAS** each pin down a triangle by a *rigid* congruence: 0 free parameters $\Rightarrow$ exactly one triangle (when the data are geometrically possible).
- **SSA** is *not* a congruence: fixing a, b, A leaves the position of the third vertex to be found from $\sin B = \dfrac{b \sin A}{a}$. That equation is quadratic-like in outcome because $\sin B = \sin(180° - B)$, so it can yield 0, 1, or 2 admissible values of B.
- **AAA** fixes only the shape (angles), leaving a 1-parameter family of *similar* triangles (scale is free): infinitely many.

The count is thus a solution count of $\sin B = k$ subject to $A + B < 180°$: two solutions when $0 < k < 1$ and both B values keep the angle sum valid; one when $k = 1$ or when the obtuse B' is rejected; none when $k > 1$.

Area by several methods, and area ratios

The same triangle's area can be written many ways; equating them yields identities.

$$K = \tfrac{1}{2}ab\sin C = \sqrt{s(s-a)(s-b)(s-c)} = rs = \frac{abc}{4R},$$

where $s = \frac{a+b+c}{2}$, r is the inradius, and R is the circumradius. Useful ratio facts:

- A **median** splits a triangle into two triangles of **equal area** (equal bases $\frac{a}{2}$, shared height).
- A cevian dividing the base in ratio $m : n$ splits the area in the **same ratio** $m : n$.
- Two triangles sharing an angle θ have areas in ratio $\dfrac{K_1}{K_2} = \dfrac{a_1 b_1}{a_2 b_2}$ (from $\frac{1}{2}ab\sin\theta$).
- Similar triangles with side ratio k have area ratio k^2.

One area, four ways

For $a = 7$, $b = 8$, $c = 9$ we found Heron's area $K = \sqrt{720} \approx 26.83$. Then:

$$r = \frac{K}{s} = \frac{26.83}{12} \approx 2.2, \qquad R = \frac{abc}{4K} = \frac{504}{4(26.83)} \approx 4.7.$$

SAS cross-check: first $\cos C = \dfrac{7^2 + 8^2 - 9^2}{2(7)(8)} = 0.2857 \Rightarrow \sin C \approx 0.9583$, so $K = \frac{1}{2}(7)(8)(0.9583) \approx 26.8$. ✓ All four formulas agree.

Parallelogram law: $\|\vec{u} \pm \vec{v}\|$ from magnitudes and the angle

If θ is the angle between $\vec{u}$ and $\vec{v}$, then because $\|\vec{u} \pm \vec{v}\|^2 = (\vec{u} \pm \vec{v}) \cdot (\vec{u} \pm \vec{v})$,

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\|\vec{v}\|\cos\theta, \qquad \|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta.$$

These are just the Law of Cosines on the triangle formed by the two vectors. Adding them gives the **parallelogram law** $\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$ (the two diagonals determine the sides).

Resultant and difference without components

Two forces of magnitude $\|\vec{u}\| = 6$ and $\|\vec{v}\| = 10$ meet at an angle of $\theta = 60°$.

$$\|\vec{u} + \vec{v}\| = \sqrt{6^2 + 10^2 + 2(6)(10)\cos 60°} = \sqrt{36 + 100 + 60} = \sqrt{196} = 14.0,$$

$$\|\vec{u} - \vec{v}\| = \sqrt{36 + 100 - 60} = \sqrt{76} \approx 8.7.$$

Check: $14^2 + 8.7^2 \approx 196 + 75.7 = 271.7$ and $2(6^2) + 2(10^2) = 272$, confirming the parallelogram law (rounding). The direction of $\vec{u} + \vec{v}$ relative to $\vec{u}$ is $\phi = \tan^{-1}\dfrac{10\sin 60°}{6 + 10\cos 60°} = \tan^{-1}\dfrac{8.66}{11} \approx 38.2°$.

Vector proofs: perpendicularity and right triangles

The dot product turns geometry into algebra. Key tools:

- Two vectors are **perpendicular** iff $\vec{u} \cdot \vec{v} = 0$.
- Triangle PQR has a **right angle at** Q iff $\overrightarrow{QP} \cdot \overrightarrow{QR} = 0$.
- To show a quadrilateral's diagonals are perpendicular, form them as vectors and show their dot product is 0.
- An angle inscribed in a semicircle is right: if O is the center and P is on the circle, then $\overrightarrow{PA} \cdot \overrightarrow{PB} = 0$ where AB is a diameter.

Prove a triangle is right

Let $P = (1, 2)$, $Q = (4, 3)$, $R = (3, 6)$. Test the angle at Q:

$$\overrightarrow{QP} = \langle 1 - 4,\ 2 - 3 \rangle = \langle -3, -1 \rangle, \qquad \overrightarrow{QR} = \langle 3 - 4,\ 6 - 3 \rangle = \langle -1, 3 \rangle.$$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = (-3)(-1) + (-1)(3) = 3 - 3 = 0,$$

so the angle at Q is exactly 90° and $\triangle PQR$ is right. (Equivalently, verify $\|PQ\|^2 + \|QR\|^2 = \|PR\|^2$: $10 + 10 = 20$. ✓)

Projection and work in context

A wagon is pulled by a rope with force $\vec{F} = \langle 25, 15 \rangle$ N while it rolls along level ground in the direction $\vec{d} = \langle 30, 0 \rangle$ m.

$$W = \vec{F} \cdot \vec{d} = 25(30) + 15(0) = 750 \text{ joules}.$$

Only the component of $\vec{F}$ along the motion does work: $\operatorname{comp}_{\vec{d}} \vec{F} = \dfrac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|} = \dfrac{750}{30} = 25$ N, and indeed 25 N $\times$ 30 m $= 750$ J. The vertical part $\langle 0, 15 \rangle$ (perpendicular to the motion) does no work. If instead the rope makes angle α with the ground and pulls with force F, then $W = Fd \cos \alpha$.

Circumradius and inradius links. From $K = \dfrac{abc}{4R}$ and the Law of Sines, $2R = \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (so the common ratio in the Law of Sines *is* the diameter of the circumscribed circle). From $K = rs$, the inradius is $r = \dfrac{K}{s}$. Together they give handy identities such as $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$ and $a = 2R \sin A$. Rounding convention as above: lengths and radii to the nearest tenth, angles to the nearest tenth of a degree.

10 Polar Coordinates and Complex Numbers

Complex plane, trig form, DeMoivre, roots, polar points, equations, and curves

10.1 Complex Numbers: Operations and the Complex Plane

The essentials

A **complex number** has the form $z = a + bi$ where a is the *real part*, b is the *imaginary part*, and $i^2 = -1$.

- **Add / subtract:** combine real parts and imaginary parts separately.
- **Multiply:** FOIL, then replace $i^2 = -1$.
- **Conjugate:** $\bar{z} = a - bi$. To **divide**, multiply top and bottom by the conjugate of the denominator.
- **Complex plane:** plot $z = a + bi$ as the point (a, b); the horizontal axis is real (Re), the vertical axis is imaginary (Im).

Worked example: division

Simplify $\dfrac{3+2i}{1-4i}$. Multiply by the conjugate $1 + 4i$:

$$\frac{3+2i}{1-4i} \cdot \frac{1+4i}{1+4i} = \frac{3+12i+2i+8i^2}{1^2+4^2} = \frac{3+14i-8}{17} = \frac{-5+14i}{17} = -\frac{5}{17} + \frac{14}{17}i.$$

Powers of i cycle every 4: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. To simplify i^n, divide n by 4 and use the remainder.

10.2 Modulus, Argument, and Trigonometric (Polar) Form

Modulus and argument

For $z = a + bi$:

$$r = |z| = \sqrt{a^2 + b^2} \qquad \tan\theta = \frac{b}{a} \ \text{ (choose } \theta \text{ in the quadrant of } (a, b)\text{)}.$$

The **trigonometric (polar) form** is

$$z = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta,$$

where r is the modulus and θ is the argument. To go back to rectangular form, evaluate $a = r\cos\theta$ and $b = r\sin\theta$.

Rectangular → trigonometric form

Write $z = -\sqrt{3} + i$ in trig form. Here $a = -\sqrt{3},\ b = 1$, so

$$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2, \qquad \text{reference angle} = \tan^{-1}\frac{1}{\sqrt{3}} = 30^\circ.$$

The point $(-\sqrt{3}, 1)$ is in Quadrant II, so $\theta = 180^\circ - 30^\circ = 150^\circ$. Thus

$$z = 2(\cos 150^\circ + i \sin 150^\circ) = 2 \operatorname{cis} 150^\circ.$$

Quadrant check. A calculator's $\tan^{-1}$ only returns angles in $(-90^\circ, 90^\circ)$. Always sketch (a, b) and add 180° (Quadrants II, III) as needed.

10.3 Products, Quotients, and DeMoivre's Theorem

Multiply, divide, and raise to powers

Let $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$.

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) && \text{(multiply moduli, add arguments)} \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) && \text{(divide moduli, subtract arguments)} \end{aligned}$$

DeMoivre's Theorem: for any positive integer n,

$$z^n = \left[r \operatorname{cis} \theta\right]^n = r^n \operatorname{cis}(n\theta) = r^n(\cos n\theta + i \sin n\theta).$$

DeMoivre power (full)

Compute $(1 + i\sqrt{3})^4$. First convert: $r = \sqrt{1+3} = 2$ and $\theta = 60^\circ$ (Quadrant I), so $1 + i\sqrt{3} = 2 \operatorname{cis} 60^\circ$. By DeMoivre,

$$(1 + i\sqrt{3})^4 = 2^4 \operatorname{cis}(4 \cdot 60^\circ) = 16 \operatorname{cis} 240^\circ.$$

Convert back: $16(\cos 240^\circ + i \sin 240^\circ) = 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -8 - 8\sqrt{3}\, i$.

Add or subtract 360° from the resulting angle so your final argument is a standard angle in $[0^\circ, 360^\circ)$ before converting back.

10.4 The nth Roots of a Complex Number

Root formula

Every nonzero complex number $z = r\,\mathrm{cis}\,\theta$ has exactly n distinct n**th roots**:

$$z_k = \sqrt[n]{r}\ \mathrm{cis}\left(\frac{\theta + 360^\circ k}{n}\right), \qquad k = 0, 1, 2, \ldots, n-1.$$

All n roots lie on a circle of radius $\sqrt[n]{r}$, equally spaced $\dfrac{360^\circ}{n}$ apart.

All nth roots (full)

Find the three cube roots of $8i$. Write $8i = 8\,\mathrm{cis}\,90^\circ$, so $\sqrt[3]{r} = \sqrt[3]{8} = 2$ and the angles are $\dfrac{90^\circ + 360^\circ k}{3}$ for $k = 0, 1, 2$:

$$\theta = 30^\circ,\ 150^\circ,\ 270^\circ.$$

$$\begin{aligned}
z_0 &= 2\,\mathrm{cis}\,30^\circ = 2\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right) = \sqrt{3} + i,\\
z_1 &= 2\,\mathrm{cis}\,150^\circ = 2\left(-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right) = -\sqrt{3} + i,\\
z_2 &= 2\,\mathrm{cis}\,270^\circ = 2(0 - i) = -2i.
\end{aligned}$$

Spacing shortcut. Find the first root ($k = 0$), then add $\dfrac{360^\circ}{n}$ repeatedly to the argument to get the rest.

10.5 Polar Coordinates: Points and Conversions

Plotting and converting points

A polar point (r, θ) is located by rotating θ from the positive x-axis, then moving r along that ray (if $r < 0$, move *backward*, i.e. 180° opposite).

$$\textbf{Polar} \to \textbf{Rect:} \quad x = r\cos\theta, \quad y = r\sin\theta.$$

$$\textbf{Rect} \to \textbf{Polar:} \quad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}\ \text{(quadrant of } (x, y)\text{)}.$$

Multiple representations: $(r, \theta) = (r, \theta + 360^\circ) = (-r, \theta + 180^\circ)$ all name the same point.

Distance and midpoint come from the coordinates.

(diagram in the online version)

Polar $\leftrightarrow$ rectangular (full both ways)

Polar to rectangular: convert $(6, 150°)$.

$$x = 6\cos 150° = 6\left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}, \qquad y = 6\sin 150° = 6\left(\tfrac{1}{2}\right) = 3.$$

So $(6, 150°) \to (-3\sqrt{3},\, 3)$.

Rectangular to polar: convert $(-1, \sqrt{3})$.

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2, \qquad \text{ref angle} = \tan^{-1}\frac{\sqrt{3}}{1} = 60°.$$

The point is in Quadrant II, so $\theta = 180° - 60° = 120°$; thus $(-1, \sqrt{3}) \to (2, 120°)$.

A polar point is *not* unique. Adding $360°$ or flipping the sign of r (with a $180°$ shift) gives another valid name for the same location.

10.6 Converting Equations Between Polar and Rectangular Form

Substitution toolkit

Use $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$.

- **Polar → Rect:** multiply through by r to create r^2, $r\cos\theta$, or $r\sin\theta$, then substitute.
- **Rect → Polar:** replace x, y; solve for r. Useful: $r\cos\theta = x$, $r\sin\theta = y$, so a vertical line $x = c$ becomes $r = c\sec\theta$ and a horizontal line $y = c$ becomes $r = c\csc\theta$.

Equation conversion (both directions)

Polar to rectangular: convert $r = 4\sin\theta$. Multiply by r: $r^2 = 4r\sin\theta$, so

$$x^2 + y^2 = 4y \;\Rightarrow\; x^2 + (y-2)^2 = 4,$$

a circle centered at $(0, 2)$ with radius 2.

Rectangular to polar: convert $y = 3$. Substitute $y = r\sin\theta$: $r\sin\theta = 3$, so $r = \dfrac{3}{\sin\theta} = 3\csc\theta$.

The circle $r = a\cos\theta$ passes through the origin with center $\left(\frac{a}{2}, 0\right)$; the circle $r = a\sin\theta$ has center $\left(0, \frac{a}{2}\right)$. Both have radius $\frac{|a|}{2}$.

10.7 Graphs of Polar Equations

Recognize the family

- **Circle:** $r = a$ (center origin), $r = a\cos\theta$ or $r = a\sin\theta$ (through origin).
- **Cardioid:** $r = a \pm a\cos\theta$ or $r = a \pm a\sin\theta$ (heart shape; $a = b$).
- **Limaçon:** $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. Ratio $\frac{a}{b}$: < 1 inner loop, $= 1$ cardioid, $1 < \frac{a}{b} < 2$ dimpled, ≥ 2 convex.
- **Rose:** $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$. Petal length $|a|$; n **odd** $\Rightarrow n$ **petals**, n **even** $\Rightarrow 2n$ **petals**.
- **Lemniscate:** $r^2 = a^2\cos 2\theta$ or $r^2 = a^2 \sin 2\theta$ (figure-eight).

Rose petal count trap: the number of petals depends on the parity of n, not on n itself. $r = \cos 3\theta$ has 3 petals; $r = \cos 2\theta$ has 4 petals; $r = \cos 4\theta$ has 8 petals.

10.8 Going Deeper: Advanced Polar & Complex Ideas

Roots of unity: sums and products

The nth **roots of unity** are the solutions of $z^n = 1$. Writing $\omega = \operatorname{cis}\dfrac{360^\circ}{n}$, they are the n equally spaced points

$$1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}, \qquad \omega^k = \operatorname{cis}\left(\frac{360^\circ k}{n}\right).$$

Two identities fall out of the geometry:

- **Sum is zero:** $\displaystyle\sum_{k=0}^{n-1} \omega^k = 1 + \omega + \cdots + \omega^{n-1} = 0$ for $n \geq 2$ (finite geometric series with ratio $\omega \neq 1$ and $\omega^n = 1$). The roots balance about the origin.
- **Product identity:** $\displaystyle\prod_{k=1}^{n-1} \left(1 - \omega^k\right) = n$. (Factor $z^n - 1 = \prod_{k=0}^{n-1}(z - \omega^k)$, divide by $z - 1$, then let $z \to 1$.)

Worked example: all fifth roots of unity, and their sum

Solve $z^5 = 1$. Here $r = 1$ and $\theta = 0^\circ$, so $\sqrt[5]{1} = 1$ and the arguments are $\dfrac{0^\circ + 360^\circ k}{5} = 72^\circ k$ for

$k = 0, 1, 2, 3, 4$. All five roots:

$$z_0 = \operatorname{cis} 0^\circ = 1, \qquad z_1 = \operatorname{cis} 72^\circ, \qquad z_2 = \operatorname{cis} 144^\circ,$$
$$z_3 = \operatorname{cis} 216^\circ, \qquad z_4 = \operatorname{cis} 288^\circ.$$

They sit at the vertices of a regular pentagon on the unit circle. Their sum pairs conjugates:

$$1 + 2\cos 72^\circ + 2\cos 144^\circ = 1 + 2(0.309) + 2(-0.809) = 0,$$

confirming $\sum_{k=0}^{4} z_k = 0$.

The identity $z^n + z^{-n} = 2\cos n\theta$

If $z = \operatorname{cis}\theta$ lies on the unit circle then $\dfrac{1}{z} = z^{-1} = \operatorname{cis}(-\theta) = \cos\theta - i\sin\theta$. By DeMoivre, $z^n = \operatorname{cis} n\theta$ and $z^{-n} = \operatorname{cis}(-n\theta)$, so adding and subtracting gives

$$z^n + z^{-n} = 2\cos n\theta, \qquad z^n - z^{-n} = 2i\sin n\theta.$$

The special case $z + \dfrac{1}{z} = 2\cos\theta$ is the workhorse: raising it to a power and expanding lets you rewrite $\cos^m\theta$ as a sum of terms $\cos k\theta$ (Fourier/power-reduction identities).

Worked example: DeMoivre proves the triple-angle formulas

Expand $(\cos\theta + i\sin\theta)^3$ two ways. By DeMoivre it equals $\cos 3\theta + i\sin 3\theta$. By the binomial theorem (writing $c = \cos\theta,\ s = \sin\theta$),

$$(c + is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = c^3 - 3cs^2 + i\,(3c^2 s - s^3).$$

Matching real and imaginary parts:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta,$$
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta,$$

using $\sin^2\theta = 1 - \cos^2\theta$ (and $\cos^2\theta = 1 - \sin^2\theta$) to simplify.

Loci in the complex plane: the Apollonius circle

Treat $z = x + iy$ as the point (x, y) and read $|z - a|$ as the distance from z to the point a.

- $|z - a| = k$ is a **circle** of radius k centered at a.
- $|z - a| = |z - b|$ is the **perpendicular bisector** of the segment ab.
- $|z - a| = k|z - b|$ with $k \neq 1$ is the **Apollonius circle**: the set of points whose distances to a and b are in a fixed ratio k.

To find it, square both sides and expand with $|z-a|^2 = (x-a_1)^2 + (y-a_2)^2$; the x^2 and y^2 terms survive with equal coefficients, so the result is always a circle.

Worked example: the locus $|z-1| = \frac{1}{2}|z-4|$

Square: $|z-1|^2 = \frac{1}{4}|z-4|^2$, i.e. $(x-1)^2 + y^2 = \frac{1}{4}\big[(x-4)^2 + y^2\big]$. Multiply by 4:

$$4(x-1)^2 + 4y^2 = (x-4)^2 + y^2.$$

Expand: $4x^2 - 8x + 4 + 4y^2 = x^2 - 8x + 16 + y^2$. The $-8x$ terms cancel, leaving

$$3x^2 + 3y^2 = 12 \implies x^2 + y^2 = 4,$$

a circle of radius 2 centered at the origin (drawn above).

Sum of the roots of $z^n = w$. The equation $z^n - w = 0$ has no z^{n-1} term, so by Vieta's formulas the n roots always *sum to* 0 (for $n \geq 2$), and their product is $(-1)^{n+1}w$. Geometrically the roots are n points equally spaced on a circle of radius $\sqrt[n]{|w|}$, so their centroid is the origin.

Area enclosed by a polar curve

A thin wedge of angle $d\theta$ at radius r is a circular sector of area $\frac{1}{2}r^2\,d\theta$. Summing wedges, the area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta = \frac{1}{2}\int_\alpha^\beta \big[f(\theta)\big]^2\,d\theta.$$

Choosing limits: for one rose petal, integrate between the two consecutive θ where $r = 0$; for a full cardioid, use 0 to 360° once around. For the area *between* two curves, subtract: $A = \frac{1}{2}\int (r_{\text{out}}^2 - r_{\text{in}}^2)\,d\theta$.

Worked example: area of one petal and a curve intersection

Area of one petal of $r = 2\cos 2\theta$. A petal runs from $\theta = -45°$ to $\theta = 45°$ (where $r = 0$):

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} (2\cos 2\theta)^2\,d\theta = \int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\,d\theta = \Big[\theta + \tfrac{\sin 4\theta}{4}\Big]_{-\pi/4}^{\pi/4} = \frac{\pi}{2},$$

using $4\cos^2 2\theta = 2(1 + \cos 4\theta)$.

Intersection of $r = 1 + \cos\theta$ and $r = 3\cos\theta$. Set the radii equal:

$$1 + \cos\theta = 3\cos\theta \;\Rightarrow\; 2\cos\theta = 1 \;\Rightarrow\; \cos\theta = \tfrac{1}{2} \;\Rightarrow\; \theta = 60°,\ 300°,$$

giving $r = \frac{3}{2}$: the points $\left(\frac{3}{2}, 60°\right)$ and $\left(\frac{3}{2}, 300°\right)$. **Also check the pole separately:** the cardioid reaches $r = 0$ at $\theta = 180°$ and the circle reaches $r = 0$ at $\theta = 90°$, so both pass through the origin even though no single θ solves the equation there.

Polar intersection trap. Setting $r_1(\theta) = r_2(\theta)$ can *miss* intersections, because the same point has many names (r, θ). Always test the pole and try replacing (r, θ) with $(-r, \theta + 180°)$ or $(r, \theta + 360°)$ in one equation before solving.

11 Systems, Matrices, and Determinants

Substitution, elimination, Gaussian elimination, matrix algebra, inverses, determinants, Cramer's Rule, partial fractions, and linear programming

11.1 Solving Systems by Substitution and Elimination

The idea

A **solution** of a system is an ordered pair (or triple) that satisfies *every* equation at once. Two main hand methods:

- **Substitution:** solve one equation for a variable, then plug into the other. Best when a variable already has coefficient 1, or for **nonlinear** systems.
- **Elimination:** add multiples of the equations to cancel a variable. Best for linear systems with awkward coefficients.

A linear system may have **one** solution (lines cross), **no** solution (parallel lines; you reach a false statement like $0 = 5$), or **infinitely many** (same line; you reach a true statement like $0 = 0$). A nonlinear system (line and circle, two conics, …) can have $0, 1, 2$, or more solutions.

Elimination (linear)

Solve $\begin{cases} 3x + 2y = 16 \\ 2x - 5y = -2 \end{cases}$

Multiply eq. 1 by 5 and eq. 2 by 2 to cancel y:

$$15x + 10y = 80, \qquad 4x - 10y = -4.$$

Add: $19x = 76 \Rightarrow x = 4$. Back-substitute into eq. 1: $12 + 2y = 16 \Rightarrow y = 2$.

Check: $2(4) - 5(2) = 8 - 10 = -2$. ✓ Solution: $(4, 2)$.

Substitution (nonlinear: line meets circle)

Solve $\begin{cases} x^2 + y^2 = 25 \\ y = x + 1 \end{cases}$

Substitute $y = x + 1$: $x^2 + (x+1)^2 = 25 \Rightarrow 2x^2 + 2x - 24 = 0 \Rightarrow x^2 + x - 12 = 0$.

Factor: $(x+4)(x-3) = 0 \Rightarrow x = -4$ or $x = 3$, giving $y = -3$ or $y = 4$.

Solutions: $(-4, -3)$ and $(3, 4)$. **Check** $(3, 4)$: $9 + 16 = 25$. ✓

Tip: After finding a variable, always back-substitute into the *original* equation and verify *both*. For nonlinear systems, each x must be paired with the y from the equation you substituted into.

11.2 Three-Variable Linear Systems: Gaussian Elimination

Row-echelon form

Write the system as an **augmented matrix** and use **elementary row operations** to reach *row-echelon form* (leading 1's stepping to the right, zeros below), then **back-substitute**.

- Swap two rows: $R_i \leftrightarrow R_j$.
- Scale a row: $R_i \to kR_i$ $(k \neq 0)$.
- Add a multiple of one row to another: $R_i \to R_i + kR_j$.

Reduced row-echelon form additionally clears entries *above* each leading 1 (Gauss–Jordan).

Full Gaussian elimination

Solve $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 2 \end{cases}$

Augmented matrix, then eliminate below the first pivot with $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 2 & -1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 1 & -2 & -4 \end{bmatrix}.$$

Swap $R_2 \leftrightarrow R_3$, then $R_3 \to R_3 + 3R_2$:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & -7 & -21 \end{bmatrix}.$$

Back-substitute: $-7z = -21 \Rightarrow z = 3$; $y - 2(3) = -4 \Rightarrow y = 2$; $x + 2 + 3 = 6 \Rightarrow x = 1$.

Check in eq. 2: $2(1) - 2 + 3 = 3$. ✓ Solution: $(1, 2, 3)$.

Tip: A row of the form $[\,0\ 0\ 0 \mid c\,]$ with $c \neq 0$ means **no solution**. A row $[\,0\ 0\ 0 \mid 0\,]$ means a **free variable** and infinitely many solutions.

11.3 Matrices and Their Operations

Adding, scaling, multiplying

For matrices of the *same size*, **add/subtract entrywise**; **scalar multiply** multiplies every entry.

Matrix product AB is defined only when A is $m \times n$ and B is $n \times p$ (inner dimensions match); the result is $m \times p$. Entry (i, j) of AB is the **dot product** of row i of A with column j of B. In general $AB \neq BA$.

Combined operations and a product

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$.

$2A - B = \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}$.

$AB = \begin{bmatrix} 1(3) + 2(-2) & 1(1) + 2(4) \\ 0(3) + (-1)(-2) & 0(1) + (-1)(4) \end{bmatrix} = \begin{bmatrix} -1 & 9 \\ 2 & -4 \end{bmatrix}$.

Tip: Check dimensions *before* multiplying: $(m \times n)(n \times p) = (m \times p)$. If the inner numbers differ, the product does not exist.

11.4 The Inverse of a Matrix

Inverses

A^{-1} satisfies $AA^{-1} = A^{-1}A = I$. For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

provided $\det A = ad - bc \neq 0$. If $\det A = 0$ the matrix is **singular** (no inverse). To solve $A\mathbf{x} = \mathbf{b}$, use $\mathbf{x} = A^{-1}\mathbf{b}$.

Find a 2×2 inverse and solve a system

Solve $\begin{cases} 2x + 3y = 7 \\ x + 4y = 6 \end{cases}$ using an inverse.

Coefficient matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $\det A = 2(4) - 3(1) = 5$.

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

Then $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 28 - 18 \\ -7 + 12 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Check: $2(2) + 3(1) = 7$ and $2 + 4(1) = 6$. ✓ Solution $(2, 1)$.

Tip: Verify an inverse cheaply by confirming $AA^{-1} = I$, or at least that $\det A \cdot \det(A^{-1}) = 1$.

11.5 Determinants and Cramer's Rule

Determinants

2×2: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$.

3×3 (expansion along row 1):

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Cramer's Rule: for $A\mathbf{x} = \mathbf{b}$ with $D = \det A \neq 0$, each variable is $x_k = D_k/D$, where D_k replaces column k of A with $\mathbf{b}$.

3×3 determinant and Cramer's Rule

Solve $\begin{cases} x + y + z = 6 \\ 2x - y + z = 3 \\ x + 2y - z = 2 \end{cases}$ for x by Cramer.

$D = \det \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} = 1(1 - 2) - 1(-2 - 1) + 1(4 + 1) = -1 + 3 + 5 = 7.$

$$D_x = \det \begin{bmatrix} 6 & 1 & 1 \\ 3 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} = 6(1-2) - 1(-3-2) + 1(6+2) = -6+5+8 = 7.$$

So $x = D_x/D = 7/7 = 1$ (matching the Gaussian result above).

Tip: Remember the alternating $+ - +$ sign pattern across the top row. If $D = 0$, Cramer's Rule fails—the system has either no solution or infinitely many.

11.6 Partial Fraction Decomposition

Breaking a rational function apart

For a *proper* fraction (numerator degree $<$ denominator degree), factor the denominator and write one term per factor:

- Distinct linear $(ax+b)$: term $\dfrac{A}{ax+b}$.
- Repeated linear $(ax+b)^2$: terms $\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$.
- Irreducible quadratic (ax^2+bx+c): term $\dfrac{Ax+B}{ax^2+bx+c}$.

If *improper*, do polynomial division first. Solve for the constants by clearing denominators.

Full partial-fraction decomposition

Decompose $\dfrac{5x-1}{x^2-x-2}$.

Factor: $x^2 - x - 2 = (x-2)(x+1)$. Write $\dfrac{5x-1}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$.

Clear: $5x - 1 = A(x+1) + B(x-2)$. Let $x = 2$: $9 = 3A \Rightarrow A = 3$. Let $x = -1$: $-6 = -3B \Rightarrow B = 2$.

$$\frac{5x-1}{x^2-x-2} = \frac{3}{x-2} + \frac{2}{x+1}.$$

Check: $\dfrac{3(x+1)+2(x-2)}{(x-2)(x+1)} = \dfrac{5x-1}{(x-2)(x+1)}$. ✓

Tip: Substituting the *roots* of the factors instantly isolates constants. For repeated or quadratic factors, also compare coefficients of like powers.

11.7 Systems of Inequalities and Linear Programming

Feasible regions and optimization

Graph each inequality and shade the overlap—the **feasible region**. In **linear programming** you maximize or minimize a linear **objective** $z = ax + by$ over that region. The optimum (if it exists) always occurs at a **corner (vertex)**, so evaluate z at every vertex and pick the best.

Linear-programming optimization

Maximize $z = 3x + 2y$ subject to $x \geq 0,\ y \geq 0,\ x + y \leq 4,\ x + 3y \leq 6$.

Vertices: $(0,0), (4,0), (3,1), (0,2)$. The lines cross at $x + y = 4,\ x + 3y = 6 \Rightarrow y = 1,\ x = 3$.

$$z(0,0) = 0, \quad z(4,0) = 12, \quad z(3,1) = 11, \quad z(0,2) = 4.$$

Maximum $z = 12$ at $(4,0)$.

Tip: Always list the vertices explicitly—including where two boundary lines intersect—then test each. A bounded feasible region guarantees both a max and a min exist.

11.8 Going Deeper: Advanced Systems & Matrices

Matrix powers, recurrences, and the Cayley–Hamilton shortcut

Every square matrix satisfies its own **characteristic equation**. For a 2×2 matrix A with trace $t = \operatorname{tr} A = a + d$ and determinant $\Delta = \det A$, the **Cayley–Hamilton theorem** gives

$$A^2 - t\,A + \Delta\,I = \mathbf{0} \qquad \Longleftrightarrow \qquad A^2 = t\,A - \Delta\,I.$$

Two powerful consequences:

- **Reduce high powers:** A^2 (and hence $A^3, A^4, \dots$) is always a *linear* combination $\alpha A + \beta I$. So $A^n = c_n A + d_n I$ for scalars found by a **recurrence**, never a full multiplication.
- **Instant inverse:** rearrange to $A\big(tI - A\big) = \Delta I$, so if $\Delta \neq 0$,

$$A^{-1} = \frac{1}{\Delta}\big(t\,I - A\big).$$

The scalar recurrence itself mirrors the matrix: if $A^n = c_n A + d_n I$ then $A^{n+1} = c_n A^2 + d_n A = (tc_n + d_n)A - \Delta c_n I$, so $c_{n+1} = tc_n + d_n$ and $d_{n+1} = -\Delta c_n$.

Worked: Cayley–Hamilton inverse and a reduced power

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, so $t = 6$ and $\Delta = 2(4) - 3(1) = 5$.

Inverse. $A^{-1} = \frac{1}{5}(6I - A) = \frac{1}{5}\begin{bmatrix} 6-2 & -3 \\ -1 & 6-4 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$, matching the adjugate formula.

Power via recurrence. From $A^2 = 6A - 5I$:

$$A^3 = A \cdot A^2 = 6A^2 - 5A = 6(6A - 5I) - 5A = 31A - 30I.$$

So $A^3 = 31\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - 30\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 93 \\ 31 & 94 \end{bmatrix}$ — no repeated matrix multiplication needed.

Special determinant identities and singular conditions

Some determinants have closed forms worth memorizing:

- **Vandermonde:** $\det\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)$, i.e. the product of all differences $x_j - x_i$ ($i < j$).
- **Identity plus all-ones ($I + aJ$):** if J is the $n \times n$ matrix of all 1's, then $\det(I + aJ) = 1 + na$.
- **Block triangular:** $\det\begin{bmatrix} A & B \\ \mathbf{0} & D \end{bmatrix} = \det A \cdot \det D$ (the off-diagonal block B does not matter).

Singular ($\det = 0$) exactly when rows/columns are linearly dependent: one row is a scalar multiple of another, a row is a sum of others, or a row (or column) is all zeros. A singular matrix has no inverse, and $A\mathbf{x} = \mathbf{b}$ then has either no solution or infinitely many.

Worked: three determinant identities

Vandermonde with $x_1 = 1$, $x_2 = 2$, $x_3 = 4$:

$$\det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{bmatrix} = (2-1)(4-1)(4-2) = 1 \cdot 3 \cdot 2 = 6.$$

$I + 2J$, size 3: formula gives $1 + 3(2) = 7$. Directly, $I + 2J = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$, and expanding confirms $\det = 7$. ✓

Block triangular: $\det\begin{bmatrix} 2 & 5 & 9 & 9 \\ 0 & 3 & 9 & 9 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix} = (2 \cdot 3)(1 \cdot 4) = 24$ (product of the two 2×2 diagonal blocks, which are themselves triangular).

Nonlinear systems: intersecting conics

When both equations are conics, **eliminate the squared terms** rather than a single variable. Add or subtract multiples so that x^2 (or y^2) cancels, leaving a simpler relation; substitute back. Two conics can meet in up to **four** points, so expect several ordered pairs and pair each recovered value with the correct sign from the equation you used.

Worked: ellipse meets hyperbola

Solve $\begin{cases} x^2 + 2y^2 = 6 \\ x^2 - y^2 = 3 \end{cases}$

Subtract eq. 2 from eq. 1 to kill x^2: $3y^2 = 3 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.

Substitute into eq. 2: $x^2 - 1 = 3 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

Solutions (four): $(2, 1)$, $(2, -1)$, $(-2, 1)$, $(-2, -1)$. **Check** $(2, 1)$: $4 + 2 = 6$ and $4 - 1 = 3$. ✓

Partial fractions with repeated and irreducible factors

Two cases beyond the distinct-linear setup demand extra terms:

- **Repeated linear** $(x - r)^k$: include *one term per power*, $\dfrac{A_1}{x-r} + \dfrac{A_2}{(x-r)^2} + \cdots + \dfrac{A_k}{(x-r)^k}$.
- **Irreducible quadratic** $x^2 + bx + c$ (no real roots): use a *linear* numerator $\dfrac{Bx + C}{x^2 + bx + c}$.

Root-substitution alone will not find every constant here, so **clear denominators and match coefficients** of like powers of x, giving a small linear system for the unknowns.

Worked: an irreducible quadratic factor

Decompose $\dfrac{3x^2 + 2x + 1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 + 1}$.

Clear denominators: $3x^2 + 2x + 1 = A(x^2 + 1) + (Bx + C)(x - 1)$.

Root shortcut, $x = 1$: $6 = 2A \Rightarrow A = 3$.

Match coefficients: $x^2:\ 3 = A + B \Rightarrow B = 0$; constant: $1 = A - C \Rightarrow C = 2$; check x^1: $2 = -B + C = 2$. ✓

$$\frac{3x^2 + 2x + 1}{(x-1)(x^2+1)} = \frac{3}{x-1} + \frac{2}{x^2+1}.$$

Linear programming: unbounded regions and an edge of optima

The corner principle assumes a **bounded** feasible region. Two degenerate cases:

- **Unbounded region:** if the objective can grow without limit along an open direction, there is **no maximum** (though a minimum may still exist at a vertex, or vice versa). Always check whether the region is closed in the direction you are optimizing.
- **Edge of optima:** when the objective line $z = ax + by$ is *parallel* to a boundary edge, *every* point on that edge is optimal — infinitely many solutions, all giving the same z. You detect this when two adjacent vertices tie for the best value.

Worked: an unbounded feasible region

Consider $x \geq 0,\ y \geq 0,\ x + y \geq 2$ (region opens outward to the upper right).

Maximize $z = x + y$: moving farther from the origin increases z without bound, so **no maximum exists** — the problem is unbounded.

Minimize $z = x + y$ over the *same* region: the constraint forces $x + y \geq 2$, and every point of the edge from $(2, 0)$ to $(0, 2)$ gives $z = 2$. So the **minimum is** $z = 2$, attained along an *entire edge* (an edge of optima), not a single corner.

12 Conic Sections and Parametric Equations

Parabolas, ellipses, hyperbolas, classifying by completing the square, parametric curves, and polar conics

12.1 Parabolas: Focus, Directrix, and Latus Rectum

Standard forms of a parabola (vertex (h, k))

A parabola is the set of points equidistant from a fixed **focus** and a fixed line, the **directrix**. The number p is the signed distance from vertex to focus.

$$(x-h)^2 = 4p(y-k) \quad \text{opens up/down,} \qquad (y-k)^2 = 4p(x-h) \quad \text{opens right/left.}$$

- **Vertical** axis: focus $(h,\ k+p)$, directrix $y = k - p$. Opens up if $p > 0$, down if $p < 0$.
- **Horizontal** axis: focus $(h+p,\ k)$, directrix $x = h - p$. Opens right if $p > 0$, left if $p < 0$.
- **Latus rectum** (focal chord through the focus, perpendicular to the axis) has length $|4p|$.
- **Eccentricity** of every parabola is $e = 1$.

Completing the square: $x^2 - 6x - 8y + 1 = 0$

Group the squared variable and complete the square:

$$\begin{aligned} x^2 - 6x &= 8y - 1 \\ x^2 - 6x + 9 &= 8y - 1 + 9 \\ (x-3)^2 &= 8(y+1). \end{aligned}$$

So $4p = 8 \Rightarrow p = 2$; vertex $(3, -1)$; opens **up**. Focus $(3, -1+2) = (3, 1)$; directrix $y = -1 - 2 = -3$; latus rectum $= |4p| = 8$.

From standard form: $y^2 = 12x$

Horizontal axis, vertex $(0, 0)$. $4p = 12 \Rightarrow p = 3$, opens **right**. Focus $(3, 0)$; directrix $x = -3$; latus rectum $= 12$.

Tip: The squared variable tells you the axis. If x is squared, the parabola opens up/down; if y is squared, it opens left/right. The focus always sits *inside* the curve, the directrix outside.

12.2 Ellipses: Center, Axes, Foci, Eccentricity

Standard forms of an ellipse (center (h, k), $a > b > 0$)

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{(major axis horizontal)}, \qquad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \quad \text{(major axis vertical)}.$$

- a = semi-major axis (largest denominator, a^2); b = semi-minor axis.
- **Vertices** are a from center along the major axis; **co-vertices** are b from center along the minor axis.
- **Foci** lie on the major axis, c from center, where $c^2 = a^2 - b^2$.
- **Eccentricity** $e = \dfrac{c}{a}$, with $0 < e < 1$ (nearer 0 is more circular).

Completing the square: $4x^2 + 9y^2 - 16x + 18y - 11 = 0$

Group by variable, factor out leading coefficients, then complete each square:

$$\begin{gathered}4(x^2-4x)+9(y^2+2y)=11\\4(x^2-4x+4)+9(y^2+2y+1)=11+16+9\\4(x-2)^2+9(y+1)^2=36\\\frac{(x-2)^2}{9}+\frac{(y+1)^2}{4}=1.\end{gathered}$$

Center $(2,-1)$; $a^2=9,\ b^2=4$ so $a=3,\ b=2$; major axis horizontal. $c^2=9-4=5\Rightarrow c=\sqrt{5}$.

Vertices $(5,-1),(-1,-1)$; co-vertices $(2,1),(2,-3)$; foci $(2\pm\sqrt{5},\ -1)$; $e=\dfrac{\sqrt{5}}{3}\approx 0.75$.

Tip: The larger denominator is always a^2, and it names the direction of the major axis: under x means horizontal, under y means vertical. Add the amounts you completed to *both* sides, remembering to multiply by the factored-out coefficient.

12.3 Hyperbolas: Vertices, Foci, Asymptotes, Eccentricity

Standard forms of a hyperbola (center (h,k))

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1\quad\text{(opens left/right)},\qquad\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1\quad\text{(opens up/down)}.$$

- The *positive* term names the **transverse axis**; a^2 is its denominator (it need not be the larger one).
- **Vertices** are a from center along the transverse axis; **foci** are c from center with $c^2=a^2+b^2$.
- **Asymptotes:** $y-k=\pm\dfrac{b}{a}(x-h)$ for the left/right form; $y-k=\pm\dfrac{a}{b}(x-h)$ for the up/down form.
- **Eccentricity** $e=\dfrac{c}{a}$, with $e>1$.

Completing the square: $4x^2-9y^2+8x+36y-68=0$

Watch the sign on the y group when factoring out -9:

$$\begin{aligned} 4(x^2+2x)-9(y^2-4y) &= 68 \\ 4(x^2+2x+1)-9(y^2-4y+4) &= 68+4-36 \\ 4(x+1)^2-9(y-2)^2 &= 36 \\ \frac{(x+1)^2}{9}-\frac{(y-2)^2}{4} &= 1. \end{aligned}$$

Center $(-1,2)$; $a^2=9,\ b^2=4$ so $a=3,\ b=2$; opens left/right. $c^2=9+4=13 \Rightarrow c=\sqrt{13}$. Vertices $(2,2),(-4,2)$; foci $(-1\pm\sqrt{13},\,2)$; asymptotes $y-2=\pm\frac{2}{3}(x+1)$; $e=\frac{\sqrt{13}}{3}\approx 1.20$.

Tip: For a hyperbola c is the *largest* of the three ($c>a$ and $c>b$) because $c^2=a^2+b^2$. Do not assume $a>b$: for a hyperbola a is simply whichever denominator sits under the positive term.

12.4 Classifying Conics from the General Form

The general second-degree equation

With no xy term, $Ax^2+Cy^2+Dx+Ey+F=0$ classifies by comparing A and C:

- **Circle:** $A=C$ (and same sign), e.g. $A=C\neq 0$.
- **Parabola:** exactly one of A, C is 0 (only one variable is squared).
- **Ellipse:** A and C have the *same sign* but $A\neq C$.
- **Hyperbola:** A and C have *opposite signs* ($AC<0$).

Classify each equation

- $3x^2+3y^2-12x+6=0$: $A=C=3\Rightarrow$ **circle**.
- $x^2-8y+2x=0$: only x is squared $\Rightarrow$ **parabola**.
- $2x^2+5y^2-4=0$: same sign, $2\neq 5\Rightarrow$ **ellipse**.
- $9x^2-4y^2+18x=0$: opposite signs $\Rightarrow$ **hyperbola**.

Tip: Classify *before* completing the square. The sign comparison of A and C instantly names the conic; completing the square then locates its center and features.

12.5 Parametric Equations

Curves defined by a parameter

A **parametric** curve gives x and y each as a function of a third variable t:

$$x = f(t), \qquad y = g(t).$$

As t increases the point (x, y) traces the curve in a definite **direction** (its orientation).

- **Graph by a table:** choose values of t, compute (x, y), and connect in order of increasing t.
- **Eliminate the parameter:** solve one equation for t (or use a trig identity) and substitute to get a relation in x and y.

Eliminate the parameter (polynomial): $x = t^2,\ y = t$

From the second equation $t = y$. Substitute: $x = y^2$. The curve is a **sideways parabola** opening right, traced upward as t increases. Building a table:

Eliminate the parameter (trig): $x = 3\cos t,\ y = 2\sin t$

Solve for the trig functions: $\cos t = \dfrac{x}{3}$ and $\sin t = \dfrac{y}{2}$. Apply $\cos^2 t + \sin^2 t = 1$:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \quad \Longrightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = 1.$$

This is an **ellipse** with $a = 3,\ b = 2$, traced counterclockwise.

Writing a parametrization

To parametrize the line segment from $(1, 2)$ to $(5, -4)$ for $0 \le t \le 1$, use

$$x = 1 + (5 - 1)t = 1 + 4t, \qquad y = 2 + (-4 - 2)t = 2 - 6t.$$

At $t = 0$ you are at $(1, 2)$; at $t = 1$ at $(5, -4)$.

Tip: For trig parametrizations reach for the identity $\cos^2 t + \sin^2 t = 1$ (or $\sec^2 t - \tan^2 t = 1$ for hyperbolas) rather than solving for t. Always state the direction and any restriction on the resulting graph.

12.6 Polar Equations of Conics (Introduction)

Eccentricity form

With a focus at the pole, a conic has polar equation

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta},$$

where e is the eccentricity and $d > 0$ is the distance from focus to directrix.

- $e = 1 \Rightarrow$ **parabola**, $\quad 0 < e < 1 \Rightarrow$ **ellipse**, $\quad e > 1 \Rightarrow$ **hyperbola**.
- cos means the directrix is vertical; sin means it is horizontal.

Identify $r = \dfrac{6}{2 + \cos\theta}$

Put the constant term to 1 by dividing numerator and denominator by 2:

$$r = \frac{6}{2 + \cos\theta} = \frac{3}{1 + \frac{1}{2}\cos\theta}.$$

So $e = \frac{1}{2} < 1$: an **ellipse**. From $ed = 3$ and $e = \frac{1}{2}$ we get $d = 6$; the directrix is vertical.

Tip: Before reading off e, force the denominator into the form $1 \pm e(\cos \text{ or } \sin)\theta$ by dividing through by the leading constant. Only then is the coefficient of $\cos\theta$ or $\sin\theta$ equal to e.

12.7 Going Deeper: Advanced Conics & Parametrics

The line–conic tangency condition

Substitute a line $y = mx + c$ into a conic and collect a quadratic in x. The line is **tangent** exactly when that quadratic has a *double root*, i.e. its discriminant is 0 (secant if the discriminant is positive, misses if negative). Carrying this out once gives memorable tangency conditions for lines of slope m:

- **Ellipse** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$: tangent iff $c^2 = a^2m^2 + b^2$.
- **Hyperbola** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$: tangent iff $c^2 = a^2m^2 - b^2$.
- **Parabola** $y^2 = 4px$: tangent iff $c = \dfrac{p}{m}$ (line $y = mx + \frac{p}{m}$).

These same relations let you **find a conic** from tangency, point, or focus data: each condition gives one equation in the unknowns $a^2, b^2, p, \ldots$, and you solve the resulting system.

Worked: tangent lines of slope 2 to $\frac{x^2}{9} + \frac{y^2}{4} = 1$

Here $a^2 = 9,\ b^2 = 4,\ m = 2$. The tangency condition $c^2 = a^2m^2 + b^2$ gives

$$c^2 = 9(2)^2 + 4 = 40 \quad \Longrightarrow \quad c = \pm 2\sqrt{10}.$$

So the two tangent lines of slope 2 are $y = 2x + 2\sqrt{10}$ and $y = 2x - 2\sqrt{10}$. (Check: substituting either into the ellipse yields a perfect-square quadratic in x.)

Worked: build a conic from a focus and a tangency condition

Find the ellipse centered at the origin with horizontal major axis, a focus at $(3, 0)$, that is tangent to the line $y = x + 5$.

$$\begin{aligned} \text{focus condition:} \quad & c = 3 \Rightarrow a^2 - b^2 = c^2 = 9, \\ \text{tangency } (m = 1,\ c = 5): \quad & c^2 = a^2m^2 + b^2 \Rightarrow 25 = a^2 + b^2. \end{aligned}$$

Adding and subtracting: $a^2 = 17,\ b^2 = 8$. The ellipse is $\frac{x^2}{17} + \frac{y^2}{8} = 1$. Since $a^2 - b^2 = 9$, the foci are indeed $(\pm 3, 0)$, confirming the fit.

Reflection property (why conics focus light and sound):

- **Parabola:** rays traveling parallel to the axis reflect *through the focus* (and vice versa) — the principle behind satellite dishes and headlights.
- **Ellipse:** a ray leaving one focus reflects *to the other focus* — "whispering galleries."
- **Hyperbola:** a ray aimed at one focus reflects *away from the other focus*.

Focal chords and the latus rectum

A **focal chord** is any chord through a focus. The **latus rectum** is the focal chord perpendicular to the major/transverse axis; its half-length ℓ (the *semi-latus rectum*) controls the polar form below.

$$\text{parabola } y^2 = 4px: \ \text{L.R.} = |4p|; \qquad \text{ellipse \& hyperbola: L.R.} = \frac{2b^2}{a}, \quad \ell = \frac{b^2}{a}.$$

For an ellipse $\ell = a(1 - e^2)$ and for a hyperbola $\ell = a(e^2 - 1)$, so knowing a and e recovers the full shape. Every focal chord of a conic has semi-latus rectum ℓ as its harmonic-mean radius.

Classifying a general conic by $B^2 - 4AC$

The *full* second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

may include a cross term Bxy, which **rotates** the axes. The rotation-invariant **discriminant** $B^2 - 4AC$ classifies the curve without ever finding the rotation angle:

- $B^2 - 4AC < 0 \Rightarrow$ **ellipse** (circle if also $A = C,\ B = 0$),
- $B^2 - 4AC = 0 \Rightarrow$ **parabola**,
- $B^2 - 4AC > 0 \Rightarrow$ **hyperbola**.

When $B = 0$ this reduces to the earlier sign test on A and C. **Shortest distance to a conic:** the closest point of a smooth conic to an external line lies where a tangent is *parallel* to that line, so slide the tangency condition to the tangent line nearest the given one, then measure the gap between the two parallel lines.

Worked: classify rotated conics

- $xy = 1$: here $A = C = 0,\ B = 1$, so $B^2 - 4AC = 1 > 0 \Rightarrow$ **hyperbola** (rotated 45°).
- $5x^2 - 4xy + 8y^2 - 36 = 0$: $B^2 - 4AC = (-4)^2 - 4(5)(8) = 16 - 160 = -144 < 0 \Rightarrow$ **ellipse**.
- $x^2 + 2\sqrt{3}\,xy + 3y^2 - 8x = 0$: $B^2 - 4AC = (2\sqrt{3})^2 - 4(1)(3) = 12 - 12 = 0 \Rightarrow$ **parabola**.

Worked: shortest distance from a line to an ellipse

Find the shortest distance from the line $x + y = 6$ to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. The line has slope $m = -1$. The parallel tangents satisfy $c^2 = a^2m^2 + b^2 = 9 + 4 = 13$, so $c = \pm\sqrt{13}$; the one nearer the line is $y = -x + \sqrt{13}$, i.e. $x + y = \sqrt{13}$. The gap between the parallel lines $x + y - 6 = 0$ and $x + y - \sqrt{13} = 0$ is

$$d = \frac{|6 - \sqrt{13}|}{\sqrt{1^2 + 1^2}} = \frac{6 - \sqrt{13}}{\sqrt{2}} \approx \frac{2.394}{1.414} \approx 1.69.$$

Worked: eliminate the parameter (rational parametrization)

Consider $x = t + \dfrac{1}{t},\ y = t - \dfrac{1}{t}$ for $t \neq 0$. Rather than solving for t, combine the equations:

$x+y=2t$ and $x-y=\dfrac{2}{t}$, so their product removes t:

$$(x+y)(x-y)=(2t)\left(\frac{2}{t}\right)=4 \quad\Longrightarrow\quad x^2-y^2=4.$$

This is a **hyperbola** with $a=b=2$. Because $x=t+\frac{1}{t}$ satisfies $|x|\geq 2$, only the outer branches are traced: $t>0$ gives the right branch, $t<0$ the left. The same trick with $x=a\sec t,\ y=b\tan t$ uses $\sec^2 t-\tan^2 t=1$ to yield $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

Polar eccentricity form, revisited: in $r=\dfrac{\ell}{1\pm e\cos\theta}$ the numerator is the *semi-latus rectum* $\ell=ed$, so once you normalize the denominator to $1\pm e(\cos \text{ or } \sin)\theta$ you can read off both e (the coefficient) and ℓ (the numerator), then recover $a=\dfrac{\ell}{1-e^2}$ for an ellipse or $a=\dfrac{\ell}{e^2-1}$ for a hyperbola. **Pitfalls:** always normalize the leading denominator constant to 1 before reading e; when a cross term is present, classify with B^2-4AC (not the A-vs-C test); and for rational or trig parametrizations always state the branch or orientation restriction the algebra hides.

13 Conic Sections

Slice a cone, get every curve: circle, ellipse, parabola, hyperbola

13.1 Where Conics Come From: Slicing a Cone

One cone, four curves

Take a DOUBLE cone (two ice-cream cones tip to tip, extending forever) and cut it with a plane. The angle of the cut decides everything: a horizontal cut gives a **circle**; tilt the plane a little and the circle stretches into an **ellipse**.

(diagram in the online version)

The borderline cut: a parabola

Tilt the plane until it is exactly PARALLEL to the cone's slant side and the curve never closes up — that single borderline angle produces the **parabola**. It is the knife-edge between the closed curves (ellipses) and the doubly-open one (hyperbola).

(diagram in the online version)

Cutting both nappes: a hyperbola

Tilt past the parallel position (or cut straight down) and the plane slices BOTH halves of the double cone, producing the two branches of a **hyperbola**. The two branches are one curve — the same plane meeting the same cone, twice.

(diagram in the online version)

Why "conic sections" is literal

Circle, ellipse, parabola, hyperbola are not four unrelated graphs — they are the SAME construction (plane meets cone) at four tilts. That is why their equations are all degree-two: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ covers every one of them.

13.2 The Ellipse

Anatomy of an ellipse

Standard form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (with $a > b$): the long semi-axis is a, the short one b, and the two **foci** sit at $(\pm c, 0)$ with $c^2 = a^2 - b^2$. The defining property: every point of the ellipse has distances to the two foci summing to $2a$ (a loop of string around two pins traces one).

x/25 + y/9 = 1

$a = 5$, $b = 3$, so $c^2 = 25 - 9 = 16$ and the foci are $(\pm 4, 0)$. Check the string property at $(5, 0)$: distances 1 and 9 sum to $10 = 2a$ ✓. Eccentricity $e = \frac{c}{a} = \frac{4}{5}$ measures how stretched it is ($e = 0$ is a circle).

13.3 The Parabola

Focus and directrix

$x^2 = 4py$: focus $(0, p)$, directrix $y = -p$, and every point of the curve is EQUIDISTANT from the focus and the directrix. Satellite dishes are parabolas because incoming parallel rays all bounce to the focus.

y = x/8

Rewrite as $x^2 = 8y$: $4p = 8$, so $p = 2$ — focus $(0, 2)$, directrix $y = -2$. The point $(4, 2)$: distance to focus $= 4$, distance to directrix $= 4$ ✓.

13.4 The Hyperbola

Asymptotes are the skeleton

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ opens left-right with foci at $(\pm c, 0)$, $c^2 = a^2 + b^2$ (note the PLUS — foci sit outside), and asymptotes $y = \pm\frac{b}{a}x$: the branches hug these lines forever without touching. Defining property: the DIFFERENCE of the focal distances is constant ($2a$).

x/9 y/16 = 1

$a = 3$, $b = 4$: $c = 5$, foci $(\pm 5, 0)$, asymptotes $y = \pm\frac{4}{3}x$. At the vertex $(3, 0)$ the focal distances are 2 and 8: difference $6 = 2a$ ✓.

14 Course Review

Functions, polynomials, exponentials, all of trigonometry, polar & complex numbers, matrices, conics, sequences, and an introduction to limits

14.1 Functions and Their Graphs

Definition, domain & range

A **function** f assigns each input x (the **domain**) exactly one output $y = f(x)$; the outputs form the **range**. Passes the **vertical line test**.

- Polynomial: domain $(-\infty, \infty)$.
- Radical $\sqrt{g(x)}$: require $g(x) \geq 0$.
- Rational $\frac{p(x)}{q(x)}$: require $q(x) \neq 0$.

Symmetry: even if $f(-x) = f(x)$ (y-axis); odd if $f(-x) = -f(x)$ (origin).

Difference quotient, rate of change, composition, inverses

Difference quotient: $\frac{f(x+h) - f(x)}{h}$, $h \neq 0$. **Average rate of change** on $[a, b]$: $\frac{f(b) - f(a)}{b - a}$ (secant slope).

Combinations: $(f \pm g)(x) = f(x) \pm g(x)$, $(fg)(x) = f(x)g(x)$, $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$.

Composition: $(f \circ g)(x) = f\big(g(x)\big)$ (not commutative). Domain: those x in dom g with $g(x)$ in dom f.

Inverse: exists iff f is one-to-one (horizontal line test). Swap x, y and solve; $f\big(f^{-1}(x)\big) = x$ and $f^{-1}\big(f(x)\big) = x$; graphs reflect over $y = x$.

Parent functions & transformations of $y = a\,f(bx - c) + d$

Parents: $c,\ x,\ |x|,\ \sqrt{x},\ x^2,\ x^3,\ \dfrac{1}{x},\ \lfloor x \rfloor$.

- $+d$ up / $-d$ down; $f(x-c)$ right c, $f(x+c)$ left (horizontal moves oppose the sign).
- $a < 0$: reflect over x-axis; $|a| > 1$ vertical stretch, $0 < |a| < 1$ shrink.
- $b < 0$: reflect over y-axis; $|b| > 1$ horizontal shrink, $0 < |b| < 1$ stretch (factor: $f\big(b(x - \frac{c}{b})\big)$).

Domain, composition, inverse

$f(x) = \dfrac{\sqrt{x+2}}{x-3}$: need $x \geq -2$ and $x \neq 3$, so domain $[-2, 3) \cup (3, \infty)$.

$f(x) = 2x + 6 \Rightarrow$ swap $x = 2y + 6 \Rightarrow f^{-1}(x) = \dfrac{x-6}{2}$. Check: $2 \cdot \frac{x-6}{2} + 6 = x$. ✓

Tip: Horizontal transformations and shifts always act *opposite* to their sign; vertical ones act as written. Restrict the domain (e.g. $x \geq 0$ for x^2) to force a one-to-one function before inverting.

14.2 Polynomial and Rational Functions

End behavior & zeros

For leading term $a_n x^n$: if n even, both ends same way ($a_n > 0$ up/up, $a_n < 0$ down/down); if n odd, ends opposite ($a_n > 0$ down/up). A zero of **multiplicity** m: graph crosses if m odd, touches (turns) if m even. A degree-n polynomial has at most $n - 1$ turning points.

Division theorems & rational zeros

Division algorithm: $f(x) = d(x)q(x) + r(x)$.

Remainder Theorem: dividing f by $(x - k)$ leaves remainder $f(k)$.

Factor Theorem: $(x - k)$ is a factor $\iff f(k) = 0$.

Rational Zero Test: any rational zero of a polynomial with integer coefficients has the form $\dfrac{p}{q}$, $p \mid$ (constant term), $q \mid$ (leading coefficient).

Fundamental Thm of Algebra: degree n ($n \geq 1$) has exactly n complex zeros (with multiplicity). Complex zeros of a *real* polynomial come in conjugate pairs $a \pm bi$.

Rational functions $f(x) = \dfrac{N(x)}{D(x)}$

- **Vertical asymptote** at zeros of D not cancelled by N; a cancelled factor gives a **hole**.
- **Horizontal asymptote:** $\deg N < \deg D \Rightarrow y = 0$; equal degree $\Rightarrow y = \dfrac{\text{lead } N}{\text{lead } D}$; $\deg N > \deg D \Rightarrow$ none.
- **Slant asymptote** when $\deg N = \deg D + 1$: the quotient from long division.

Synthetic division by $(x-2)$

Divide $x^3 - 4x^2 + x + 6$ by $(x-2)$:

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 1 & 6 \\ & & 2 & -4 & -6 \\ \hline & 1 & -2 & -3 & 0 \end{array}$$

Remainder 0, so $f(2) = 0$ and $x^3 - 4x^2 + x + 6 = (x-2)(x^2 - 2x - 3) = (x-2)(x-3)(x+1)$.

Tip: A remainder of 0 in synthetic division means both "$(x-k)$ is a factor" and "$f(k) = 0$." Use it to peel off one zero at a time until the quotient is quadratic, then factor or use the quadratic formula.

14.3 Exponential and Logarithmic Functions

Exponential functions and e

$f(x) = a\,b^x$ $(b > 0,\ b \neq 1)$: horizontal asymptote $y = 0$, domain $(-\infty, \infty)$, range $(0, \infty)$; grows if $b > 1$, decays if $0 < b < 1$. The natural base $e \approx 2.71828$.

Compound interest: $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$; continuous: $A = Pe^{rt}$.

Logarithms & properties

$y = \log_b x \iff b^y = x$ (so log is the inverse of b^x); $\ln x = \log_e x$. Domain $(0, \infty)$.

$$\log_b(MN) = \log_b M + \log_b N \qquad \log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b(M^p) = p \log_b M \qquad \log_b b = 1, \quad \log_b 1 = 0$$

Inverse: $b^{\log_b x} = x$, $\log_b b^x = x$. **Change of base:** $\log_b x = \dfrac{\ln x}{\ln b} = \dfrac{\log x}{\log b}$.

Solving exponential & logarithmic equations

$3e^{2x} = 24 \Rightarrow e^{2x} = 8 \Rightarrow 2x = \ln 8 \Rightarrow x = \frac{1}{2}\ln 8 = \ln 2\sqrt{2}$.

$\log_2(x) + \log_2(x-2) = 3 \Rightarrow \log_2\big(x(x-2)\big) = 3 \Rightarrow x^2 - 2x = 8 \Rightarrow (x-4)(x+2) = 0$. Reject $x = -2$ (domain), so $x = 4$.

Tip: Condense to a single log before exponentiating, or take a log of both sides to bring an exponent down. Always check solutions against the domain $x > 0$ of every logarithm; discard extraneous roots.

14.4 Trigonometric Functions

Angles, radians & arc length

$180° = \pi$ rad. **Arc length:** $s = r\theta$; **area of sector:** $A = \frac{1}{2}r^2\theta$; **angular/linear speed:** $v = r\omega$ (θ in radians). Coterminal: add multiples of $360°$ or 2π.

The six functions & unit circle

On the unit circle a terminal point is $(\cos\theta, \sin\theta)$; $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, and $\csc, \sec, \cot$ are the reciprocals of $\sin, \cos, \tan$. Right triangle: $\sin = \dfrac{\text{opp}}{\text{hyp}}$, $\cos = \dfrac{\text{adj}}{\text{hyp}}$, $\tan = \dfrac{\text{opp}}{\text{adj}}$.

First-quadrant exact values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	und.

Signs by quadrant: **ASTC** — All, Sine, Tangent, Cosine positive in QI–QIV.

Graphs & inverse trig

For $y = a\sin(bx - c) + d$ (same for cos): **amplitude** $|a|$, **period** $\dfrac{2\pi}{b}$, **phase shift** $\dfrac{c}{b}$, midline $y = d$. For tan the period is $\dfrac{\pi}{b}$.

Inverse ranges: $\arcsin x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arccos x \in [0, \pi]$, $\arctan x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Reading a sine graph

$y = 3\sin\left(2x - \frac{\pi}{2}\right) + 1$: amplitude 3, period $\dfrac{2\pi}{2} = \pi$, phase shift $\dfrac{\pi/2}{2} = \dfrac{\pi}{4}$ right, midline $y = 1$ (so y ranges from -2 to 4).

Tip: Use a reference angle (acute angle to the x-axis) plus ASTC to get exact values in any quadrant. Memorize QI values; every other value is one of them with a sign attached.

14.5 Analytic Trigonometry

Fundamental identities

Reciprocal/Quotient: $\csc = \frac{1}{\sin}$, $\sec = \frac{1}{\cos}$, $\cot = \frac{1}{\tan} = \frac{\cos}{\sin}$, $\tan = \frac{\sin}{\cos}$.
Pythagorean: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$.
Even/Odd: $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.
Cofunction: $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$, $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$.

Sum/difference, double, half, power-reducing, product-to-sum

$$\begin{aligned}
\sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B \\
\cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B \\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \\
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \\
\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

Power-reducing: $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$, $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$.
Half-angle: $\sin\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 - \cos\theta}{2}}$, $\cos\dfrac{\theta}{2} = \pm\sqrt{\dfrac{1 + \cos\theta}{2}}$, $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}$.
Product-to-sum: $\sin A\sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$, $\cos A\cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$, $\sin A\cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$.

Exact value & solving

$\cos 15° = \cos(45° - 30°) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$2\cos^2 x - \cos x - 1 = 0 \Rightarrow (2\cos x + 1)(\cos x - 1) = 0 \Rightarrow \cos x = -\frac{1}{2}$ or 1; on $[0, 2\pi)$: $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$.

Tip: To verify an identity, work *one side only* — convert to sin / cos, get a common denominator, and apply a Pythagorean identity. To solve, factor to one function and remember to add $+2\pi n$ (sin, cos) or $+\pi n$ (tan) for all solutions.

14.6 Triangle Trigonometry & Vectors

Laws of Sines and Cosines, area

Law of Sines: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ (cases AAS, ASA, SSA).

Ambiguous case (SSA): given a, b, A with A acute and $h = b \sin A$: no triangle if $a < h$; one if $a = h$ or $a \geq b$; two if $h < a < b$.

Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$ (cases SAS, SSS).

Area: $\frac{1}{2}ab\sin C$; **Heron:** $\sqrt{s(s-a)(s-b)(s-c)}$ with $s = \dfrac{a+b+c}{2}$.

Vectors, dot product & projection

$\mathbf{v} = \langle v_1, v_2 \rangle$, magnitude $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$, direction angle θ with $\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta \rangle$. Unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|}$; standard form $v_1\mathbf{i} + v_2\mathbf{j}$.

Dot product: $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$. Angle: $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$. Orthogonal $\iff \mathbf{u}\cdot\mathbf{v} = 0$.

Projection: $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

Law of Cosines (SAS)

$a = 5,\ b = 8,\ C = 60°$: $c^2 = 25 + 64 - 2(5)(8)\cos 60° = 89 - 80 \cdot \frac{1}{2} = 49$, so $c = 7$. Area $= \frac{1}{2}(5)(8)\sin 60° = 10\sqrt{3}$.

Tip: Choose the Law of Cosines when you know all three sides (SSS) or two sides and the included angle (SAS); otherwise use the Law of Sines. In the SSA case always test whether a *second* triangle (supplementary angle) also fits.

14.7 Polar Coordinates & Complex Numbers

Polar & rectangular conversion

Point (r, θ). **Polar $\to$ rectangular:** $x = r\cos\theta,\ y = r\sin\theta$. **Rectangular $\to$ polar:** $r = \sqrt{x^2+y^2},\ \tan\theta = \dfrac{y}{x}$ (choose θ by the quadrant of (x, y)).

Common curves: circle $r = a$; line $\theta = \alpha$; rose $r = a\cos(n\theta)$ (n petals if odd, $2n$ if even); limaçon $r = a \pm b\cos\theta$; cardioid when $a = b$.

Complex numbers & trig form

$z = a + bi,\ |z| = r = \sqrt{a^2+b^2}$. **Trig (polar) form:** $z = r(\cos\theta + i\sin\theta)$ with $\tan\theta = \dfrac{b}{a}$.

Product/Quotient: $z_1z_2 = r_1r_2\big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big]$, $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\big[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big]$.

DeMoivre: $z^n = r^n\big(\cos n\theta + i\sin n\theta\big)$.

nth roots: $\sqrt[n]{r}\left[\cos\dfrac{\theta+2\pi k}{n} + i\sin\dfrac{\theta+2\pi k}{n}\right]$, $k = 0, 1, \ldots, n-1$ (equally spaced by $\dfrac{2\pi}{n}$).

DeMoivre's Theorem

$\big(1+i\big)^8$: here $r = \sqrt{2},\ \theta = 45°$, so $(1+i)^8 = (\sqrt{2})^8\big(\cos 360° + i\sin 360°\big) = 16(1+0i) = 16$.

Tip: Convert to trig form before raising to a power or taking a root — DeMoivre turns messy multiplication into multiplying moduli and *adding* angles. The n distinct nth roots all share modulus $\sqrt[n]{r}$ and sit at equal angular spacing.

14.8 Systems, Matrices & Determinants

Solving systems & Gaussian elimination

Methods: substitution, elimination, or row-reduce the augmented matrix to **row-echelon form** (leading 1's, zeros below) then back-substitute. Row operations: swap rows, multiply a row by a nonzero constant, add a multiple of one row to another. A system is *inconsistent* (no solution) if a row gives $0 =$ nonzero; *dependent* (infinitely many) if a variable is free.

Matrix operations, inverse, determinants, Cramer

Add/subtract entrywise (same size). **Product** AB: rows of A dotted with columns of B; defined when (cols of A) = (rows of B); not commutative. Identity I: $AI = IA = A$.

2×2 inverse: for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det A = ad - bc$ and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (exists iff $\det A \neq 0$).

3×3 determinant by cofactor expansion. **Cramer's Rule:** $x_i = \dfrac{\det A_i}{\det A}$, where A_i replaces column i with the constants (requires $\det A \neq 0$).

Partial fraction decomposition

For a proper rational $\dfrac{N(x)}{D(x)}$ (divide first if improper), split by the factors of D:

- Linear $(ax+b)$: term $\dfrac{A}{ax+b}$; repeated $(ax+b)^k$: one term per power up to k.
- Irreducible quadratic (ax^2+bx+c): term $\dfrac{Bx+C}{ax^2+bx+c}$.

Clear denominators and match coefficients (or substitute convenient x-values) to solve.

2×2 inverse and Cramer

$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $\det A = 2(3) - 1(5) = 1$, so $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

For $\begin{cases} 2x + y = 5 \\ 5x + 3y = 13 \end{cases}$: $x = \dfrac{\det \begin{bmatrix} 5 & 1 \\ 13 & 3 \end{bmatrix}}{1} = \dfrac{15-13}{1} = 2$, $y = \dfrac{\det \begin{bmatrix} 2 & 5 \\ 5 & 13 \end{bmatrix}}{1} = 26 - 25 = 1$.

Tip: An inverse (and a unique solution via Cramer) exists only when $\det A \neq 0$. If $\det A = 0$ the system is either inconsistent or dependent — fall back on row reduction to tell which.

14.9 Conic Sections & Parametric Equations

Standard forms (center/vertex (h,k))

Circle: $(x-h)^2 + (y-k)^2 = r^2$.

Parabola: $(x-h)^2 = 4p(y-k)$ opens up/down; $(y-k)^2 = 4p(x-h)$ opens left/right. Focus is p from vertex; directrix is the opposite side; $|4p|$ is the latus rectum.

Ellipse: $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, $a>b$: major axis length $2a$, minor $2b$, foci c from center with $c^2=a^2-b^2$.

Hyperbola: $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$: vertices a from center, foci with $c^2=a^2+b^2$, asymptotes $y-k=\pm\dfrac{b}{a}(x-h)$.

Eccentricity & parametric equations

Eccentricity $e=\dfrac{c}{a}$: circle $e=0$; ellipse $0<e<1$; parabola $e=1$; hyperbola $e>1$.

Parametric: $x=f(t),\ y=g(t)$ traces a curve with orientation as t increases. **Eliminate the parameter** by solving for t (or using $\cos^2 t+\sin^2 t=1$) to get a relation in x, y.

Complete the square; eliminate the parameter

$x^2+4y^2-4x=0 \Rightarrow (x-2)^2+4y^2=4 \Rightarrow \dfrac{(x-2)^2}{4}+y^2=1$: ellipse, center $(2,0)$, $a=2,\ b=1$.

$x=2\cos t,\ y=3\sin t \Rightarrow \left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=\cos^2 t+\sin^2 t=1$: ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$.

Tip: Read the conic from the signs: two squared terms *added* with equal denominators = circle, unequal = ellipse; *subtracted* = hyperbola; only *one* squared term = parabola. Complete the square to reach standard form.

14.10 Sequences, Series & the Binomial Theorem

Arithmetic & geometric

Arithmetic (common difference d): $a_n=a_1+(n-1)d$; sum $S_n=\dfrac{n}{2}(a_1+a_n)=\dfrac{n}{2}\big[2a_1+(n-1)d\big]$.

Geometric (common ratio r): $a_n=a_1r^{n-1}$; sum $S_n=a_1\dfrac{1-r^n}{1-r}$.

Infinite geometric: converges iff $|r|<1$, then $S=\dfrac{a_1}{1-r}$.

Sigma notation, induction, counting

Sigma: $\displaystyle\sum_{i=1}^{n} a_i=a_1+a_2+\cdots+a_n$. Useful: $\displaystyle\sum_{i=1}^{n} i=\frac{n(n+1)}{2}$.

Mathematical induction: prove a base case ($n = 1$), then assume $P(k)$ and prove $P(k+1)$; concludes $P(n)$ for all n.

Counting: permutations ${}_nP_r = \dfrac{n!}{(n-r)!}$ (order matters); combinations ${}_nC_r = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ (order does not).

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

The $(k+1)$th term is $\dbinom{n}{k} a^{n-k} b^k$. Coefficients form Pascal's triangle; $\dbinom{n}{k} = \dbinom{n}{n-k}$.

Series values & expansion

Infinite geometric $3 + 1 + \frac{1}{3} + \cdots$: $r = \frac{1}{3}$, so $S = \dfrac{3}{1 - \frac{1}{3}} = \dfrac{3}{\frac{2}{3}} = \dfrac{9}{2}$.

$(x+2)^4 = x^4 + 4x^3(2) + 6x^2(4) + 4x(8) + 16 = x^4 + 8x^3 + 24x^2 + 32x + 16.$

Tip: Check for a common *difference* (arithmetic) versus a common *ratio* (geometric) before choosing a formula. An infinite geometric series has a finite sum only when $|r| < 1$.

14.11 Introduction to Limits

Meaning & limit laws

$\lim_{x \to c} f(x) = L$ means $f(x)$ approaches L as $x \to c$. It exists iff the one-sided limits agree: $\lim_{x \to c^-} f = \lim_{x \to c^+} f$. Limits ignore the actual value $f(c)$.

For constants/sums/products/quotients the limit distributes:

$$\lim(f \pm g) = \lim f \pm \lim g, \quad \lim(fg) = \lim f \cdot \lim g, \quad \lim \frac{f}{g} = \frac{\lim f}{\lim g} \ (\lim g \neq 0).$$

Polynomials/rationals: substitute c when the denominator is nonzero.

Indeterminate forms, infinite limits, limits at infinity

$\frac{0}{0}$ **form:** factor and cancel, rationalize, or simplify, then substitute.

Infinite limits: if $f \to \pm\infty$ near c there is a vertical asymptote $x = c$.

Limits at infinity (end behavior of rationals): compare degrees like horizontal asymptotes — lower top $\to 0$, equal $\to$ ratio of leading coefficients, higher top $\to \pm\infty$.

Continuity & the derivative

f is **continuous** at c if all hold: (1) $f(c)$ defined, (2) $\lim_{x\to c} f(x)$ exists, (3) $\lim_{x\to c} f(x) = f(c)$.

Limit definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

the instantaneous rate of change / slope of the tangent line (the difference quotient as $h \to 0$).

A $\frac{0}{0}$ limit and a derivative

$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$

For $f(x) = x^2$: $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$

Tip: Try direct substitution first. If you get $\frac{0}{0}$, the factor causing it cancels — factor, rationalize, or simplify, then substitute again. A nonzero-over-zero result signals an infinite limit (vertical asymptote), not 0.