# MathCastle

## Geometry

Complete course booklet • 12 topics

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## Contents

# 1 Core Ideas in Plain Terms

*Shapes and reasoning, in everyday language*

## 1.1 Angles and Lines

**Angles in a shape add to a fixed total**

The **Triangle Angle-Sum Theorem** says the three angles of any triangle add to 180°; a quadrilateral adds to 360° — both come from the polygon angle-sum formula $(n-2)\cdot 180°$. So a missing angle is just the total minus the ones you know. When a line crosses two parallel lines, it makes matching (equal) angles — the key to "angle chasing."

*(diagram in the online version)*

**Find the third angle**

A triangle has angles 50° and 60°; by the Triangle Angle-Sum Theorem the third is $180° - 110° = 70°$.

## 1.2 Triangles and the Pythagorean Theorem

**Right triangles: a + b = c**

In a right triangle, the two short sides (legs) and the longest side (hypotenuse) satisfy the **Pythagorean theorem**: $a^2 + b^2 = c^2$. A few side-triples come up constantly — 3-4-5, 5-12-13 — so recognizing them saves time.

*(diagram in the online version)*

**Find the hypotenuse**

Legs 3 and 4: by the Pythagorean theorem, $c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

## 1.3 Similarity, Area, and Volume

**Same shape, scaled size**

Two figures are **similar** if one is a scaled copy of the other — all angles equal, all sides in the same ratio. If you scale lengths by $k$, areas scale by $k^2$ and volumes by $k^3$. That is why doubling a shape's size quadruples its area.

*(diagram in the online version)*

**Area of a triangle**

Using the triangle-area formula $A = \frac{1}{2}bh$: base 6, height 4 gives $A = \frac{1}{2} \cdot 6 \cdot 4 = 12$.

### 1.4 Going Deeper: Circles and Coordinates

**Inscribed angles are half the central angle**

An angle drawn from the circle's edge sees an arc at half the angle drawn from the center. Two consequences worth knowing cold: every inscribed angle in a semicircle is $90°$, and inscribed angles on the same arc are equal.

*(diagram in the online version)*

**Use it**

If arc $AB$ measures $80°$, an inscribed angle standing on $AB$ measures $40°$ — no matter where on the far side of the circle its vertex sits.

**Distance and midpoint are Pythagoras in disguise**

The distance between two points is $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ — the hypotenuse of the right triangle their coordinates make. From $(0, 0)$ to $(3, 4)$: $\sqrt{9 + 16} = 5$. The midpoint just averages the coordinates.

**Power of a Point (multi-step)**

Chords $AB$ and $CD$ meet at $P$ inside a circle, with $AP = 4$, $PB = 6$, and $CP = 3$. The **Power of a Point** theorem says $AP \cdot PB = CP \cdot PD$. Step 1: $AP \cdot PB = 4 \cdot 6 = 24$. Step 2: $PD = 24 \div 3 = 8$.

*(diagram in the online version)*

**Ptolemy's theorem (multi-step)**

Quadrilateral $ABCD$ is inscribed in a circle with diagonal $AC = 25$ a diameter, and sides $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$. **Ptolemy's theorem** says $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Step 1: $7 \cdot 20 + 15 \cdot 24 = 140 + 360 = 500$. Step 2: $BD = 500 \div 25 = 20$.

### 1.5 Problem-Solving Playbook

**Add a line**

When a figure gives you nothing to compute with, **add an auxiliary line**: a radius to a tangent point, an altitude, a diagonal. Each new line creates triangles — and triangles bring the Pythagorean theorem, similarity, and area formulas into play.

**Worked: Heron's formula**

A triangle has sides 13, 14, 15. **Heron's formula**: with $s = \dfrac{13+14+15}{2} = 21$, the area is $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84$.

# 2 Transformations

*Slides, flips, turns, and resizes on the coordinate plane*

## 2.1 Transformation Vocabulary

A **transformation** moves or changes a figure according to a rule.

- **Pre-image:** the original figure (points $A$, $B$, $C$).
- **Image:** the new figure after the transformation (points $A'$, $B'$, $C'$, read "$A$ prime").
- **Mapping:** the pairing of each pre-image point with its image point. We write $A \to A'$.
- **Rigid motion** (also called an **isometry**): a transformation that keeps the same size and shape. Translations, reflections, and rotations are rigid.
- **Non-rigid** transformation: changes the size. A **dilation** is non-rigid.

A rigid motion moves triangle $ABC$ to triangle $A'B'C'$ without stretching it. Every side keeps its length and every angle keeps its measure, so the two triangles are **congruent**.

**Tip:** If you can slide, flip, or turn a figure onto its image, the two are congruent. If you must also grow or shrink it, they are only *similar*.

## 2.2 Translations (Slides)

A **translation** slides every point the same distance in the same direction. If you move $a$ units horizontally and $b$ units vertically:

$$(x, y) \to (x+a,\ y+b).$$

The pair $\langle a, b\rangle$ is the **translation vector**. Positive $a$ is right, negative is left; positive $b$ is up, negative is down.

*A translation slides.*

*(diagram in the online version)*

Translate triangle $A(1,1)$, $B(4,1)$, $C(1,3)$ by the rule $(x,y) \to (x+2,\ y+1)$.

$$A'(3,2),\quad B'(6,2),\quad C'(3,4).$$

**Tip:** "Right/left" changes only $x$; "up/down" changes only $y$. A slide never turns or flips the figure.

### 2.3 Reflections (Flips)

A **reflection** flips a figure over a **line of reflection**, producing a mirror image. Coordinate rules:

$$\begin{aligned}
\text{over the } x\text{-axis:}\quad & (x,y) \to (x,\ -y)\\
\text{over the } y\text{-axis:}\quad & (x,y) \to (-x,\ y)\\
\text{over } y = x:\quad & (x,y) \to (y,\ x)\\
\text{over } y = -x:\quad & (x,y) \to (-y,\ -x)
\end{aligned}$$

*A reflection flips across a line.*

*(diagram in the online version)*

Reflect triangle $A(1,1)$, $B(4,1)$, $C(1,3)$ over the $x$-axis using $(x,y) \to (x,-y)$.

$$A'(1,-1),\quad B'(4,-1),\quad C'(1,-3).$$

Reflect $P(3,5)$ over the line $y = x$ using $(x,y) \to (y,x)$: the coordinates swap, giving $P'(5,3)$. Over $y = -x$, the rule $(x,y) \to (-y,-x)$ gives $P''(-5,-3)$.

**Tip:** A point on the line of reflection does not move. Watch the signs: over the $x$-axis only $y$ changes sign; over the $y$-axis only $x$ changes sign.

### 2.4 Rotations (Turns) About the Origin

A **rotation** turns a figure about a fixed point (the **center**). Rotating **counterclockwise** about the origin:

$$\begin{aligned} 90^\circ: &\quad (x, y) \to (-y,\, x) \\ 180^\circ: &\quad (x, y) \to (-x,\, -y) \\ 270^\circ: &\quad (x, y) \to (y,\, -x) \end{aligned}$$

A $90^\circ$ *clockwise* turn equals a $270^\circ$ counterclockwise turn (and vice versa). A $180^\circ$ turn is the same either direction.

*A rotation turns about a point.*

*(diagram in the online version)*

Rotate triangle $A(1, 1)$, $B(4, 1)$, $C(1, 3)$ by $90^\circ$ counterclockwise using $(x, y) \to (-y, x)$.

$$A'(-1, 1), \quad B'(-1, 4), \quad C'(-3, 1).$$

**Tip:** For $90^\circ$ turns, swap the coordinates first, then fix signs. Counterclockwise is the positive direction in math.

### 2.5 Compositions of Transformations

A **composition** applies one transformation and then another, in order. Do the *first* transformation, write the image, then apply the *second* transformation to that image.

Reflect $(1, 2)$ over the $x$-axis, *then* translate by $(x, y) \to (x + 3, y)$.

$$\begin{aligned} \text{Step 1 (reflect):} &\quad (1, 2) \to (1, -2) \\ \text{Step 2 (translate):} &\quad (1, -2) \to (4, -2) \end{aligned}$$

The final image is $(4, -2)$.

Two reflections over the $x$-axis and then the $y$-axis have the same effect as one 180° rotation about the origin, because $(x, y) \to (x, -y) \to (-x, -y)$.

**Tip:** Order matters! Reflecting then rotating usually gives a different image than rotating then reflecting. Always finish step 1 completely before starting step 2.

### 2.6 Dilations (Resizing) About the Origin

A **dilation** centered at the origin multiplies every coordinate by the same **scale factor** $k$:

$$(x, y) \to (kx,\ ky).$$

- $k > 1$: an **enlargement** (the figure grows).
- $0 < k < 1$: a **reduction** (the figure shrinks).
- $k = 1$: the figure is unchanged.

A dilation is *not* a rigid motion, but it keeps the same shape, so the image is **similar** to the pre-image.

Dilate triangle $A(1, 1)$, $B(2, 1)$, $C(1, 2)$ by scale factor $k = 2$ using $(x, y) \to (2x, 2y)$.

$$A'(2, 2), \quad B'(4, 2), \quad C'(2, 4).$$

**Tip:** The scale factor of the *lengths* is $k$, but the *area* scales by $k^2$. Doubling the sides ($k = 2$) makes the area 4 times as large.

### 2.7 Symmetry

**Line (reflectional) symmetry:** a figure has line symmetry if a reflection over some line maps the figure exactly onto itself. That line is a **line of symmetry**.

**Rotational symmetry:** a figure has rotational symmetry if a rotation of less than 360° about its center maps it onto itself. The **order** is how many times it matches in one full turn; the **angle** of rotation is 360° ÷ order.

A regular hexagon has 6 lines of symmetry. Its rotational symmetry has **order** 6, with smallest angle $360° \div 6 = 60°$.

A square has 4 lines of symmetry (two diagonals and two through opposite side midpoints) and rotational symmetry of order 4 (90°). A non-square rectangle has only 2 lines of symmetry and rotational symmetry of order 2 (180°).

**Tip:** Every regular polygon with $n$ sides has exactly $n$ lines of symmetry and rotational symmetry of order $n$.

### 2.8 Congruence vs. Similarity

**Rigid Motions vs. Dilations**

- **Rigid motions** (translations, reflections, rotations, and their compositions) preserve size and shape. The image is **congruent** to the pre-image.
- **Dilations** (scale factor $k \neq 1$) preserve shape but change size. The image is **similar**, but not congruent, to the pre-image.

Two figures are **congruent** if a sequence of rigid motions maps one onto the other. They are **similar** if a sequence of rigid motions *and* dilations does so.

**Tip:** Congruent $\Rightarrow$ similar (with ratio 1), but similar does not always mean congruent. Ask: "Did the size change?" If yes, it is similar only.

### 2.9 Going Deeper: Advanced Transformations

**Composing Two Reflections**

Reflecting twice in a row is always a *single* rigid motion. Which one depends on the two mirror lines:

- **Parallel mirrors** $\to$ a **translation**. The shift is *perpendicular* to the mirrors, in the direction from the first mirror to the second, and its distance is $2d$, where $d$ is the gap between the mirrors.
- **Intersecting mirrors** $\to$ a **rotation** about the point where they cross. If the angle from the first mirror to the second is $\theta$, the rotation angle is $2\theta$ (same turning direction).

Either way, two flips restore the original **orientation**, so the result is an "even" (direct) isometry.

*A reflection flips across a line.*

*(diagram in the online version)*

**Parallel mirrors give a translation.** Reflect $P(0,2)$ over the line $x=1$, then over $x=3$. The mirrors are $d=2$ apart, so the net move is a translation of $2d=4$ units to the right.

$$\begin{aligned}\text{Reflect over } x=1: &\quad (0,2)\to(2,2)\\ \text{Reflect over } x=3: &\quad (2,2)\to(4,2)\end{aligned}$$

Net effect: $(0,2)\to(4,2)$, i.e. $(x,y)\to(x+4,y)$.

**Intersecting mirrors give a rotation.** Reflect $P(3,1)$ over the $x$-axis, then over the line $y=x$. These mirrors cross at the origin, and the angle from the $x$-axis to $y=x$ is $45°$, so the result is a rotation of $2(45°)=90°$ about the origin.

$$\begin{aligned}\text{Reflect over } x\text{-axis}: &\quad (3,1)\to(3,-1)\\ \text{Reflect over } y=x: &\quad (3,-1)\to(-1,3)\end{aligned}$$

Indeed $(3,1)\to(-1,3)$ is exactly the $90°$ counterclockwise rule $(x,y)\to(-y,x)$.

### Rotations About Any Center & General Lines

**Rotation about a center** $(a,b)$: shift the center to the origin, rotate, then shift back.

$$(x,y)\xrightarrow{\text{subtract}}(x-a,\ y-b)\xrightarrow{\text{rotate}}\cdots\xrightarrow{\text{add}}(\text{result}).$$

For a $90°$ counterclockwise turn about $(a,b)$ this collapses to

$$(x,y)\to\big(a-(y-b),\ \ b+(x-a)\big).$$

**Reflection over a general line** $y=mx+c$: it still sends each point to the opposite side at equal perpendicular distance. A clean way is to compose known moves — translate so the line passes through the origin, rotate the line onto the $x$-axis, reflect over the $x$-axis, then undo the rotation and translation.

**Rotate about a non-origin center.** Rotate $P(4,1)$ by $90°$ counterclockwise about the center $O(2,1)$.

$$\begin{aligned} \text{Subtract center:} \quad & (4,1)-(2,1)=(2,0) \\ \text{Rotate } 90^\circ \ (x,y)\to(-y,x): \quad & (2,0)\to(0,2) \\ \text{Add center back:} \quad & (0,2)+(2,1)=(2,3) \end{aligned}$$

So $P'(2,3)$: the point swings a quarter turn around $O$, not around the origin.

### Transformations as Matrices (Preview)

Every rotation, reflection, and dilation *about the origin* can be written as multiplication by a $2\times 2$ matrix acting on the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$:

$$\text{Rotate } 90^\circ \text{ CCW}: \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{Reflect over } x\text{-axis}: \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\text{Reflect over } y=x: \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{Dilation, factor } k: \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

For example, the $90^\circ$ rotation of $(3,1)$ is

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix}=\begin{bmatrix} -1 \\ 3 \end{bmatrix},$$

matching $(x,y)\to(-y,x)$. A *translation* is not linear, so it needs a separate added vector $\begin{bmatrix} a \\ b \end{bmatrix}$ (or a $3\times 3$ "homogeneous" matrix you will meet later). **Composing** transformations then corresponds to *multiplying* their matrices.

### Glide Reflections

A **glide reflection** is a reflection over a line followed by a translation *parallel* to that line (the order does not matter). It is the fourth basic rigid motion of the plane, alongside translations, rotations, and reflections.

- It *reverses* orientation (like a single reflection), so it is an "odd" isometry.
- It has *no* fixed points and no fixed line pointwise — the classic footprint pattern of a walking person is a glide reflection.

Here $T$ is reflected over the $x$-axis and slid 2 units right to land on $T'$.

### Symmetry Groups & Order of Rotational Symmetry

The set of all rigid motions that map a figure onto itself forms its **symmetry group**.

- **Cyclic group** $C_n$: only rotational symmetry — $n$ rotations (including the 360° "do nothing"), and *no* lines of symmetry. A pinwheel or "S" shape is like this.
- **Dihedral group** $D_n$: rotational *and* reflective symmetry — $n$ rotations plus $n$ reflections, giving $2n$ symmetries in all.

A regular $n$-gon has symmetry group $D_n$: order-$n$ rotational symmetry (smallest angle $360°/n$) *and* $n$ lines of symmetry, for $2n$ total symmetries. An equilateral triangle is $D_3$ (6 symmetries); a square is $D_4$ (8 symmetries).

**Dilation — area & orientation:** Under a dilation with scale factor $k$, lengths scale by $|k|$, *areas* scale by $k^2$, and angles are unchanged. When $k > 0$ orientation is preserved; a *negative* $k$ acts like a positive dilation of factor $|k|$ combined with a 180° rotation about the center, which also preserves orientation. So a dilation never produces a mirror image — only a resized, similar copy.

# 3 Points, Lines & Planes

*Foundations: undefined terms, segments, midpoints & distance*

## 3.1 Undefined Terms: Point, Line, Plane

**The Three Building Blocks**

In geometry we start with three ideas so basic that we do not even define them. We describe them instead.

- **Point** — a single exact location. It has no size. We draw a dot and name it with a capital letter, like $P$.
- **Line** — a straight path of points that goes on forever in both directions. Name it by any two of its points with a line symbol, $\overleftrightarrow{AB}$, or by a lowercase letter, $\ell$.
- **Plane** — a perfectly flat surface that goes on forever. Name it with a capital script letter or by three points that are not all on one line, like plane $ABC$.

**Collinear** points lie on the same line. **Coplanar** points lie on the same plane.

**Naming a line and its points**

The points $A$, $B$, and $C$ below are *collinear* because they all lie on one line. That line can be named $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$, or simply $\ell$.

**Collinear or not?**

In the figure, $D$, $E$, and $F$ lie on line $m$, so they are collinear. Point $G$ is off the line, so $D$, $E$, and $G$ are *not* collinear. All four points, however, lie on the same flat page, so they are **coplanar**.

**Tip:** Any *two* points determine exactly one line. Any *three* non-collinear points determine exactly one plane.

### 3.2 Segments, Rays, and Opposite Rays

**Segments and Rays**

- A **segment** $\overline{AB}$ is the part of a line between two **endpoints** $A$ and $B$, including both endpoints. It has a definite length.
- A **ray** $\overrightarrow{AB}$ starts at endpoint $A$ and goes forever through $B$. Order matters: $\overrightarrow{AB}$ and $\overrightarrow{BA}$ point opposite ways.
- **Opposite rays** are two rays with the *same endpoint* that together form a straight line.

Notation: $\overline{AB}$ names the segment (a figure); $AB$ with no bar means its *length* (a number).

*An angle measures a turn.*

*(diagram in the online version)*

**Reading a figure of rays**

Here $\overrightarrow{BA}$ and $\overrightarrow{BC}$ share endpoint $B$ and form a straight line, so they are **opposite rays**. The segment $\overline{AC}$ is the piece between $A$ and $C$.

**Remember:** A bar means a *figure* ($\overline{AB}$, $\overrightarrow{AB}$, $\overleftrightarrow{AB}$); no bar means a *number* ($AB =$ the length).

### 3.3 Measuring Segments & the Segment Addition Postulate

**Segment Addition Postulate**

If point $B$ lies *on* $\overline{AC}$ (between $A$ and $C$), then the two shorter lengths add to the whole:

$$AB + BC = AC.$$

This lets you find any one of the three lengths when you know the other two.

**Adding segment lengths**

Point $B$ is between $A$ and $C$ with $AB = 6$ and $BC = 4$. Then

$$AC = AB + BC = 6 + 4 = 10.$$

**Solving for a missing length**

If $AC = 15$ and $AB = 9$ with $B$ between $A$ and $C$, then

$$BC = AC - AB = 15 - 9 = 6.$$

**Tip:** The point named in the middle of the sentence ("$B$ is between $A$ and $C$") is the one that splits the segment. It appears on *both* short pieces.

### 3.4 Congruent Segments & Midpoints

**Congruence and Midpoint**

Two segments are **congruent** ($\overline{AB} \cong \overline{CD}$) when they have equal length ($AB = CD$). We mark congruent segments with matching tick marks.

The **midpoint** $M$ of $\overline{AB}$ is the point exactly halfway between $A$ and $B$. It splits the segment into two congruent halves:

$$AM = MB = \tfrac{1}{2}\,AB.$$

**Using the midpoint**

$M$ is the midpoint of $\overline{AB}$ and $AB = 14$. Then each half is

$$AM = MB = \tfrac{1}{2}(14) = 7.$$

The tick marks show the two halves are congruent.

**Remember:** A midpoint makes two *equal* halves, so if you know the whole, halve it; if you know a half, double it.

### 3.5 Midpoint Formula & Distance Formula

**Formulas on the Coordinate Plane**

For points $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\textbf{Midpoint } M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\textbf{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The midpoint *averages* the coordinates. The distance formula is the Pythagorean Theorem in disguise.

**Midpoint of two points**

Find the midpoint of $A(1, 1)$ and $B(4, 3)$.

$$M = \left(\frac{1+4}{2}, \frac{1+3}{2}\right) = \left(\frac{5}{2}, 2\right) = (2.5,\ 2).$$

**Distance between two points**

Find the distance from $A(1, 1)$ to $B(4, 5)$.

$$AB = \sqrt{(4-1)^2 + (5-1)^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5.$$

This is the famous 3-4-5 right triangle: the horizontal leg is 3, the vertical leg is 4, and the segment is the hypotenuse.

**Tip:** In the distance formula the subtractions are squared, so it does not matter which point you call first — a negative squared is positive. Just be careful and neat with the arithmetic.

## 3.6 Segment Bisectors & Basic Constructions

**Segment Bisector**

A **segment bisector** is any line, ray, or segment that passes through the *midpoint* of a segment, cutting it into two congruent halves. A line that bisects a segment *and* meets it at a right angle is a **perpendicular bisector**.

**A perpendicular bisector**

Line $k$ passes through $M$, the midpoint of $\overline{PQ}$, and meets it at $90°$. So $PM = MQ$ (equal tick marks) and $k$ is the perpendicular bisector of $\overline{PQ}$.

**Two Classic Compass Constructions**

**Copying a segment $\overline{AB}$:** Draw a ray with endpoint $C$. Open your compass to the length $AB$. Without changing that opening, place the point on $C$ and swing a small arc across the ray; label the crossing $D$. Then $\overline{CD} \cong \overline{AB}$.

**Bisecting a segment $\overline{AB}$:** Open the compass wider than half of $AB$. From $A$, draw arcs above and below the segment; from $B$ with the *same* opening, draw two more arcs. Connect the two crossing points. That line is the perpendicular bisector, and where it meets $\overline{AB}$ is the midpoint.

**Remember:** Good constructions never measure with a ruler — they use only a compass and straightedge. Keep the compass opening *unchanged* for each step.

## 3.7 Going Deeper: Advanced Ideas

**The Section (Partition) Formula**

The midpoint splits a segment in the ratio $1 : 1$. To find a point $P$ that divides $\overline{AB}$ in *any* ratio $m : n$ (so that $AP : PB = m : n$), with $A(x_1, y_1)$ and $B(x_2, y_2)$, use the **section formula**:

$$P = \left( \frac{m\,x_2 + n\,x_1}{m+n}, \quad \frac{m\,y_2 + n\,y_1}{m+n} \right).$$

Notice the "cross" pattern: the weight $m$ (nearest to $B$) multiplies the $B$-coordinates, and $n$ multiplies the $A$-coordinates. Setting $m = n = 1$ recovers the midpoint formula, since each coordinate becomes a plain average.

**Dividing a segment in the ratio $2 : 3$**

Find the point $P$ on $\overline{AB}$ with $A(1, 2)$ and $B(6, 7)$ such that $AP : PB = 2 : 3$. Here $m = 2$ and $n = 3$, so $m + n = 5$:

$$P = \left( \frac{2(6) + 3(1)}{5}, \quad \frac{2(7) + 3(2)}{5} \right) = \left( \frac{15}{5}, \frac{20}{5} \right) = (3,\ 4).$$

Check: $P$ is closer to $A$ than to $B$, as expected when the first part of the ratio (2) is the smaller one.

**Distance and Midpoint in Three Dimensions**

A point in space needs three coordinates, $A(x_1, y_1, z_1)$. The plane formulas simply grow a $z$-term:

$$\textbf{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\textbf{Midpoint } M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

This is still just the Pythagorean Theorem, applied twice: once in the base plane and once up the vertical direction.

**Distance between two points in space**

Find the distance from $A(1, 2, 3)$ to $B(3, 5, 15)$.

$$AB = \sqrt{(3-1)^2 + (5-2)^2 + (15-3)^2} = \sqrt{4 + 9 + 144} = \sqrt{157}.$$

Because 157 is not a perfect square, we leave the answer as the exact radical $\sqrt{157}$ (about 12.53).

**Counting Lines, Planes, and Regions**

When points and lines are in "general position" (no needless overlaps), the counts follow neat patterns. For $n$ points with *no three collinear*, the number of distinct lines through pairs of them is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

For $n$ points with *no four coplanar*, the number of distinct planes through triples of them is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$

And $n$ lines drawn in a plane, no two parallel and no three meeting at one point, cut the plane into

$$1 + n + \binom{n}{2}$$

regions — each new line adds one more region than the line before it.

**How many lines and planes from 5 points?**

Take 5 points, no three collinear and no four coplanar. The number of lines is

$$\binom{5}{2} = \frac{5 \cdot 4}{2} = 10,$$

and the number of planes is

$$\binom{5}{3} = \frac{5 \cdot 4 \cdot 3}{6} = 10.$$

**Reflection for Shortest Paths**

To find the shortest path from point $A$ to a line $\ell$ and then on to point $B$ (both $A$ and $B$ on the same side of $\ell$), **reflect** one point across the line. If $A'$ is the mirror image of $A$ across $\ell$, then for any point $X$ on the line $AX = A'X$. So the bent path $A \to X \to B$ has the same length as $A' \to X \to B$, which is shortest when $A'$, $X$, and $B$ are *collinear*. Draw the straight segment $\overline{A'B}$; where it crosses $\ell$ is the best point $X$.

**The Idea of a Locus**

A **locus** is the set of *all* points that satisfy a given condition, and nothing else. Two rules of thumb turn everyday shapes into loci:

- The locus of points a fixed distance $r$ from a point $O$ is a **circle** of radius $r$ centered at $O$.
- The locus of points *equidistant* from two fixed points $A$ and $B$ is the **perpendicular bisector** of $\overline{AB}$.

To describe a locus, ask: what shape do *all* the qualifying points trace out?

**A coordinate proof using the locus idea**

Prove that the point equidistant from $A(0, 0)$ and $B(6, 0)$ with $y$-coordinate 0 is the midpoint. A point $P(x, 0)$ is equidistant when $AP = BP$, so $AP^2 = BP^2$:

$$x^2 = (x - 6)^2 = x^2 - 12x + 36.$$

Subtracting $x^2$ from both sides gives $0 = -12x + 36$, so $x = 3$. Thus $P(3, 0)$, which is exactly the midpoint $\left(\frac{0+6}{2}, \frac{0+0}{2}\right)$. This is a **coordinate proof**: by placing the figure on convenient axes, an algebra step settles a geometric claim.

**Remember:** Coordinate proofs work best when you place a key point at the origin and a key segment along an axis — the zeros make the distance and midpoint arithmetic short and clean.

# 4 Angles

*Naming, classifying, and solving for angle measures*

## 4.1 Naming Angles and Their Parts

**What is an angle?**

An **angle** is formed by two **rays** that share a common endpoint. That shared endpoint is the **vertex**, and the two rays are the **sides** of the angle. We measure the "opening" between the sides in **degrees** (°).

There are three ways to name an angle:

- **Three points:** $\angle AOB$ or $\angle BOA$ — the *vertex letter always goes in the middle.*
- **The vertex alone:** $\angle O$ — only when there is no confusion about which angle you mean.
- **A number:** $\angle 1$ — when a small number is written inside the angle.

*An angle measures a turn.*

*(diagram in the online version)*

**Naming the angle below**

The rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ meet at the vertex $O$. We can call this angle $\angle AOB$, $\angle BOA$, or simply $\angle O$.

**Protractor tip.** To measure an angle, place the protractor's center hole on the **vertex** and line up one side with the 0° mark. Read where the *other* side crosses the scale. Use the scale that starts at 0° on the side you lined up — most protractors have two rows of numbers!

## 4.2 Classifying Angles

**Five kinds of angles**

- **Acute:** greater than 0° and less than 90°.
- **Right:** exactly 90° (marked with a small square).
- **Obtuse:** greater than 90° and less than 180°.
- **Straight:** exactly 180° (a straight line).
- **Reflex:** greater than 180° and less than 360°.

*An angle measures a turn.*

*(diagram in the online version)*

**Remember.** A right angle is your reference. If an angle looks "smaller than the corner of a page," it is acute; if it looks "more open than a corner," it is obtuse.

## 4.3 Adjacent Angles & the Angle Addition Postulate

**Adjacent angles**

Two angles are **adjacent** when they share a common vertex and a common side, but do *not* overlap (no interior points in common). Think of two angles sitting side by side.

*An angle measures a turn.*

*(diagram in the online version)*

**Angle Addition Postulate**

If point $B$ is in the interior of $\angle AOC$, then the two small angles add up to the big angle:

$$m\angle AOB + m\angle BOC = m\angle AOC.$$

**Adding adjacent angles**

Ray $\overrightarrow{OB}$ lies inside $\angle AOC$. If $m\angle AOB = 35^\circ$ and $m\angle BOC = 40^\circ$, then

$$m\angle AOC = 35^\circ + 40^\circ = 75^\circ.$$

**Tip.** The Angle Addition Postulate works both ways: you can add two pieces to get the whole, or subtract a known piece from the whole to find the missing piece.

## 4.4 Angle Bisectors

**What is a bisector?**

A ray that **bisects** an angle divides it into **two congruent (equal) angles**. If $\overrightarrow{OB}$ bisects $\angle AOC$, then

$$m\angle AOB = m\angle BOC = \tfrac{1}{2}\, m\angle AOC.$$

*An angle measures a turn.*

*(diagram in the online version)*

**Using a bisector**

$\overrightarrow{OB}$ bisects $\angle AOC$, and $m\angle AOC = 80°$. Then each half is

$$m\angle AOB = m\angle BOC = \tfrac{1}{2}(80°) = 40°.$$

**Remember.** "Bisect" means "cut into two equal parts." The two halves are always *congruent*, so you can set them equal to each other when solving with algebra.

### 4.5 Complementary & Supplementary Angles

**Two special sums**

- **Complementary** angles have measures that add to **90**°.
- **Supplementary** angles have measures that add to **180**°.

The angles do *not* have to be next to each other — only their measures matter.

*An angle measures a turn.*

*(diagram in the online version)*

**Finding a complement and a supplement**

The **complement** of 30° is $90° - 30° = 60°$.

The **supplement** of 30° is $180° - 30° = 150°$.

Together the two angles above form a right angle, so they are complementary.

**Memory trick.** **C** comes before **S** in the alphabet, and 90 comes before 180: **C**omplementary = 90°, **S**upplementary = 180°. (Or: "**C**orner" is 90°, "**S**traight" is 180°.)

### 4.6 Linear Pairs & Vertical Angles

**Linear pairs**

When two angles are **adjacent** and their outer sides form a **straight line**, they make a **linear pair**. The angles of a linear pair are always **supplementary** (they add to 180°).

*An angle measures a turn.*

*(diagram in the online version)*

**A linear pair**

$\angle 1$ and $\angle 2$ below form a linear pair, so $m\angle 1 + m\angle 2 = 180°$. If $m\angle 1 = 110°$, then $m\angle 2 = 180° - 110° = 70°$.

**Vertical angles**

When two straight lines cross, they form two pairs of **vertical angles** — the angles "across" from each other. **Vertical angles are always congruent (equal).**

**Vertical angles are equal**

Two lines cross at $O$. Angles $\angle 1$ and $\angle 3$ are vertical (so $m\angle 1 = m\angle 3$), and $\angle 2$ and $\angle 4$ are vertical (so $m\angle 2 = m\angle 4$). If $m\angle 1 = 65°$, then $m\angle 3 = 65°$, while $\angle 2$ and $\angle 4$ each measure $180° - 65° = 115°$.

**Key facts.** Linear pair $\Rightarrow$ angles are *supplementary* (180°). Vertical angles $\Rightarrow$ angles are *congruent* (equal). These two rules solve most crossing-line puzzles!

### 4.7 Finding Unknown Angles with Algebra

**Set up an equation**

When an angle relationship gives you a total (90°, 180°, or a bisected angle), write an equation, then solve for the variable. Finally, *substitute back* to find each actual angle measure.

*An angle measures a turn.*

*(diagram in the online version)*

**Complementary angles with algebra**

Two complementary angles measure $x$ and $2x$. Find each angle.

$$\begin{aligned} x + 2x &= 90 \\ 3x &= 90 \\ x &= 30 \end{aligned}$$

So the angles are $x = 30°$ and $2x = 60°$. **Check:** $30° + 60° = 90°$. ✓

**Vertical angles with algebra**

Two vertical angles measure $(3x + 10)°$ and $(x + 50)°$. Because vertical angles are congruent, set them equal:

$$\begin{aligned} 3x + 10 &= x + 50 \\ 2x &= 40 \\ x &= 20 \end{aligned}$$

Each angle measures $3(20) + 10 = 70°$ (and $20 + 50 = 70°$). ✓

**Always check!** After solving, plug your answer back in. Complementary angles should total 90°, supplementary and linear pairs should total 180°, and vertical angles should come out equal.

## 4.8 Going Deeper: Advanced Angle Ideas

### Angle chasing: work one step at a time

**Angle chasing** means finding an unknown angle by filling in *other* angles one at a time, using the rules you already know, until the one you want is forced. At every step, ask which single rule turns something you know into something new:

- angles on a straight line add to 180° (linear pair);
- angles around a point add to 360°;
- vertical angles are equal;
- a bisector splits an angle into two equal halves.

Mark each angle on the figure as soon as you find it — one known angle usually unlocks the next.

*An angle measures a turn.*

*(diagram in the online version)*

### Two counting rules that act as constraints

These two facts are the "budget" every angle problem must respect:

$$\text{along a line: } \sum \text{angles} = 180°, \qquad \text{around a point: } \sum \text{angles} = 360°.$$

If three rays leave a single point $O$ and go all the way around, the three angles between neighboring rays must total 360°. If several angles sit in a row on one side of a straight line, they must total 180°. Turning a picture into one of these equations is the whole game.

### Angle chasing around a point

Four rays leave point $O$. Going around, the angles between neighbors are $90°$, $x$, $130°$, and $2x$. Find $x$ and the two unknown angles.

Everything around a point must add to $360°$:

$$\begin{aligned} 90 + x + 130 + 2x &= 360 \\ 3x + 220 &= 360 \\ 3x &= 140 \\ x &= \tfrac{140}{3} \approx 46.7. \end{aligned}$$

So $x \approx 46.7°$ and $2x \approx 93.3°$. **Check:** $90 + 46.7 + 130 + 93.3 = 360°$. ✓

### The Exterior Angle Theorem

Extend one side of a triangle to make an **exterior angle**. That exterior angle equals the **sum of the two remote (non-adjacent) interior angles**:

$$m\angle \text{ext} = m\angle A + m\angle B.$$

This follows from two facts working together: the three interior angles add to $180°$, and the exterior angle forms a linear pair ($180°$) with the interior angle beside it. Subtracting the shared interior angle from both leaves exactly the two remote angles.

### Using the exterior angle theorem

In the triangle below, the two remote interior angles are $50°$ and $65°$, and $\angle 4$ is an exterior angle. Instead of finding the third interior angle first, add the remote angles directly:

$$m\angle 4 = 50° + 65° = 115°.$$

**Check the long way:** the interior angle beside $\angle 4$ is $180 - 115 = 65°$, and $50 + 65 + 65 = 180°$. ✓

### Bisectors and the angle between them

Angle bisectors set up equal pieces you can chain together. A useful pattern: if two rays bisect neighboring angles that sit on a straight line, the angle *between the two bisectors* is always $90°$. Reason: the four half-angles come in two equal pairs, and all four together fill the $180°$ line, so the middle two halves total exactly half of $180°$. Whenever a problem mentions a bisector, replace the whole angle by "two equal halves" and the algebra usually falls out.

### A bisector condition with algebra

$\overrightarrow{OB}$ bisects $\angle AOC$. The two halves are given as $m\angle AOB = (4x-5)^\circ$ and $m\angle BOC = (2x+15)^\circ$. Because a bisector makes the halves equal, set them equal:

$$\begin{aligned} 4x-5 &= 2x+15 \\ 2x &= 20 \\ x &= 10. \end{aligned}$$

Each half is $4(10)-5 = 35^\circ$, so the whole angle is $m\angle AOC = 70^\circ$. ✓

### Clock hands as angles

A clock face is a circle of $360^\circ$ split into 12 hours, so each hour mark is $360/12 = 30^\circ$ apart. The two hands move at different, *steady* speeds:

$$\text{minute hand: } \tfrac{360^\circ}{60\text{ min}} = 6^\circ \text{ per minute,} \qquad \text{hour hand: } \tfrac{30^\circ}{60\text{ min}} = 0.5^\circ \text{ per minute.}$$

The minute hand gains on the hour hand at $6-0.5 = 5.5^\circ$ per minute. To find the angle between the hands at a given time, compute each hand's position from the top (12) and subtract. If the gap is more than $180^\circ$, use $360^\circ$ minus that gap for the smaller angle.

### The angle at 3:40, and when the hands overlap

**Part 1 — angle at 3:40.** Measure each hand clockwise from the 12:

$$\begin{aligned} \text{minute hand} &= 40 \times 6 = 240^\circ, \\ \text{hour hand} &= 3 \times 30 + 40 \times 0.5 = 90 + 20 = 110^\circ. \end{aligned}$$

The gap is $240 - 110 = 130^\circ$, which is already less than $180^\circ$, so the angle between the hands is $\mathbf{130}^\circ$.

**Part 2 — when do the hands overlap after 12:00?** At 12:00 both hands sit at $0^\circ$. The minute hand closes the gap at $5.5^\circ$ per minute, and it must make up a full $360^\circ$ to lap the hour hand and line up again:

$$t = \frac{360}{5.5} = \frac{360}{11/2} = \frac{720}{11} \approx 65.45 \text{ minutes.}$$

So the hands first overlap again at about 1:05$\frac{5}{11}$. In general, consecutive overlaps are exactly $\dfrac{720}{11} = \dfrac{360}{5.5}$ minutes apart, which is why the hands line up only 11 times every 12 hours — not 12.

### An angle condition that leads to a quadratic

An angle measures $x$ degrees. Its *supplement* is numerically equal to the *square* of its complement divided by 10. Find $x$.

The complement is $(90 - x)$ and the supplement is $(180 - x)$, so

$$180 - x = \frac{(90 - x)^2}{10}.$$

Multiply through by 10 and expand:

$$\begin{aligned} 1800 - 10x &= (90 - x)^2 \\ 1800 - 10x &= 8100 - 180x + x^2 \\ 0 &= x^2 - 170x + 6300. \end{aligned}$$

Factor (or use the quadratic formula): $x^2 - 170x + 6300 = (x - 50)(x - 120)$, so

$$x = 50 \qquad \text{or} \qquad x = 120.$$

**Reject the extra root.** A complement only exists for an angle *less than* $90°$, and $x = 120$ has no complement (and a negative $90 - x$). So we discard $x = 120$ and keep $\boxed{x = 50°}$. **Check:** complement $= 40$, supplement $= 130$, and $40^2/10 = 1600/10 = 130$. ✓

**Advanced checklist.** (1) Chase angles one rule at a time and mark each result on the figure. (2) Treat "on a line $= 180°$" and "around a point $= 360°$" as equations. (3) An exterior angle equals the two remote interior angles. (4) Replace any bisected angle by two equal halves. (5) For clocks, use $6°$/min and $0.5°$/min, with overlaps every $\frac{720}{11}$ minutes. (6) When algebra gives two roots, keep only the one that makes every angle valid (complements need angles below $90°$; all measures must be positive).

# 5 Parallel & Perpendicular Lines

*Everything you need to name angles, find them, and work with slopes*

## 5.1 Kinds of Lines: Parallel, Perpendicular, and Skew

### The three relationships

**Parallel lines** lie in the same flat surface (plane) and never meet, no matter how far you stretch them. We write $\ell \parallel m$.

**Perpendicular lines** cross to form a perfect square corner, a $90°$ angle. We write $\ell \perp m$.

**Skew lines** do *not* lie in the same plane. They never meet, but they are not parallel either — think of an airplane's path crossing high above a straight road.

*Two parallel lines cut by a transversal.*

*(diagram in the online version)*

**Transversal**

A **transversal** is a line that crosses two (or more) other lines. When a transversal cuts two lines it creates **eight angles** — four at each crossing — and those angles come in special matching pairs.

**Quick check:** Parallel lines stay the exact same distance apart forever. Perpendicular is a *special* case of intersecting — the angle just happens to be exactly 90°.

### 5.2 The Eight Angles and Their Pairs

When a transversal $t$ crosses lines $m$ and $n$, number the angles like this.

*An angle measures a turn.*

*(diagram in the online version)*

**Name the four pairs**

**Corresponding angles** sit in the *same corner* at each crossing (one up, one down). Pairs: $(1, 5)$, $(2, 6)$, $(3, 7)$, $(4, 8)$.

**Alternate interior angles** are *between* the two lines and on *opposite* sides of the transversal. Pairs: $(3, 6)$, $(4, 5)$.

**Alternate exterior angles** are *outside* the two lines and on *opposite* sides of the transversal. Pairs: $(1, 8)$, $(2, 7)$.

**Same-side interior angles** (also called co-interior or consecutive interior) are *between* the lines and on the *same* side of the transversal. Pairs: $(3, 5)$, $(4, 6)$.

**Memory helper:** *Interior* means *inside* (between the two lines: angles 3, 4, 5, 6). *Exterior* means *outside* (angles 1, 2, 7, 8). *Alternate* always means *opposite sides* of the transversal; *same-side* means the *same* side.

**Naming a pair**

Look at the diagram above. What is the relationship between angle 4 and angle 5?

Both angles are *between* lines $m$ and $n$ (interior), and angle 4 is on the right of $t$ while angle 5 is on the left (opposite sides). **They are alternate interior angles.**

### 5.3 Angle Relationships When the Lines Are Parallel

Everything above works for any two lines. The magic happens when the two lines are *parallel.*

*An angle measures a turn.*

*(diagram in the online version)*

**The parallel-line angle rules**

If two **parallel** lines are cut by a transversal, then:

- **Corresponding angles are congruent** (equal).
- **Alternate interior angles are congruent.**
- **Alternate exterior angles are congruent.**
- **Same-side interior angles are supplementary** (they add to $180°$).

Also, **vertical angles** (the ones directly across an X) are always congruent, and a straight line gives a **linear pair** that adds to $180°$.

**One angle unlocks them all**

In the figure, $m \parallel n$ and angle $1 = 70°$. Find every other angle.

Angle $1 = 70°$. Its *vertical* angle is 4, so angle $4 = 70°$. Angles 2 and 3 form linear pairs with angle 1, so each is $180° - 70° = 110°$.

Now jump to the bottom crossing using the parallel rules: angle 5 corresponds to angle 1, so angle $5 = 70°$. Then angle $8 = 70°$ (vertical to 5), and angles 6 and 7 are $110°$ each.

**Result:** angles $1, 4, 5, 8 = 70°$ and angles $2, 3, 6, 7 = 110°$.

**Same-side interior are supplementary**

Suppose $m \parallel n$ and one same-side interior angle measures $115°$. Its partner satisfies $115° + x = 180°$, so $x = 65°$.

**Shortcut:** With parallel lines, every one of the eight angles is either equal to the angle you know or is its supplement ($180°$ minus it). There are really only *two* different numbers in the whole picture!

### 5.4 Finding Unknown Angles with Algebra

**Turn a relationship into an equation**

Decide whether the two labeled angles are **congruent** (equal) or **supplementary** (add to

180°). Then write the matching equation and solve for the variable. Finally, plug back in to find the actual angle measures.

*An angle measures a turn.*

*(diagram in the online version)*

**Congruent angles (equal)**

Lines $m \parallel n$. Two *corresponding* angles measure $(3x + 15)°$ and $(5x - 25)°$. Find $x$ and each angle.

Corresponding angles are congruent, so set them equal:

$$\begin{aligned} 3x + 15 &= 5x - 25 \\ 40 &= 2x \\ x &= 20 \end{aligned}$$

Each angle is $3(20)+15 = 75°$ (and $5(20)-25 = 75°$ as a check). $x = 20$**, both angles** $= 75°$**.**

**Supplementary angles (add to 180)**

Lines $m \parallel n$. Two *same-side interior* angles measure $(2x + 20)°$ and $(3x - 10)°$. Find $x$ and each angle.

$$\begin{aligned} (2x + 20) + (3x - 10) &= 180 \\ 5x + 10 &= 180 \\ 5x &= 170 \\ x &= 34 \end{aligned}$$

The angles are $2(34) + 20 = 88°$ and $3(34) - 10 = 92°$. Check: $88 + 92 = 180$. ✓

**Watch the word:** "congruent" ⇒ set the expressions *equal.* "Supplementary" ⇒ set their *sum equal to* 180.

## 5.5 Proving Lines Are Parallel (the Converses)

The rules above run *backwards* too. If you can find one of the special angle relationships, you can *conclude* the lines are parallel.

*Two parallel lines cut by a transversal.*

*(diagram in the online version)*

**The converse statements**

Two lines cut by a transversal are **parallel** if any one of these is true:

- a pair of **corresponding** angles is congruent, *or*
- a pair of **alternate interior** angles is congruent, *or*
- a pair of **alternate exterior** angles is congruent, *or*
- a pair of **same-side interior** angles is supplementary.

**Are these lines parallel?**

A transversal crosses two lines. A pair of alternate interior angles measure $63°$ and $63°$. Because these alternate interior angles are *congruent*, the converse rule tells us the two lines **are parallel**.

If instead they had measured $63°$ and $70°$, they would *not* be congruent, so the lines would **not** be parallel.

**Careful:** Same-side interior angles do the opposite — the lines are parallel when those two angles *add to* $180°$, not when they are equal.

### 5.6 Perpendicular Lines and Right Angles

**Perpendicular means right angles**

When two lines are **perpendicular** ($\ell \perp m$), they cross to make four $90°$ angles. A small square drawn in the corner is the symbol for a right angle. If *one* of the four angles is $90°$, all four are.

*An angle measures a turn.*

*(diagram in the online version)*

**Splitting a right angle**

A right angle is split by a ray into two smaller angles of $x°$ and $52°$. Since the whole angle is $90°$:

$$x + 52 = 90 \quad \Rightarrow \quad x = 38°.$$

**Remember:** Perpendicular is the same idea whether you are looking at drawn lines (right angles) or graphed lines (slopes that are negative reciprocals — next section).

## 5.7 Slopes of Parallel and Perpendicular Lines

On the coordinate plane, **slope** measures steepness: slope $= \dfrac{\text{rise}}{\text{run}} = \dfrac{\text{change in } y}{\text{change in } x}$.

*Two parallel lines cut by a transversal.*

*(diagram in the online version)*

**The slope rules**

**Parallel lines have equal slopes.** If line 1 has slope $m$, then any line parallel to it also has slope $m$.

**Perpendicular lines have slopes that are negative reciprocals.** Flip the fraction and change the sign: if one slope is $m$, the perpendicular slope is $-\dfrac{1}{m}$. A quick test: two slopes are perpendicular exactly when their *product is* $-1$.

**Read off the slopes**

Line $A$ has slope $\dfrac{3}{4}$.

- A line **parallel** to $A$ has slope $\dfrac{3}{4}$ (same).
- A line **perpendicular** to $A$ has slope $-\dfrac{4}{3}$ (flip $\frac{3}{4}$ to $\frac{4}{3}$, change the sign). Check: $\dfrac{3}{4} \cdot \left(-\dfrac{4}{3}\right) = -1$. ✓

**Parallel, perpendicular, or neither?**

Line $A$ has slope $-2$ and line $B$ has slope $\dfrac{1}{2}$. Their product is $-2 \cdot \dfrac{1}{2} = -1$, so the lines are **perpendicular**.

**Special cases:** A *horizontal* line has slope 0; a *vertical* line has an undefined slope. A horizontal line and a vertical line are perpendicular to each other.

## 5.8 Writing Equations of Parallel and Perpendicular Lines

We use **slope-intercept form** $y = mx + b$ and **point-slope form** $y - y_1 = m(x - x_1)$.

*Two parallel lines cut by a transversal.*

*(diagram in the online version)*

**The three-step recipe**

1. Find the slope of the given line (read $m$ from $y = mx + b$).
2. Choose the new slope: *same* $m$ for parallel, *negative reciprocal* for perpendicular.
3. Plug that slope and the given point into point-slope form, then simplify to $y = mx + b$.

**A parallel line through a point**

Write the equation of the line through $(1, 4)$ that is **parallel** to $y = 2x + 1$.

The given slope is 2; parallel means the new slope is also 2. Point-slope:

$$y - 4 = 2(x - 1) \Rightarrow y - 4 = 2x - 2 \Rightarrow y = 2x + 2.$$

Check the point: $2(1) + 2 = 4$. ✓

**A perpendicular line through a point**

Write the equation of the line through $(4, 3)$ that is **perpendicular** to $y = 2x + 1$.

The given slope is 2; the perpendicular slope is $-\dfrac{1}{2}$. Point-slope:

$$y - 3 = -\tfrac{1}{2}(x - 4) \Rightarrow y - 3 = -\tfrac{1}{2}x + 2 \Rightarrow y = -\tfrac{1}{2}x + 5.$$

Check the point: $-\frac{1}{2}(4) + 5 = 3$. ✓

**Final tip:** Always test your equation by plugging the given point back in. If it makes the equation true, your line really does pass through that point — and the slope guarantees it is parallel or perpendicular as required.

## 5.9 Going Deeper: Advanced Ideas

Up to now we have *used* the parallel-line rules. Here we see *why* they are true, chase angles through several transversals at once, and push the slope ideas into real coordinate computations: distances, feet of perpendiculars, and equations of lines.

*An angle measures a turn.*

*(diagram in the online version)*

**Why the transversal theorems are true**

The whole system rests on *one* assumed fact, the **Corresponding Angles Postulate**: if two parallel lines are cut by a transversal, corresponding angles are congruent. Everything else is a short deduction from it, using two facts you already know:

- **Vertical angles** (across an X) are congruent.
- A **linear pair** (two angles on a straight line) is supplementary.

Chain them together: a corresponding angle is congruent to the one directly across from it (vertical), which *is* the alternate interior angle — so **alternate interior angles are congruent**. Swap the vertical step for a linear pair and you get **same-side interior angles are supplementary**. No new assumptions are needed.

### Worked proof: alternate interior angles are congruent

Using the eight-angle numbering from earlier, suppose $m \parallel n$. Prove $\angle 3 \cong \angle 6$.

The angle you *start* with ($\angle 3$) hops to its corresponding partner ($\angle 7$), then across the X to its vertical partner ($\angle 6$). Two congruences in a row give the result. The same two-step hop, run backwards, proves the *converse* used to show lines are parallel.

### The auxiliary parallel line trick (zigzag chases)

When a path *bends* between two parallel lines, there is no single transversal to use. The fix: **draw a new line through the bend point, parallel to the two given lines.** This splits the bend angle into two pieces, and each piece is now an alternate-interior angle with one of the original lines. Add (or subtract) the pieces to finish. This "build your own parallel line" move turns an impossible-looking figure into two ordinary transversal problems.

### Worked zigzag angle chase

Lines $r$ and $s$ are parallel. A path goes from $A$ on $r$ down to a corner $P$, then to $B$ on $s$. The path makes a $35°$ angle with $r$ at $A$ and a $25°$ angle with $s$ at $B$. Find $\angle APB$.

Draw the dashed line through $P$ parallel to $r$ and $s$. It splits $\angle APB$ into an upper piece and a lower piece.

- Upper piece $= 35°$: alternate interior angles with $A$'s angle across transversal $AP$.
- Lower piece $= 25°$: alternate interior angles with $B$'s angle across transversal $BP$.

So $\angle APB = 35° + 25° = \mathbf{60°}$. (Whenever the bend points *away* from the region between the parallels you add the pieces; if the path stays on one side you subtract them instead.)

### Distance between two parallel lines

Parallel lines never meet, so they stay a fixed distance apart — but that distance is measured *perpendicular* to the lines, not vertically. For two lines written as

$$y = mx + b_1 \qquad \text{and} \qquad y = mx + b_2 \quad \text{(same slope } m\text{)},$$

the perpendicular distance between them is

$$d = \frac{|b_2 - b_1|}{\sqrt{1+m^2}}.$$

The numerator is the vertical gap between the intercepts; dividing by $\sqrt{1+m^2}$ tilts that vertical gap into the true perpendicular gap. (If both lines are given in the form $ax+by+c=0$ with matching $a, b$, the distance is $|c_2 - c_1|/\sqrt{a^2+b^2}$.)

### Worked distance between parallels

Find the distance between $y = 2x + 1$ and $y = 2x + 6$.

Same slope $m = 2$, so they are parallel. Here $b_1 = 1$ and $b_2 = 6$:

$$d = \frac{|6-1|}{\sqrt{1+2^2}} = \frac{5}{\sqrt{5}} = \sqrt{5} \approx 2.24.$$

Notice this is *less* than the vertical gap of 5 — the perpendicular route between two slanted lines is always the shortest one.

### The foot of a perpendicular

The **foot of the perpendicular** from a point $P$ to a line $\ell$ is the single point $F$ on $\ell$ where the perpendicular dropped from $P$ lands. It is the closest point on $\ell$ to $P$, and the distance $PF$ is the true distance from the point to the line. To find it:

1. Take the slope of $\ell$ and use its negative reciprocal as the slope of the perpendicular through $P$.
2. Write that perpendicular line in point-slope form.
3. Solve the two equations together — the solution is the foot $F$.

### Worked foot of a perpendicular

Find the foot of the perpendicular from $P = (4, 1)$ to the line $\ell :\ y = x + 1$, and the distance from $P$ to $\ell$.

Line $\ell$ has slope 1, so the perpendicular has slope $-1$. Through $P = (4, 1)$:

$$y - 1 = -1(x - 4) \ \Rightarrow\ y = -x + 5.$$

Set the two lines equal to find where they cross:

$$\begin{aligned} x + 1 &= -x + 5 \\ 2x &= 4 \\ x &= 2, \qquad y = 2 + 1 = 3. \end{aligned}$$

The foot is $F = (2, 3)$. The distance is

$$PF = \sqrt{(4-2)^2 + (1-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2} \approx 2.83.$$

No other point on $\ell$ is closer to $P$ than $F$.

**Why the slope criteria are consistent:** a line with slope $\frac{b}{a}$ points in direction $(a, b)$. Rotating that direction a quarter turn ($90°$) sends $(a, b) \mapsto (-b, a)$, whose slope is $\dfrac{a}{-b} = -\dfrac{a}{b}$ — exactly the *negative reciprocal*, and their product is $-1$. So the algebra (negative reciprocals) and the geometry (a right angle) are the *same* statement. Likewise, two lines have *equal slopes* precisely when they point the same direction, which is precisely when they never cross — the definition of parallel.

# 6 Triangles & Congruence

*Classifying, angle rules, congruence shortcuts, and a first proof*

## 6.1 Classifying Triangles

**Two ways to sort a triangle**

Every triangle can be described in **two** ways at once: by its **sides** and by its **angles**.

**By sides:**

- **Scalene** — all three sides different lengths.
- **Isosceles** — at least two sides equal.
- **Equilateral** — all three sides equal.

**By angles:**

- **Acute** — all three angles less than $90°$.
- **Right** — exactly one $90°$ angle.
- **Obtuse** — one angle greater than $90°$.
- **Equiangular** — all three angles equal ($60°$ each).

*An angle measures a turn.*

*(diagram in the online version)*

**Example: name a triangle two ways**

The triangle below has a square corner mark (a $90^\circ$ angle) and its two short sides carry matching tick marks, meaning they are equal.

**By sides:** two equal sides $\Rightarrow$ **isosceles**. **By angles:** one $90^\circ$ angle $\Rightarrow$ **right**.

So this is an **isosceles right triangle**.

**Tip:** "Equilateral" and "equiangular" are partners — if all three sides are equal, then all three angles are equal ($60^\circ$), and the other way around too.

## 6.2 The Triangle Angle-Sum Theorem

**The angles always add to $180^\circ$**

In **any** triangle, the three interior angles add up to exactly $180^\circ$.

$$\angle A + \angle B + \angle C = 180^\circ$$

If you know two angles, subtract their sum from $180^\circ$ to find the third.

*An angle measures a turn.*

*(diagram in the online version)*

**Example: find the missing angle**

Add the two known angles: $65^\circ + 40^\circ = 105^\circ$. Then $x = 180^\circ - 105^\circ = \boxed{75^\circ}$.

**Tip:** A triangle can have *at most one* right or obtuse angle. Two right angles alone would already use up $180^\circ$, leaving nothing for the third!

## 6.3 The Exterior Angle Theorem

**An outside angle equals two inside angles**

If you extend one side of a triangle, you create an **exterior angle**. That exterior angle equals the **sum of the two remote (far-away) interior angles**.

$$\text{exterior angle} = \text{remote angle}_1 + \text{remote angle}_2$$

*An angle measures a turn.*

*(diagram in the online version)*

**Example: use the exterior angle**

Side $AB$ is extended to point $D$. The two remote interior angles are $A = 70°$ and $C = 50°$.

The exterior angle $y = 70° + 50° = \boxed{120°}$.

**Tip:** The exterior angle and its neighbor interior angle always add to 180° (a straight line). That is a handy way to check your work.

## 6.4 Congruent Figures and Corresponding Parts

**Same shape, same size**

Two figures are **congruent** ($\cong$) if they have exactly the same shape and size — one could be slid, turned, or flipped onto the other perfectly. When we write $\triangle ABC \cong \triangle DEF$, the **order of the letters** tells us which parts match:

$$A \leftrightarrow D, \quad B \leftrightarrow E, \quad C \leftrightarrow F.$$

These matching pieces are called **corresponding parts**.

**Example: read off corresponding parts**

Given $\triangle ABC \cong \triangle DEF$, the matching sides and angles are:

$$\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \overline{AC} \cong \overline{DF},$$

$$\angle A \cong \angle D, \quad \angle B \cong \angle E, \quad \angle C \cong \angle F.$$

So if $BC = 9$, then $EF = 9$ too, because they correspond.

**CPCTC** stands for "**C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent." Once you prove two triangles congruent, *every* pair of matching parts is automatically congruent.

## 6.5 Triangle Congruence Shortcuts

**Five ways to prove congruence**

You do not need to check all six parts (3 sides + 3 angles). Any **one** of these shortcuts is enough:

- **SSS** — three pairs of sides.
- **SAS** — two sides and the *included* angle (the angle *between* them).
- **ASA** — two angles and the *included* side.
- **AAS** — two angles and a *non-included* side.
- **HL** — (right triangles only) the hypotenuse and one leg.

*An angle measures a turn.*

*(diagram in the online version)*

**Example: a congruent pair marked with SAS**

Matching tick marks show equal sides; the arc shows the equal included angle. Two sides and the angle *between* them match, so this is **SAS**.

Therefore $\triangle ABC \cong \triangle DEF$ by **SAS**.

**The two that do NOT work: SSA** (two sides and a non-included angle) is unreliable — it can produce two different triangles (the "ambiguous case"). **AAA** (three angles) only guarantees the *same shape*, not the same *size* — think of a small triangle and a giant one with equal angles.

### 6.6 Isosceles and Equilateral Triangles

**Base Angles Theorem**

In an **isosceles** triangle, the two sides that are equal are the *legs*, and the third side is the *base*. The theorem says:

*The angles opposite the equal sides (the base angles) are congruent.*

And the reverse is true too: if two angles are equal, the sides opposite them are equal.

*An angle measures a turn.*

*(diagram in the online version)*

**Example: find the vertex angle**

The two legs are marked equal, so the two base angles are equal (50° each).

Vertex angle $= 180° - (50° + 50°) = \boxed{80°}$.

**Equilateral facts:** every equilateral triangle is also isosceles, is equiangular, and has three $60°$ angles. Its perimeter is simply $3\times$ (one side).

### 6.7 The Triangle Inequality

**Can these lengths make a triangle?**

Three lengths form a triangle only if the two shorter ones can "reach across." The rule:

*The sum of any two sides must be greater than the third side.*

A quick check: add the two **smallest** sides. If that sum beats the largest side, you have a triangle.

*An angle measures a turn.*

*(diagram in the online version)*

**Example: test some lengths**

**Lengths $5, 7, 9$:** smallest two are $5 + 7 = 12$, and $12 > 9$. **Yes**, a triangle.

**Lengths $2, 3, 6$:** smallest two are $2 + 3 = 5$, but $5 < 6$. **No** triangle — the short sides cannot reach.

**Range for a third side:** if two sides are 8 and 5, the third side $s$ must satisfy

$$8 - 5 < s < 8 + 5, \quad \text{that is} \quad 3 < s < 13.$$

**Tip:** The third side is always *between* the difference and the sum of the other two: $|a - b| < s < a + b$.

### 6.8 A First Two-Column Proof

**What a proof looks like**

A **two-column proof** lists **Statements** on the left and the **Reason** for each on the right. You start from what is *Given* and end at what you want to *Prove*, one logical step at a time.

*An angle measures a turn.*

*(diagram in the online version)*

**Example: prove two triangles congruent**

**Given:** $\overline{AB} \cong \overline{CB}$, and $\overline{BM}$ bisects $\angle ABC$ (so $\angle ABM \cong \angle CBM$). **Prove:** $\triangle ABM \cong \triangle CBM$.

Notice the pattern: **Side** ($AB$), **Angle** (at $B$), **Side** ($BM$) — that is SAS!

**Tip:** The **Reflexive Property** ($\overline{BM} \cong \overline{BM}$) is your best friend in proofs. Whenever two triangles *share* a side or an angle, that shared part is congruent to itself.

## 6.9 Going Deeper: Advanced Triangle Ideas

You have met the everyday rules of triangles. Now we push further into the ideas that professional geometers actually use — special points inside a triangle, a clever "inner" triangle, a theorem that slices sides in a neat ratio, proofs that add a line of their own, and the full story of which lengths can form a triangle. Sketch along and enjoy the extra depth.

*An angle measures a turn.*

*(diagram in the online version)*

### The four centers of a triangle

Every triangle has four famous "centers," each built from a different set of special lines. All three lines of a kind always meet at a single point (they are *concurrent*).

- **Centroid** — where the three **medians** meet (a median joins a vertex to the *midpoint* of the opposite side). It is the triangle's balance point, and it always sits *inside*.
- **Incenter** — where the three **angle bisectors** meet. It is the center of the inscribed circle that just touches all three sides, and it is always *inside*.
- **Circumcenter** — where the three **perpendicular bisectors** of the sides meet. It is the center of the circle through all three vertices; it may fall outside for an obtuse triangle.
- **Orthocenter** — where the three **altitudes** (perpendicular heights from each vertex) meet. It too can land outside an obtuse triangle.

**The centroid's 2:1 rule:** the centroid divides each median so that the piece from the *vertex* is **twice** as long as the piece to the *midpoint*. So the vertex-to-centroid part is $\frac{2}{3}$ of the whole median, and the centroid-to-midpoint part is $\frac{1}{3}$.

### Example: use the centroid's 2:1 ratio

In $\triangle ABC$ below, $M$ is the midpoint of $\overline{BC}$, so $\overline{AM}$ is a median. The three medians meet at the centroid $G$. Suppose the whole median $AM = 18$. Find $AG$ and $GM$.

The centroid splits the median in the ratio $2:1$ from the vertex. So the vertex part is $\frac{2}{3}$ of 18 and the midpoint part is $\frac{1}{3}$ of 18:

$$AG = \tfrac{2}{3} \times 18 = \boxed{12}, \qquad GM = \tfrac{1}{3} \times 18 = \boxed{6}.$$

Check: $12 + 6 = 18$, and 12 is indeed twice 6.

### The medial triangle and the Midsegment Theorem

Join the **midpoints** of the three sides of a triangle and you get a smaller triangle inside, called the **medial triangle**. Each of its sides is a **midsegment** of the original. The **Midsegment Theorem** says a midsegment is:

- **parallel** to the third side of the triangle, and
- exactly **half** as long as that side.

Because all three sides are halved, the medial triangle has the same shape as the original, half the perimeter, and one quarter of the area.

### Example: a midsegment length

$P$, $Q$, $R$ are the midpoints of the sides of $\triangle ABC$. Side $\overline{BC}$ has length 10. Find the midsegment $\overline{PQ}$ that joins the midpoints of $\overline{AB}$ and $\overline{AC}$.

$\overline{PQ}$ joins the midpoints of the two sides meeting at $A$, so it is the midsegment parallel to $\overline{BC}$. By the Midsegment Theorem it is half of $BC$:

$$PQ = \tfrac{1}{2} \times 10 = \boxed{5}, \quad \text{and} \quad \overline{PQ} \parallel \overline{BC}.$$

### The Angle Bisector Theorem

Draw the bisector of one angle of a triangle down to the opposite side. It cuts that side into two pieces whose lengths are in the **same ratio as the two sides** of the angle. In $\triangle ABC$, if the bisector from $A$ meets $\overline{BC}$ at $D$, then

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

The bisector divides the far side in proportion to the two nearby sides.

### Example: split a side with the Angle Bisector Theorem

In $\triangle ABC$, the bisector from $A$ meets $\overline{BC}$ at $D$. The two sides are $AB = 8$ and $AC = 6$, and the whole base is $BC = 14$. Find $BD$ and $DC$.

By the theorem, $\frac{BD}{DC} = \frac{AB}{AC} = \frac{8}{6} = \frac{4}{3}$. So the base of 14 splits into $4 + 3 = 7$ equal shares, each of length $14/7 = 2$:

$$BD = 4 \times 2 = \boxed{8}, \qquad DC = 3 \times 2 = \boxed{6}.$$

Check: $8 + 6 = 14$, and $8 : 6 = 4 : 3$ as required.

### Congruence proofs with an auxiliary line

Sometimes a proof seems stuck because there are not yet two triangles to compare. The trick is to **add your own line** — called an **auxiliary line** — that creates the triangles you need. Popular choices are an *angle bisector*, a *median*, an *altitude*, or a segment joining two existing points. Draw it, justify why it exists, and the congruence shortcuts (SSS, SAS, ASA, AAS, HL) do the rest.

### Example: prove the base angles of an isosceles triangle are equal

**Given:** $\triangle ABC$ with $\overline{AB} \cong \overline{AC}$. **Prove:** $\angle B \cong \angle C$.

**Auxiliary line:** draw the bisector of $\angle A$, meeting $\overline{BC}$ at $D$. This single extra segment splits the triangle into two triangles we can compare.

The auxiliary bisector turned one triangle into two matching ones, and CPCTC finished the job.

### The full triangle-inequality range

For three sides $a$, $b$, $c$ *all three* sums must beat the remaining side:

$$a + b > c, \qquad b + c > a, \qquad a + c > b.$$

Turned around, if two sides are $a$ and $b$, the third side $s$ is squeezed between their **difference** and their **sum**:

$$|a - b| \ < \ s \ < \ a + b.$$

Both ends are *strict* — landing exactly on $|a - b|$ or $a + b$ would flatten the triangle into a straight line.

### Example: count the integer third sides

Two sides of a triangle are 8 and 5. How many **whole-number** lengths are possible for the third side $s$?

Apply the range with $a = 8$, $b = 5$:

$$|8 - 5| < s < 8 + 5 \quad \Longrightarrow \quad 3 < s < 13.$$

Since the ends are strict, $s$ cannot equal 3 or 13. The allowed integers are

$$s \in \{4, 5, 6, 7, 8, 9, 10, 11, 12\}.$$

Counting them (or using $12 - 4 + 1$) gives $\boxed{9}$ possible integer lengths.

**Exterior angles in an angle chase:** in a longer figure, keep reusing "exterior angle = sum of the two remote interior angles." Each time you meet an extended side, you can jump straight to a new angle without first finding the interior one — a fast shortcut when hunting an unknown angle several steps away.

# 7 Similarity

*Ratios, similar polygons & triangles, indirect measurement, midsegments & scaling*

## 7.1 Ratios, Proportions, and Scale Factor

**The Language of Comparison**

- A **ratio** compares two quantities by division. The ratio of 6 to 4 can be written $6 : 4$, "6 to 4", or $\frac{6}{4} = \frac{3}{2}$.
- A **proportion** is an equation saying two ratios are equal, like $\frac{a}{b} = \frac{c}{d}$.
- To solve a proportion, use **cross-multiplication**: if $\frac{a}{b} = \frac{c}{d}$, then $a \cdot d = b \cdot c$.
- The **scale factor** is the number you multiply every length of a small figure by to get the matching length of the larger figure. It is just the ratio of a pair of corresponding sides.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Solving a proportion**

Solve $\dfrac{x}{6} = \dfrac{10}{4}$.

$$\frac{x}{6} = \frac{10}{4} \implies 4x = 6 \cdot 10 = 60 \implies x = 15.$$

Check: $\frac{15}{6} = 2.5$ and $\frac{10}{4} = 2.5$. ✓

**Tip:** A scale factor greater than 1 means an *enlargement*; a scale factor between 0 and 1 means a *reduction*. The scale factor is always (new length) ÷ (old length).

## 7.2 Similar Polygons

**What "Similar" Means**

Two polygons are **similar** ($\sim$) when they have the same shape but not necessarily the same size. This requires *both*:

- All pairs of **corresponding angles** are congruent (equal).
- All pairs of **corresponding sides** are proportional (same scale factor).

A **similarity statement** lists vertices in matching order. Writing $\triangle ABC \sim \triangle DEF$ tells you $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$, so $\angle A \cong \angle D$ and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Reading a similarity statement**

Here $\triangle ABC \sim \triangle DEF$. The small triangle has sides 3, 4, 5; the large one 6, 8, 10. Each large side is exactly twice its matching small side, so the scale factor from $\triangle ABC$ to $\triangle DEF$ is 2.

Corresponding sides: $\frac{AB}{DE} = \frac{4}{8} = \frac{1}{2}$, $\frac{BC}{EF} = \frac{5}{10} = \frac{1}{2}$, $\frac{AC}{DF} = \frac{3}{6} = \frac{1}{2}$. All equal, so the triangles are similar.

**Tip:** Match corresponding vertices *in order*. $\triangle ABC \sim \triangle DEF$ is not the same claim as $\triangle ABC \sim \triangle EFD$. Line up the letters, then line up the sides.

## 7.3 Triangle Similarity Criteria: AA, SSS$\sim$, SAS$\sim$

**Three Shortcuts for Triangles**

You do not need to check everything. For triangles, any one of these guarantees similarity:

- **AA** (Angle-Angle): two pairs of corresponding angles are congruent. (The third pair is then automatic.)
- **SSS$\sim$** (Side-Side-Side): all three pairs of corresponding sides are proportional.
- **SAS$\sim$** (Side-Angle-Side): two pairs of corresponding sides are proportional *and* the included angles (between those sides) are congruent.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Using AA**

The two triangles share a pair of congruent marked angles, and both have a right angle. That is two pairs of congruent angles, so by **AA** the triangles are similar.

**Using SAS∼**

Two sides of a big triangle are 10 and 6; two sides of a small triangle are 5 and 3, and the included angles are equal. Since $\frac{10}{5} = \frac{6}{3} = 2$ and the angle between them matches, the triangles are similar by **SAS∼**.

**Tip:** For SAS∼ the equal angle must be *between* the two proportional sides. An angle in the wrong spot does not count.

## 7.4 Finding Missing Sides with Similar Triangles

**Set Up a Proportion, Then Solve**

When two triangles are similar, corresponding sides are proportional. To find a missing length:

1. Write the similarity statement so vertices match.
2. Build a proportion using one *known* pair of sides and the pair that contains the unknown.
3. Cross-multiply and solve.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Solving for a missing side**

Given $\triangle ABC \sim \triangle DEF$ with $AB = 4$, $DE = 6$, and $BC = 5$, find $EF$.

$$\frac{AB}{DE} = \frac{BC}{EF} \implies \frac{4}{6} = \frac{5}{x} \implies 4x = 30 \implies x = 7.5.$$

So $EF = 7.5$.

**Tip:** Keep the same figure "on top" in every ratio. If small-triangle sides are numerators on one side of the equation, keep them numerators on the other side too.

## 7.5 Indirect Measurement (Shadows & Mirrors)

### Measuring the Unreachable

You can find a height you cannot reach by using similar triangles.

- **Shadow method:** At the same time of day, an object and its shadow form a triangle similar to a nearby object and its shadow (the sun's rays hit at the same angle). So $\dfrac{\text{height}_1}{\text{shadow}_1} = \dfrac{\text{height}_2}{\text{shadow}_2}$.
- **Mirror method:** Place a mirror on the ground. The angle of sight into the mirror equals the angle back out, creating two similar triangles.

### The shadow of a tree

A 6-ft person casts a 4-ft shadow. At the same moment a tree casts a 20-ft shadow. How tall is the tree? The two triangles (person + shadow, tree + shadow) are similar by AA, so:

$$\frac{6}{4} = \frac{h}{20} \implies 4h = 120 \implies h = 30 \text{ ft.}$$

**Tip:** Line up the ratios the same way every time — height over shadow on *both* sides. Matching units matters too; convert first if one length is in inches and another in feet.

## 7.6 The Triangle Midsegment Theorem

### The Midsegment

A **midsegment** of a triangle joins the midpoints of two sides. The **Triangle Midsegment Theorem** says a midsegment is:

- **parallel** to the third side, and
- exactly **half** the length of that third side.

This creates a small triangle similar to the whole triangle with scale factor $\frac{1}{2}$.

**Finding a midsegment length**

$M$ and $N$ are midpoints of $\overline{AB}$ and $\overline{AC}$. Since $\overline{MN}$ is a midsegment, $MN = \frac{1}{2}BC$. If $BC = 14$, then $MN = 7$.

**Tip:** The midsegment is *half* the base — so the base is *twice* the midsegment. Read the problem carefully to see which one you are given.

### 7.7 Side-Splitter & Proportional Segments

**The Side-Splitter Theorem**

If a line is drawn **parallel** to one side of a triangle and crosses the other two sides, it divides those two sides **proportionally**.

$$\text{If } \overline{DE} \parallel \overline{BC}, \quad \text{then } \frac{AD}{DB} = \frac{AE}{EC}.$$

This happens because the parallel line creates a smaller similar triangle (by AA).

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Splitting the sides**

In $\triangle ABC$, $\overline{DE} \parallel \overline{BC}$ with $AD = 3$, $DB = 6$, and $AE = 4$. Find $EC$.

$$\frac{AD}{DB} = \frac{AE}{EC} \implies \frac{3}{6} = \frac{4}{EC} \implies 3 \cdot EC = 24 \implies EC = 8.$$

**Tip:** The side-splitter compares the two *pieces* of each split side ($\frac{\text{top}}{\text{bottom}}$), not a piece to the whole. Keep top-with-top and bottom-with-bottom.

### 7.8 Scale Factor: Perimeter and Area

**How Scaling Changes Size**

If two similar figures have scale factor $k$ (each length of the big figure is $k$ times the matching length of the small one), then:

- The ratio of **perimeters** equals the scale factor: $k$.
- The ratio of **areas** equals the scale factor *squared*: $k^2$.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Perimeter and area under scaling**

Two similar rectangles have scale factor $k = 3$. The small one has perimeter 12 and area 8.

Big perimeter $= 3 \times 12 = 36$. Big area $= 3^2 \times 8 = 9 \times 8 = 72$.

**Tip:** Perimeter grows like the scale factor; area grows like its *square*. If you triple the sides, the area becomes 9 times as big — not 3 times.

## 7.9 Going Deeper: Advanced Similarity

**The Altitude-on-Hypotenuse Geometric-Mean Relations**

Drop the altitude from the right angle of a right triangle to the hypotenuse. It splits the big triangle into *two smaller triangles, each similar to the original and to each other* (all three share the same angles, by AA). Label the hypotenuse pieces $p$ and $q$, the altitude $h$, and the legs $a$ and $b$.

Matching corresponding sides across the similar triangles gives three "geometric mean" (mean proportional) relations:

$$h = \sqrt{p\,q}, \qquad a = \sqrt{p\,(p+q)}, \qquad b = \sqrt{q\,(p+q)}.$$

In words: the altitude is the geometric mean of the two hypotenuse pieces, and each leg is the geometric mean of its adjacent piece and the whole hypotenuse.

*Similar triangles: same shape, scaled sides.*

*(diagram in the online version)*

**Using the geometric-mean relations**

The altitude to the hypotenuse divides it into pieces $p = 4$ and $q = 9$. Find the altitude $h$ and the shorter leg $a$.

$$h = \sqrt{p\,q} = \sqrt{4 \cdot 9} = \sqrt{36} = 6.$$

The whole hypotenuse is $p + q = 4 + 9 = 13$, so the leg adjacent to the piece $p = 4$ is

$$a = \sqrt{p\,(p+q)} = \sqrt{4 \cdot 13} = \sqrt{52} = 2\sqrt{13} \approx 7.2.$$

Check with the Pythagorean theorem on the small left triangle: $a^2 = h^2 + p^2 = 36 + 16 = 52$. ✓

### Why the Side-Splitter Theorem Is True (A Proof)

**Claim.** In $\triangle ABC$, if $\overline{DE} \parallel \overline{BC}$ with $D$ on $\overline{AB}$ and $E$ on $\overline{AC}$, then $\dfrac{AD}{DB} = \dfrac{AE}{EC}$.

**Proof.** Because $\overline{DE} \parallel \overline{BC}$, the parallel lines cut off equal corresponding angles: $\angle ADE \cong \angle ABC$ and $\angle AED \cong \angle ACB$. Triangle $ADE$ and triangle $ABC$ also share $\angle A$. So by **AA**, $\triangle ADE \sim \triangle ABC$. Similar triangles have proportional corresponding sides, giving

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

Now flip each ratio to compare whole-to-part, then subtract 1 from both sides:

$$\frac{AB}{AD} = \frac{AC}{AE} \implies \frac{AB}{AD} - 1 = \frac{AC}{AE} - 1 \implies \frac{AB - AD}{AD} = \frac{AC - AE}{AE}.$$

Since $AB - AD = DB$ and $AC - AE = EC$, this is exactly $\dfrac{DB}{AD} = \dfrac{EC}{AE}$, and taking reciprocals gives $\dfrac{AD}{DB} = \dfrac{AE}{EC}$. ■

### Area Scales by $k^2$, Volume Scales by $k^3$

When two figures are similar with scale factor $k$ (every length of the large figure is $k$ times the matching length of the small one), the effect compounds with each dimension:

- **Lengths** (sides, perimeters, radii, heights) scale by $k^1 = k$.
- **Areas** (surface areas too) scale by $k^2$, because area multiplies two lengths.
- **Volumes** scale by $k^3$, because volume multiplies three lengths.

Read backwards, this is powerful: if you *know* the area or volume ratio, take a square or cube root to recover the length scale factor. If $\dfrac{\text{Area}_1}{\text{Area}_2} = \dfrac{9}{25}$, then $k = \sqrt{9/25} = \dfrac{3}{5}$.

### From volume ratio back to a length

Two similar cylinders hold 54 cm$^3$ and 128 cm$^3$. The smaller has radius 3 cm. Find the larger radius.

The volume ratio equals $k^3$, so

$$k^3 = \frac{128}{54} = \frac{64}{27} \implies k = \sqrt[3]{\tfrac{64}{27}} = \frac{4}{3}.$$

Radii scale by $k$, so the larger radius is $3 \times \dfrac{4}{3} = 4$ cm.

### Nested and Overlapping Similar Triangles

Similar triangles often *share* a vertex or side, so they overlap on the page. Two habits keep them straight:

- **Redraw them separately**, pulling the small triangle out of the big one and turning it so matching vertices sit in the same positions.
- **Use the whole, not the piece.** When a small triangle sits inside a big one along a shared side, the big triangle's side is the *entire* length, not just the leftover part. Mixing a piece with a whole is the most common error.

A shared angle plus a pair of parallel sides (or a pair of equal marked angles) is the usual AA setup that makes the overlap similar.

### The Angle Bisector Theorem — via Similarity

**Claim.** If a ray from vertex $A$ bisects $\angle A$ and meets $\overline{BC}$ at $D$, then $D$ divides $\overline{BC}$ in the ratio of the two adjacent sides:

$$\frac{BD}{DC} = \frac{AB}{AC}.$$

**Proof idea.** Through $C$ draw a line parallel to the bisector $\overline{AD}$, meeting line $\overline{BA}$ extended at a point $E$. Since $\overline{AD} \parallel \overline{CE}$, the side-splitter theorem in $\triangle BCE$ gives $\dfrac{BD}{DC} = \dfrac{BA}{AE}$. The parallel lines also force $\triangle ACE$ to be isosceles ($\angle AEC$ and $\angle ACE$ equal the two bisected halves of $\angle A$), so $AE = AC$. Substituting yields $\dfrac{BD}{DC} = \dfrac{AB}{AC}$. ■

### Applying the Angle Bisector Theorem

In $\triangle ABC$, $\overline{AD}$ bisects $\angle A$ with $AB = 8$, $AC = 6$, and $BC = 14$. Find $BD$ and $DC$. Let $BD = x$; then $DC = 14 - x$. The theorem gives

$$\frac{BD}{DC} = \frac{AB}{AC} \implies \frac{x}{14 - x} = \frac{8}{6}.$$

Cross-multiply: $6x = 8(14 - x) = 112 - 8x$, so $14x = 112$ and $x = 8$. Thus $BD = 8$ and $DC = 14 - 8 = 6$.

### Indirect Measurement with Two Unknowns

Sometimes a single similar-triangle setup hides *two* unknowns — for example an unknown height *and* an unknown distance. The fix is to write **two** proportions (or one proportion plus one length equation) so you have as many equations as unknowns, then solve the system.

A classic case: an observer walks toward a tall object, and you know the two viewing distances but neither the object's height nor the observer's starting gap. Set the height as one variable and the hidden distance as another, build a proportion at each observation, and eliminate.

**Two sightings, two unknowns**

A student wants the height $h$ of a flagpole. A vertical stake of height 2 m casts a shadow whose tip lines up with the top of the pole. From the first spot the stake stands $x$ m from the pole's base with a 3 m shadow; the student steps 4 m farther back and now the same-height stake needs a 5 m shadow to line up. Set up similar triangles (sight-line from shadow tip over stake top to pole top) at each position:

$$\text{First: } \frac{h}{x+3} = \frac{2}{3}, \qquad \text{Second: } \frac{h}{x+4+5} = \frac{2}{5}.$$

From the first equation, $3h = 2(x+3) = 2x+6$. From the second, $5h = 2(x+9) = 2x+18$. Subtract the first from the second to eliminate $x$:

$$5h - 3h = (2x+18) - (2x+6) \implies 2h = 12 \implies h = 6 \text{ m}.$$

Back-substitute: $3(6) = 2x+6 \Rightarrow 18 = 2x+6 \Rightarrow x = 6$ m. The flagpole is 6 m tall, and the first stake stood 6 m from its base.

**Tip:** Count your unknowns *before* solving. Two unknowns need two independent equations. Line every proportion up the same way — height over base-distance on both sides — and eliminate one variable by subtraction or substitution.

# 8 Right Triangles & Trig

*Pythagoras, special triangles, SOH-CAH-TOA, and angles of elevation*

## 8.1 The Pythagorean Theorem

**The Pythagorean Theorem**

In *any* right triangle, the two shorter sides are the **legs** ($a$ and $b$) and the longest side — always across from the right angle — is the **hypotenuse** ($c$). The theorem says:

$$a^2 + b^2 = c^2.$$

The two legs squared and added always equal the hypotenuse squared. Because $c$ is the biggest side, it sits *alone* on its own side of the equation.

*The legs and the hypotenuse.*

*(diagram in the online version)*

**Finding the hypotenuse**

The legs are 6 and 8. Find the hypotenuse $c$.

$$c^2 = 6^2 + 8^2 = 36 + 64 = 100, \qquad c = \sqrt{100} = 10.$$

**Finding a leg**

The hypotenuse is 13 and one leg is 5. Find the other leg $b$. The hypotenuse stays alone:

$$5^2 + b^2 = 13^2 \ \Rightarrow\ 25 + b^2 = 169 \ \Rightarrow\ b^2 = 144 \ \Rightarrow\ b = 12.$$

**Tip:** To find a *leg*, subtract: $b = \sqrt{c^2 - a^2}$. To find the *hypotenuse*, add: $c = \sqrt{a^2 + b^2}$. If you ever get the hypotenuse smaller than a leg, you added when you should have subtracted.

## 8.2 The Converse: Right, Acute, or Obtuse?

**Converse of the Pythagorean Theorem**

Take the three sides of a triangle and let $c$ be the *longest*. Compare $a^2 + b^2$ with $c^2$:

- If $a^2 + b^2 = c^2$, the triangle is **right**.
- If $a^2 + b^2 > c^2$, the triangle is **acute** (all angles under 90°).
- If $a^2 + b^2 < c^2$, the triangle is **obtuse** (one angle over 90°).

A bigger $c^2$ "pulls" the longest side apart into an obtuse angle.

*An angle measures a turn.*

*(diagram in the online version)*

**Classifying a triangle**

Sides 7, 8, 12. The longest is 12, so

$$7^2 + 8^2 = 49 + 64 = 113, \qquad 12^2 = 144.$$

Since $113 < 144$, we have $a^2 + b^2 < c^2$, so the triangle is **obtuse**.

**Remember:** Always square the *longest* side for $c^2$. "Sum of squares of the two short sides" versus "square of the long side" tells the whole story.

## 8.3 Pythagorean Triples

**Whole-number right triangles**

A **Pythagorean triple** is a set of three whole numbers that fit $a^2+b^2 = c^2$ exactly. Memorizing a few lets you skip the arithmetic. The common ones are:

$$3\text{-}4\text{-}5, \qquad 5\text{-}12\text{-}13, \qquad 8\text{-}15\text{-}17, \qquad 7\text{-}24\text{-}25.$$

**Any multiple of a triple is also a triple.** Multiply 3-4-5 by 2 to get 6-8-10; by 3 to get 9-12-15; by 10 to get 30-40-50.

*An angle measures a turn.*

*(diagram in the online version)*

**Spotting a triple**

The legs are 10 and 24. These are $2 \times 5$ and $2 \times 12$, so this is the 5-12-13 triple doubled. The hypotenuse is $2 \times 13 = 26$. Check: $10^2 + 24^2 = 100 + 576 = 676 = 26^2$. ✓

**Tip:** A triple only works when you match the right roles — the largest number is always the hypotenuse. In 8-15-17, the legs are 8 and 15 and the hypotenuse is 17.

## 8.4 Special Right Triangle: 45-45-90

**The 45-45-90 Triangle**

This is an isosceles right triangle: the two legs are **equal**, and each base angle is 45°. The hypotenuse is always the leg times $\sqrt{2}$:

$$\text{leg} = x, \qquad \text{leg} = x, \qquad \text{hypotenuse} = x\sqrt{2}.$$

Going *up* from a leg, multiply by $\sqrt{2}$. Going *down* from the hypotenuse, divide by $\sqrt{2}$.

*An angle measures a turn.*

*(diagram in the online version)*

**Legs equal, hypotenuse is leg times $\sqrt{2}$**

Each leg is 5. Then the hypotenuse is $5\sqrt{2}$ (about 7.07). We keep the exact radical $5\sqrt{2}$ as the answer.

**Remember:** In a 45-45-90 triangle the hypotenuse is the *only* side with a $\sqrt{2}$. Both legs match.

## 8.5 Special Right Triangle: 30-60-90

**The 30-60-90 Triangle**

The sides are always in the ratio

$$\text{short leg} = x, \qquad \text{long leg} = x\sqrt{3}, \qquad \text{hypotenuse} = 2x.$$

The **short leg** (across from 30°) is the key. The hypotenuse (across from 90°) is *twice* the short leg, and the long leg (across from 60°) is the short leg times $\sqrt{3}$.

*An angle measures a turn.*

*(diagram in the online version)*

**Using the $x$, $x\sqrt{3}$, $2x$ pattern**

The short leg (opposite 30°) is 4. Then the hypotenuse is $2 \cdot 4 = 8$ and the long leg is $4\sqrt{3}$ (about 6.93). We keep $4\sqrt{3}$ exact.

**Tip:** Always find the *short leg* first. If you are given the hypotenuse, halve it to get the short leg; then multiply the short leg by $\sqrt{3}$ for the long leg.

## 8.6 The Three Trig Ratios: SOH-CAH-TOA

**Sine, Cosine, Tangent**

Pick one of the two *acute* angles and call it $\theta$. Relative to $\theta$, name the sides:

- **opposite** — the leg across from $\theta$
- **adjacent** — the leg touching $\theta$ (that is not the hypotenuse)
- **hypotenuse** — across from the right angle

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Remember it as **SOH-CAH-TOA**.

*Opposite, adjacent and hypotenuse are named from the angle.*

*(diagram in the online version)*

**Reading the three ratios**

For angle $\theta$ at $B$: the opposite leg is 3, the adjacent leg is 4, and the hypotenuse is 5. So

$$\sin\theta = \tfrac{3}{5}, \qquad \cos\theta = \tfrac{4}{5}, \qquad \tan\theta = \tfrac{3}{4}.$$

**Remember:** "Opposite" and "adjacent" switch when you switch which angle you look at, but the hypotenuse never moves. Always decide which angle is $\theta$ *first*.

## 8.7 Using a Trig Ratio to Find a Missing Side

**Set up, then solve**

When you know one angle and one side and want another side, pick the ratio that uses your two sides (SOH, CAH, or TOA), then solve for the unknown. Your calculator must be in **degree** mode.

*Opposite, adjacent and hypotenuse are named from the angle.*

*(diagram in the online version)*

**Finding a side with cosine**

Find $x$, the side adjacent to the $40^\circ$ angle, when the hypotenuse is 10. Adjacent and hypotenuse means cosine (CAH):

$$\cos 40^\circ = \frac{x}{10} \;\Rightarrow\; x = 10\cos 40^\circ \approx 10(0.766) \approx 7.7.$$

Rounded to the nearest tenth, $x \approx 7.7$.

**When the unknown is on the bottom**

Find the hypotenuse $h$ when the side opposite $35^\circ$ is 6. Opposite and hypotenuse means sine (SOH):

$$\sin 35^\circ = \frac{6}{h} \;\Rightarrow\; h = \frac{6}{\sin 35^\circ} \approx \frac{6}{0.574} \approx 10.5.$$

When the unknown sits in the denominator, swap it with the trig value.

**Tip:** If the unknown is on *top*, multiply. If the unknown is on the *bottom*, divide the known side by the trig value.

## 8.8 Using Inverse Trig to Find a Missing Angle

**The inverse buttons**

When you know two sides but want the *angle*, use the inverse functions $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ (often the "2nd" + sin/cos/tan keys):

$$\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right), \quad \theta = \cos^{-1}\left(\frac{\text{adj}}{\text{hyp}}\right), \quad \theta = \tan^{-1}\left(\frac{\text{opp}}{\text{adj}}\right).$$

The inverse "undoes" the ratio and gives back the angle in degrees.

*An angle measures a turn.*

*(diagram in the online version)*

**Finding an angle with tangent**

The leg opposite $\theta$ is 5 and the leg adjacent is 8. Two legs means tangent (TOA):

$$\tan\theta = \frac{5}{8} = 0.625 \;\Rightarrow\; \theta = \tan^{-1}(0.625) \approx 32.0^\circ.$$

Rounded to the nearest tenth of a degree, $\theta \approx 32.0^\circ$.

**Remember:** A plain trig button turns an angle into a ratio; an inverse button ($^{-1}$) turns a ratio back into an angle. Choose the ratio SOH-CAH-TOA from the two sides you have.

## 8.9 Angles of Elevation & Depression

**Looking up and looking down**

An **angle of elevation** is measured *up* from the horizontal to an object above you. An **angle of depression** is measured *down* from the horizontal to an object below you. Because the two horizontal lines are parallel, the angle of elevation from the ground equals the angle of depression from above (alternate interior angles).

*An angle measures a turn.*

*(diagram in the online version)*

**Height from an angle of elevation**

From 50 ft away, the angle of elevation to the top of a tree is $32°$. Find the height $h$. Opposite and adjacent means tangent (TOA):

$$\tan 32° = \frac{h}{50} \;\Rightarrow\; h = 50\tan 32° \approx 50(0.625) \approx 31.2 \text{ ft}.$$

Rounded to the nearest tenth, the tree is about 31.2 ft tall.

**An angle of depression**

A lifeguard 10 ft up sees a swimmer at an angle of depression of $20°$. The angle of depression equals the angle of elevation from the swimmer, so at the tower base the angle inside the triangle is $20°$. With the height 10 opposite that angle and the horizontal distance $d$ adjacent:

$$\tan 20° = \frac{10}{d} \;\Rightarrow\; d = \frac{10}{\tan 20°} \approx \frac{10}{0.364} \approx 27.5 \text{ ft}.$$

**Tip:** Always sketch the right triangle and label horizontal, vertical, and line of sight. The elevation/depression angle sits between the *horizontal* and the *line of sight* — never at the top of a vertical side.

## 8.10 Going Deeper: Advanced Right-Triangle & Trig Ideas

**Where the 45-45-90 ratio comes from**

The ratio is not memorized magic — it *follows* from the Pythagorean Theorem. Start with an isosceles right triangle whose two equal legs are each $x$. The hypotenuse $c$ obeys

$$c^2 = x^2 + x^2 = 2x^2 \;\Rightarrow\; c = \sqrt{2x^2} = x\sqrt{2}.$$

Because both legs are equal, the two acute angles are equal, and since they must sum to $90°$ each is $45°$. That is exactly the pattern $x$, $x$, $x\sqrt{2}$.

*An angle measures a turn.*

*(diagram in the online version)*

**Where the 30-60-90 ratio comes from**

Take an equilateral triangle with every side $2x$ and every angle $60°$. Drop an altitude from the top vertex: it splits the base into two halves of length $x$ and cuts the top angle into two $30°$ pieces, making a right triangle. The short leg (opposite $30°$) is $x$ and the hypotenuse is $2x$, so the altitude (the long leg, opposite 60°) is

$$h^2 = (2x)^2 - x^2 = 4x^2 - x^2 = 3x^2 \;\Rightarrow\; h = x\sqrt{3}.$$

That gives the pattern $x$, $x\sqrt{3}$, $2x$.

**Preview: the identity $\sin^2\theta + \cos^2\theta = 1$**

In a right triangle with hypotenuse $c$, opposite side $a$, and adjacent side $b$, we have $\sin\theta = a/c$ and $\cos\theta = b/c$. Squaring and adding:

$$\sin^2\theta + \cos^2\theta = \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{a^2+b^2}{c^2} = \frac{c^2}{c^2} = 1,$$

because $a^2+b^2 = c^2$. So this famous **Pythagorean identity** is just the Pythagorean Theorem divided by $c^2$. (Here $\sin^2\theta$ means $(\sin\theta)^2$.) It holds for *every* angle, so if you know one of $\sin\theta$ or $\cos\theta$ you can find the other.

**Using the identity**

Suppose $\sin\theta = \frac{3}{5}$ for an acute angle $\theta$. Find $\cos\theta$ without drawing the triangle:

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \tfrac{9}{25} = \tfrac{16}{25} \;\Rightarrow\; \cos\theta = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}.$$

We take the positive root because $\theta$ is acute. (This is just the 3-4-5 triangle in disguise.)

**Beyond right triangles: Law of Sines & Law of Cosines**

SOH-CAH-TOA only works in right triangles. For *any* triangle with angles $A$, $B$, $C$ opposite sides $a$, $b$, $c$:

$$\text{Law of Sines:}\quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

$$\text{Law of Cosines:}\quad c^2 = a^2 + b^2 - 2ab\cos C.$$

The Law of Sines pairs each angle with its opposite side (use it when you know an angle-side pair). The Law of Cosines is the Pythagorean Theorem with a correction term $-2ab\cos C$; when $C = 90°$, $\cos C = 0$ and it collapses back to $c^2 = a^2 + b^2$.

**Law of Cosines for a non-right triangle**

A triangle has sides $a = 7$ and $b = 10$ with the included angle $C = 40°$ between them. Find the third side $c$:

$$c^2 = 7^2 + 10^2 - 2(7)(10)\cos 40° = 49 + 100 - 140(0.766) \approx 149 - 107.2 = 41.8.$$

$$c = \sqrt{41.8} \approx 6.5.$$

Rounded to the nearest tenth, $c \approx 6.5$.

**Right triangles in 3D: the space diagonal**

A rectangular box with length $\ell$, width $w$, and height $h$ has a **space diagonal** $d$ running corner to opposite corner. Use the Pythagorean Theorem *twice*: first the diagonal of the base is $\sqrt{\ell^2 + w^2}$, then that base diagonal and the height $h$ form a second right triangle:

$$d = \sqrt{\ell^2 + w^2 + h^2}.$$

The one clean formula just adds the squares of all three dimensions.

**Space diagonal of a box**

Find the space diagonal of a $3 \times 4 \times 12$ box.

$$d = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

Notice the base diagonal is $\sqrt{3^2 + 4^2} = 5$, and then $\sqrt{5^2 + 12^2} = 13$ — two nested triples (3-4-5 and 5-12-13).

**Two observers, one balloon**

Two people stand 200 ft apart on level ground, a balloon directly between them on the line joining them. One measures an angle of elevation of $50°$, the other $40°$. Find the height $h$. Let the foot of the balloon be $x$ ft from the first person, so it is $200 - x$ ft from the second. Each person sees the same height:

$$h = x \tan 50°, \qquad h = (200 - x) \tan 40°.$$

Set them equal: $x \tan 50° = (200 - x) \tan 40°$, so $1.19x = (200 - x)(0.839)$, giving $1.19x = 167.8 - 0.839x$, hence $2.03x = 167.8$ and $x \approx 82.7$. Then

$$h = 82.7 \tan 50° \approx 82.7(1.19) \approx 98.5 \text{ ft}.$$

Rounded to the nearest tenth, the balloon is about 98.5 ft high.

**The altitude on the hypotenuse: geometric means**

Drop the altitude from the right angle to the hypotenuse. It splits the hypotenuse into two pieces $p$ and $q$ and creates two smaller triangles, each *similar* to the original. Similarity gives three "geometric mean" relations:

$$(\text{altitude}) = \sqrt{p\,q}, \qquad (\text{leg}_1) = \sqrt{p\,(p+q)}, \qquad (\text{leg}_2) = \sqrt{q\,(p+q)}.$$

The altitude is the geometric mean of the two hypotenuse pieces; each leg is the geometric mean of the whole hypotenuse and the piece next to it.

**Finding the altitude by geometric mean**

The altitude to the hypotenuse splits it into pieces $p = 4$ and $q = 9$. Find the altitude $a$:

$$a = \sqrt{p\,q} = \sqrt{4 \cdot 9} = \sqrt{36} = 6.$$

As a check, the shorter leg is $\sqrt{4(4+9)} = \sqrt{52} = 2\sqrt{13}$ and the longer leg is $\sqrt{9(13)} = \sqrt{117} = 3\sqrt{13}$; indeed $(2\sqrt{13})^2 + (3\sqrt{13})^2 = 52 + 117 = 169 = 13^2$, matching the full hypotenuse $4 + 9 = 13$. ✓

**Big picture:** Almost every idea here is the Pythagorean Theorem wearing a costume — the special-triangle ratios, the identity $\sin^2\theta + \cos^2\theta = 1$, the Law of Cosines, and the 3D space diagonal all reduce to "add the squares." When a problem looks new, hunt for the right triangle hiding inside it.

# 9 Quadrilaterals & Polygons

*Vocabulary, angle formulas, special quadrilaterals, and the family tree*

## 9.1 Polygon Vocabulary

**What is a polygon?**

A **polygon** is a closed, flat shape made of straight line segments. Each segment is a **side**, each corner where two sides meet is a **vertex**, and a segment that connects two *non-adjacent* vertices is a **diagonal**.

- **Convex:** no vertex is "pushed in"; every diagonal stays inside the shape.
- **Concave:** at least one vertex is "pushed in" (it has a dent), and at least one diagonal falls outside.
- **Regular:** all sides congruent *and* all angles congruent.
- **Irregular:** not regular (sides or angles differ).

*A regular hexagon.*

*(diagram in the online version)*

**Convex vs. concave**

The pentagon on the left is convex — it bulges outward at every vertex. The arrow shape on the right is concave, because vertex $D$ is pushed inward.

**Counting diagonals**

From a single vertex you can draw $\mathbf{n-3}$ diagonals (you cannot connect a vertex to itself or to its two neighbors). Counting *every* diagonal of the whole polygon:

$$\text{number of diagonals} = \frac{n(n-3)}{2}.$$

**Diagonals of a hexagon**

A hexagon has $n = 6$ sides. From one vertex: $6 - 3 = 3$ diagonals. In all:

$$\frac{6(6-3)}{2} = \frac{6 \cdot 3}{2} = 9 \text{ diagonals.}$$

**Memory hook.** *"Concave"* has the word *cave* in it — a concave polygon has a cave (a dent). Convex shapes have no caves.

## 9.2 Sum of the Interior Angles

**The interior-angle-sum formula**

Every polygon can be split into triangles by drawing all diagonals from one vertex. A polygon with $n$ sides splits into $n - 2$ triangles, and each triangle holds 180°. So:

$$\text{sum of interior angles} = (n-2) \cdot 180^\circ.$$

*A regular hexagon.*

*(diagram in the online version)*

**Interior-angle sum of a pentagon**

A pentagon has $n = 5$ sides. Drawing diagonals from one vertex makes $5 - 2 = 3$ triangles.

$$(5-2) \cdot 180^\circ = 3 \cdot 180^\circ = 540^\circ.$$

**Finding a missing angle**

Four angles of a pentagon are $100^\circ$, $115^\circ$, $120^\circ$, and $90^\circ$. Find the fifth. The five angles must total $540^\circ$:

$$100 + 115 + 120 + 90 = 425, \qquad 540 - 425 = 115.$$

The missing angle is $115^\circ$.

**Tip.** Always find the total first with $(n-2)\cdot 180^\circ$, then subtract the angles you already know to reveal the missing one.

## 9.3 Each Interior Angle of a Regular Polygon

**Sharing the total equally**

In a **regular** polygon all $n$ interior angles are equal, so each one is the total divided by $n$:

$$\text{each interior angle} = \frac{(n-2)\cdot 180^\circ}{n}.$$

*A regular hexagon.*

*(diagram in the online version)*

**Interior angle of a regular hexagon**

A regular hexagon has $n = 6$.

$$\frac{(6-2)\cdot 180^\circ}{6} = \frac{4\cdot 180^\circ}{6} = \frac{720^\circ}{6} = 120^\circ.$$

Every interior angle of a regular hexagon is $120^\circ$.

**Watch out.** The formula $\frac{(n-2)180^\circ}{n}$ gives the measure of *one* angle only when the polygon is *regular*. In an irregular polygon the angles can be all different.

## 9.4 Sum of the Exterior Angles

**The exterior angles always total $360^\circ$**

At each vertex, extend one side; the angle between that extension and the next side is an **exterior angle**. No matter how many sides a convex polygon has,

$$\text{sum of exterior angles} = 360^\circ.$$

For a **regular** polygon each exterior angle is the same:

$$\text{each exterior angle} = \frac{360^\circ}{n}.$$

An interior angle and its exterior angle form a linear pair, so they add to 180°.

*A regular hexagon.*

*(diagram in the online version)*

**Exterior angle of a regular octagon**

A regular octagon has $n = 8$, so each exterior angle is

$$\frac{360^\circ}{8} = 45^\circ.$$

Then each interior angle is $180^\circ - 45^\circ = 135^\circ$.

**Working backward from an exterior angle**

A regular polygon has an exterior angle of 30°. How many sides?

$$\frac{360^\circ}{n} = 30^\circ \;\Rightarrow\; n = \frac{360}{30} = 12.$$

It is a regular dodecagon (12 sides).

**Fast trick.** To find the number of sides of a regular polygon, it is usually easier to use the *exterior* angle: $n = \dfrac{360^\circ}{\text{exterior angle}}$.

## 9.5 Properties of Parallelograms

**What makes a parallelogram?**

A **parallelogram** is a quadrilateral with *both* pairs of opposite sides parallel. From that one fact, four properties always follow:

- Opposite sides are **congruent**.
- Opposite angles are **congruent**.
- Consecutive (next-door) angles are **supplementary** (add to 180°).
- The diagonals **bisect each other** (cut each other in half).

*Height is perpendicular to the base.*

*(diagram in the online version)*

**Reading a parallelogram**

In parallelogram $ABCD$, side $AB \parallel DC$ and $AD \parallel BC$. Because opposite angles are equal, $m\angle A = m\angle C$ and $m\angle B = m\angle D$. Because consecutive angles are supplementary, $m\angle A + m\angle B = 180°$. If $m\angle A = 70°$, then $m\angle C = 70°$ and $m\angle B = m\angle D = 110°$.

**Diagonals bisect each other**

The diagonals of parallelogram $PQRS$ meet at $M$. Since they bisect each other, $PM = MR$ and $QM = MS$. If $PM = 5$, then $PR = 10$.

**Remember.** "Opposite parts match, next-door angles add to 180°." Diagonals bisect each other, but in a *general* parallelogram they are *not* equal and *not* perpendicular.

## 9.6 Special Parallelograms: Rectangle, Rhombus, Square

**Extra powers**

Each special parallelogram keeps *all* parallelogram properties and adds more:

- **Rectangle:** four right angles; diagonals are **congruent**.
- **Rhombus:** four congruent sides; diagonals are **perpendicular** and **bisect the angles**.
- **Square:** a rectangle *and* a rhombus — four right angles *and* four equal sides; diagonals are congruent, perpendicular, and bisect the angles.

*Height is perpendicular to the base.*

*(diagram in the online version)*

**A rhombus and a square with their diagonals**

The rhombus (left) has four equal sides and diagonals that cross at a right angle. The square (right) has everything: equal sides, right angles, and equal, perpendicular diagonals.

**Diagonal shortcut.** *Rect*angle → diagonals are equal ($R$ for *Right* angles). *Rhombus* →

diagonals are perpendicular. *Square* → both at once.

## 9.7 Trapezoids and Kites

**Trapezoid parts**

A **trapezoid** has *exactly one* pair of parallel sides, called the **bases**. The other two sides are the **legs**. The **midsegment** connects the midpoints of the legs; it is parallel to the bases and its length is the *average* of the two bases:

$$\text{midsegment} = \frac{b_1 + b_2}{2}.$$

An **isosceles trapezoid** has congruent legs; its base angles are congruent and its diagonals are congruent.

*A trapezoid has two parallel bases.*

*(diagram in the online version)*

**Midsegment of a trapezoid**

A trapezoid has bases $b_1 = 6$ and $b_2 = 10$. The midsegment is

$$\frac{6 + 10}{2} = \frac{16}{2} = 8.$$

**A kite**

A **kite** has two pairs of *consecutive* congruent sides (not opposite). One pair of opposite angles is congruent, the diagonals are **perpendicular**, and one diagonal bisects the other. Tick marks show the two pairs of equal sides.

**Don't mix them up.** A *trapezoid* has one pair of *parallel* sides; a *kite* has two pairs of *equal adjacent* sides. Both a rhombus and a kite have perpendicular diagonals, but only the rhombus has all four sides equal.

## 9.8 The Quadrilateral Family Tree

**Who is who**

Every square is a rectangle and a rhombus; every rectangle and every rhombus is a parallelogram; every parallelogram is a quadrilateral. As you climb *down* the tree, shapes gain more properties; as you climb *up*, they become more general.

**The hierarchy at a glance**

A square sits at the bottom because it inherits every property above it.

**Always, sometimes, or never?**

- A square is **always** a rectangle (it has four right angles). ✓
- A rectangle is **sometimes** a square (only when its sides are all equal).
- A parallelogram is **never** a trapezoid *under the "exactly one pair"* definition, because a parallelogram has *two* pairs of parallel sides.

**Reasoning tip.** For "always / sometimes / never," ask: does the lower shape's definition *force* the upper property? If yes → *always.* If it can happen but isn't required → *sometimes.* If the definitions clash → *never.*

## 9.9 Going Deeper: Advanced Polygon Ideas

**Cyclic quadrilaterals: opposite angles are supplementary**

A **cyclic quadrilateral** is one whose four vertices all lie on a single circle. Its defining property is beautiful and useful:

$$\text{opposite angles add to } 180^\circ.$$

So if $ABCD$ is cyclic, then $m\angle A + m\angle C = 180^\circ$ and $m\angle B + m\angle D = 180^\circ$. (The reason: an inscribed angle is half its intercepted arc, and the two opposite angles together intercept the *whole* circle, or $360^\circ$; half of that is $180^\circ$.) Every rectangle and every isosceles trapezoid is cyclic; a general parallelogram is cyclic only when it is a rectangle.

*A regular hexagon.*

*(diagram in the online version)*

**Worked example: a missing angle in a cyclic quadrilateral**

Quadrilateral $ABCD$ is inscribed in a circle. Its angles are $m\angle A = 95^\circ$ and $m\angle B = 70^\circ$. Find $m\angle C$ and $m\angle D$. Opposite angles are supplementary, so $\angle A$ pairs with $\angle C$ and $\angle B$ pairs with $\angle D$:

$$m\angle C = 180^\circ - 95^\circ = 85^\circ, \qquad m\angle D = 180^\circ - 70^\circ = 110^\circ.$$

Check: all four angles total $95 + 70 + 85 + 110 = 360^\circ$, exactly the interior-angle sum of any quadrilateral.

### Varignon's theorem: the midpoint quadrilateral

Take *any* quadrilateral and mark the midpoint of each of its four sides. Connect those midpoints in order. The result — called the **Varignon parallelogram** — is *always a parallelogram*, no matter how lopsided the original shape is.

The key idea is the triangle **midsegment**: in triangle $ABC$, the segment joining the midpoints of $AB$ and $BC$ is parallel to diagonal $AC$ and half its length. The same holds on the other side, so both pairs of opposite midpoint-sides are parallel — the definition of a parallelogram.

**Consequences of Varignon.** The Varignon parallelogram's sides are half the lengths of the original diagonals, so its perimeter equals the *sum of the diagonals* of the outer quadrilateral, and its area is exactly *half* the area of the original quadrilateral.

### The apothem and the area of a regular polygon

The **apothem** $a$ of a regular polygon is the perpendicular distance from the center to the middle of any side. Slice the polygon into $n$ congruent triangles, each with base equal to a side $s$ and height equal to the apothem. Adding the triangle areas gives

$$\text{Area} = \tfrac{1}{2}\, a\, P,$$

where $P = ns$ is the perimeter. In words: *half the apothem times the perimeter.*

### Worked example: area of a regular hexagon from its apothem

A regular hexagon has side length $s = 6$ and apothem $a = 3\sqrt{3} \approx 5.196$. Find its area. The perimeter is $P = ns = 6 \cdot 6 = 36$. Then

$$\text{Area} = \tfrac{1}{2}\, a\, P = \tfrac{1}{2}\,(3\sqrt{3})(36) = 54\sqrt{3} \approx 93.5.$$

So the hexagon covers about 93.5 square units.

### Area from coordinates: the shoelace formula

When you know the corner coordinates of a polygon, you can find its area *directly* — no base or height needed. List the vertices in order (all the way around) as $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then

$$\text{Area} = \tfrac{1}{2}\left|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_ny_1 - x_1y_n)\right|.$$

It is called the **shoelace** formula because the cross-multiplications lace diagonally like shoelaces. Go around once in a consistent direction and take the absolute value at the end.

**Worked example: quadrilateral area by the shoelace method**

Find the area of quadrilateral with vertices $A(1,1)$, $B(5,2)$, $C(4,5)$, and $D(2,4)$, taken in order. Set up the "down-then-up" products and subtract:

$$\begin{aligned} \text{down products } (x_i y_{i+1}) &: \ 1\cdot 2 + 5\cdot 5 + 4\cdot 4 + 2\cdot 1 = 2 + 25 + 16 + 2 = 45, \\ \text{up products } (y_i x_{i+1}) &: \ 1\cdot 5 + 2\cdot 4 + 5\cdot 2 + 4\cdot 1 = 5 + 8 + 10 + 4 = 27. \end{aligned}$$

Then

$$\text{Area} = \tfrac{1}{2}\,|45 - 27| = \tfrac{1}{2}\,(18) = 9 \text{ square units.}$$

**Finding $n$ from an angle condition**

Angle formulas run *both* directions: given a single interior or exterior angle of a regular polygon, you can solve for the number of sides $n$.

- From an **exterior** angle $E$: $\quad n = \dfrac{360^\circ}{E}.$
- From an **interior** angle $I$: first get the exterior angle $E = 180^\circ - I$, then $n = \dfrac{360^\circ}{E}$; or solve $\dfrac{(n-2)180^\circ}{n} = I$ directly.

A valid answer must be a whole number $n \geq 3$ — if the division is not exact, no regular polygon has that angle.

**Worked example: which regular polygon has a $150^\circ$ interior angle?**

Each interior angle is $I = 150^\circ$, so each exterior angle is

$$E = 180^\circ - 150^\circ = 30^\circ, \qquad n = \frac{360^\circ}{30^\circ} = 12.$$

It is a regular dodecagon. As a check, try the direct equation:

$$\frac{(n-2)180}{n} = 150 \ \Rightarrow\ 180n - 360 = 150n \ \Rightarrow\ 30n = 360 \ \Rightarrow\ n = 12.\checkmark$$

By contrast, an interior angle of $140^\circ$ gives $E = 40^\circ$ and $n = 9$ (a nonagon), while $155^\circ$ gives $E = 25^\circ$ and $n = 14.4$ — not a whole number, so *no* regular polygon has a $155^\circ$ interior angle.

**Justifying always / sometimes / never**

The family tree turns every classification claim into a logic question. Compare the *definitions* of the two shapes:

- **Always** — the first shape's definition *forces* the second's properties. *Every rhombus is a* *parallelogram*: having four equal sides guarantees both pairs of opposite sides are parallel.

- **Sometimes** — the property *can* hold but is not required. *A rhombus is sometimes a square*: only when its angles are also right angles.
- **Never** — the definitions *contradict*. *A square is never a (strict) trapezoid*: a square has two pairs of parallel sides, but the strict trapezoid definition demands exactly one.

The habit that never fails: state each definition, then ask whether the first *must*, *may*, or *cannot* satisfy the second.

**Big picture.** These advanced tools all come from the same two engines you already know: (1) angles built from 180° and 360°, and (2) breaking a polygon into triangles. Cyclic angles, the apothem area, and Varignon's theorem are just those engines pushed one step further.

# 10 Circles

*Parts, circumference, area, angles, tangents, chords, and equations*

## 10.1 The Parts of a Circle

**Naming the pieces**

A **circle** is the set of all points that are the same distance from one point, the **center**. We name a circle by its center: the circle below is "circle $O$," written $\odot O$.

- **Radius:** a segment from the *center* to a point *on* the circle.
- **Diameter:** a chord that passes *through the center*. It is the longest chord, and $d = 2r$.
- **Chord:** a segment whose *two endpoints are on* the circle.
- **Secant:** a *line* that cuts the circle at *two* points.
- **Tangent:** a *line* that touches the circle at *exactly one* point.

*A sector cut by a central angle.*

*(diagram in the online version)*

**A circle with its interior parts labeled**

Here $\overline{OC}$ is a radius, $\overline{AB}$ is a diameter (it goes through $O$), and $\overline{DE}$ is a chord (its endpoints are on the circle, but it misses the center).

**Secant line vs. tangent line**

A **secant** slices through the circle at two points. A **tangent** just grazes it at one point, $P$ (the *point of tangency*).

**Arcs: pieces of the circle itself**

An **arc** is a curved piece of the circle between two points.

- **Minor arc:** the *shorter* way around (less than 180°). Named with two letters, like $\overset{\frown}{AB}$.
- **Major arc:** the *longer* way around (more than 180°). Named with three letters to show which way you travel.
- **Semicircle:** exactly *half* the circle (180°); its endpoints are the ends of a diameter.

**Minor arc and major arc**

Points $A$ and $B$ split the circle into two arcs. The short (thick) path is minor arc $\overset{\frown}{AB}$; the long way around is the major arc.

**Radius vs. diameter.** The diameter is always *twice* the radius: $d = 2r$, so $r = \frac{1}{2}d$. Mixing these up is the most common circle mistake — always ask "do I have the radius or the diameter?"

### 10.2 Circumference and Arc Length

**Circumference: the distance around**

The **circumference** $C$ is the perimeter of a circle:

$$C = 2\pi r = \pi d.$$

The number $\pi$ ("pi") is about 3.14. We leave answers *in terms of* $\pi$ unless a decimal is asked for; then we use $\pi \approx 3.14$.

*A sector cut by a central angle.*

*(diagram in the online version)*

**Circumference from the radius**

A circle has radius $r = 5$ cm.

$$C = 2\pi r = 2\pi(5) = 10\pi \text{ cm} \approx 10(3.14) = 31.4 \text{ cm}.$$

**Arc length: part of the way around**

An arc is a fraction of the whole circle. If its central angle is $n°$, then

$$\text{arc length} = \frac{n}{360} \cdot 2\pi r.$$

The fraction $\frac{n}{360}$ is "how much of the full 360°" the arc covers.

**Arc length**

Find the length of an arc with central angle 90° in a circle of radius 8.

$$\text{arc length} = \frac{90}{360} \cdot 2\pi(8) = \frac{1}{4} \cdot 16\pi = 4\pi \approx 12.56.$$

**Watch the units.** Circumference and arc length are *lengths*, so they use ordinary units (cm, m, in). Later, area will use *square* units. If you see "cm$^2$" on a length answer, something went wrong.

## 10.3 Central Angles and Arc Measure

**A central angle equals its arc**

A **central angle** has its vertex at the *center* of the circle. The **measure of a minor arc equals the measure of its central angle**. A full trip around the circle is 360°, so all the central angles around the center add to 360°.

*A sector cut by a central angle.*

*(diagram in the online version)*

**Central angle = arc**

The central angle $\angle AOB$ measures 60°, so the intercepted arc $\overset{\frown}{AB}$ also measures 60°. The major arc is $360° - 60° = 300°$.

**Everything adds to $360°$.** If a circle is cut into central angles, their measures total $360°$. To find a missing central angle, subtract the known ones from $360°$.

### 10.4 Inscribed Angles

**The Inscribed Angle Theorem**

An **inscribed angle** has its vertex *on* the circle, and its two sides are chords. The

$$\text{inscribed angle} = \tfrac{1}{2}\,(\text{intercepted arc}).$$

So the intercepted arc is *twice* the inscribed angle. (Notice a central angle equals its arc, but an inscribed angle is only *half* of it.)

*A sector cut by a central angle.*

*(diagram in the online version)*

**Central angle vs. inscribed angle on the same arc**

Both angles below open onto arc $\overset{\frown}{AB}$. The central angle $\angle AOB = 90°$ equals the arc. The inscribed angle $\angle ACB$ is half of that arc: $\frac{1}{2}(90°) = 45°$.

**Angle inscribed in a semicircle is $90°$**

If a chord is a *diameter*, the arc it cuts off is a semicircle ($180°$). Any inscribed angle on that arc is $\frac{1}{2}(180°) = 90°$. So $\angle ACB$ below is a right angle.

**Two big ideas.** (1) Inscribed angle = half its arc. (2) An angle inscribed in a semicircle (sitting on a diameter) is always $90°$. This second fact turns diameters into right triangles!

### 10.5 Tangent Lines

**Tangent $\perp$ radius**

A **tangent** touches a circle at exactly one point. At that **point of tangency**, the tangent line is **perpendicular to the radius**. That right angle is the key to almost every tangent problem, because it makes a right triangle you can use with the Pythagorean Theorem.

*A tangent touches at exactly one point.*

*(diagram in the online version)*

**The tangent meets the radius at 90°**

Radius $\overline{OP}$ meets tangent line $t$ at the point of tangency $P$, forming a right angle.

**Two tangents from one outside point are equal**

From an outside point $P$, the two tangent segments to the circle are congruent: $PT_1 = PT_2$.

**Using the right triangle**

A tangent from point $P$ touches $\odot O$ at $T$. If $OP = 13$ and radius $OT = 5$, find the tangent length $PT$. Because $\angle OTP = 90°$:

$$PT = \sqrt{OP^2 - OT^2} = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12.$$

**Draw the radius.** Whenever a tangent appears, draw the radius to the point of tangency. You instantly get a 90° angle — and a right triangle where $OP$ (center to outside point) is the *hypotenuse.*

## 10.6 Chord Relationships

**Two chord facts**

- **Equal chords:** In the same circle, chords that are the same distance from the center are *congruent* (equal in length).
- **Perpendicular from the center:** A radius (or diameter) that is *perpendicular* to a chord *bisects* it — it cuts the chord into two equal halves.

*A chord joins two points on the circle.*

*(diagram in the online version)*

**The perpendicular from the center bisects the chord**

$\overline{OM} \perp \overline{AB}$, so $M$ is the midpoint of chord $\overline{AB}$ and $AM = MB$. This also makes a right triangle $OMA$: if the radius $OA = 5$ and $OM = 3$, then

$$AM = \sqrt{OA^2 - OM^2} = \sqrt{25 - 9} = \sqrt{16} = 4, \quad \text{so } AB = 2(4) = 8.$$

**Half the chord, please.** When you drop a perpendicular from the center to a chord, the right triangle uses *half* the chord as a leg and the *radius* as the hypotenuse. Don't plug in the whole chord by accident.

## 10.7 Area of a Circle and Area of a Sector

### Area of a circle

The **area** inside a circle is

$$A = \pi r^2.$$

Remember it is $r$ *squared*, then times $\pi$. Area is measured in *square* units.

*A sector cut by a central angle.*

*(diagram in the online version)*

### Area from the radius

A circle has radius $r = 5$.

$$A = \pi r^2 = \pi(5)^2 = 25\pi \approx 25(3.14) = 78.5 \text{ square units.}$$

### Area of a sector

A **sector** is a "pizza slice" bounded by two radii and an arc. If its central angle is $n°$,

$$\text{sector area} = \frac{n}{360} \cdot \pi r^2.$$

It is the same fraction $\frac{n}{360}$ of the whole circle that we used for arc length.

### Area of a sector

Find the area of a $60°$ sector in a circle of radius 6.

$$\text{sector area} = \frac{60}{360} \cdot \pi(6)^2 = \frac{1}{6} \cdot 36\pi = 6\pi \approx 18.84.$$

**Same fraction, two uses.** The fraction $\frac{n}{360}$ scales the *whole circumference* to get arc length, and scales the *whole area* to get sector area. Learn it once, use it twice!

## 10.8 The Equation of a Circle

**Center–radius form**

A circle with center $(h, k)$ and radius $r$ has the equation

$$(x-h)^2 + (y-k)^2 = r^2.$$

Read it carefully: the numbers subtracted from $x$ and $y$ give the *center*, and the right side is $r^2$ (so take the *square root* to get the radius).

*Everything follows from the radius.*

*(diagram in the online version)*

**Writing an equation**

A circle has center $(3, 3)$ and radius 2. Its equation is

$$(x-3)^2 + (y-3)^2 = 2^2 = 4.$$

**Reading center and radius from an equation**

Given $(x-2)^2 + (y+3)^2 = 16$. Rewrite $y+3$ as $y-(-3)$, so the center is $(2, -3)$. Since $r^2 = 16$, the radius is $r = \sqrt{16} = 4$.

**Mind the signs.** The center uses the *opposite* sign of what you see: $(x-2)$ means $h = +2$, and $(y+3) = (y-(-3))$ means $k = -3$. And the right side is $r^2$, *not* $r$ — always take the square root.

### 10.9 Going Deeper: Advanced Circle Ideas

**Proving the Inscribed Angle Theorem**

Why is an inscribed angle exactly *half* its arc? Look at the easy case where one side of the angle is a **diameter**. Draw radius $\overline{OB}$. Since $OB = OC = r$, triangle $OBC$ is *isosceles*, so its base angles are equal: $\angle OCB = \angle OBC$. Now $\angle AOB$ is an **exterior angle** of that triangle, so it equals the sum of the two remote interior angles:

$$\angle AOB = \angle OCB + \angle OBC = 2\,\angle OCB = 2\,\angle ACB.$$

The central angle $\angle AOB$ equals arc $\overset{\frown}{AB}$, so the inscribed angle $\angle ACB$ is half that arc. (Any inscribed angle can be split into cases like this by drawing the diameter through $C$.)

*A chord joins two points on the circle.*

*(diagram in the online version)*

### Why the perpendicular from the center bisects a chord

Section on chords stated it; here is *why*. Drop $\overline{OM} \perp \overline{AB}$ and draw radii $\overline{OA}$ and $\overline{OB}$. Then right triangles $OMA$ and $OMB$ share leg $\overline{OM}$ and have equal hypotenuses $OA = OB = r$. By **Hypotenuse–Leg**, the triangles are congruent, so $AM = MB$. The perpendicular from the center *must* split the chord in half.

### The tangent–chord angle

When a **tangent** and a **chord** meet at the point of tangency, the angle they form is *half the intercepted arc* — the same "half the arc" rule as an inscribed angle:

$$\text{tangent–chord angle} = \tfrac{1}{2}\,(\text{intercepted arc}).$$

In the picture, chord $\overline{PB}$ and tangent $t$ meet at $P$; the marked angle is half of arc $\overset{\frown}{PB}$.

### Cyclic quadrilaterals: opposite angles are supplementary

A **cyclic quadrilateral** has all four vertices *on* a circle. Each angle is inscribed, so it equals half of the arc across from it. The two arcs opposite a pair of angles together make the whole circle (360°), so each pair of *opposite* angles adds to half of that:

$$\angle A + \angle C = 180^\circ, \qquad \angle B + \angle D = 180^\circ.$$

### The Power of a Point

If two lines through a point $P$ each meet the circle, the products of the two distances along each line are *equal*. There are three versions of this one idea:

- **Two chords** crossing *inside* at $P$: $PA \cdot PB = PC \cdot PD$.
- **Two secants** from an *outside* point $P$: $PA \cdot PB = PC \cdot PD$ (each product is near point × far point).
- **Secant and tangent** from an outside point, with tangent length $PT$: $PT^2 = PA \cdot PB$.

### Power of a point: two chords

Chords $\overline{AB}$ and $\overline{CD}$ cross at $P$ inside the circle. If $PA = 4$, $PB = 6$, and $PC = 3$, find $PD$:

$$PA \cdot PB = PC \cdot PD \;\Rightarrow\; 4 \cdot 6 = 3 \cdot PD \;\Rightarrow\; PD = \frac{24}{3} = 8.$$

### Power of a point: secant and tangent

From outside point $P$, a tangent of length $PT$ and a secant hitting the circle at $A$ (near) then $B$ (far) are drawn. If $PA = 4$ and $PB = 9$, find $PT$:

$$PT^2 = PA \cdot PB = 4 \cdot 9 = 36 \;\Rightarrow\; PT = \sqrt{36} = 6.$$

### Area of a segment

A **segment** is the region between a chord and its arc — what is left of a sector after you remove the triangle formed by the two radii. So

$$\text{segment area} = (\text{sector area}) - (\text{triangle area}) = \frac{n}{360}\pi r^2 - (\text{area of the two-radius triangle}).$$

### Area of a segment

Find the area of the segment cut off by a $90^\circ$ arc in a circle of radius 4. The two radii form a right triangle with legs 4 and 4.

$$\begin{aligned} \text{sector} &= \frac{90}{360}\pi(4)^2 = \tfrac{1}{4}\cdot 16\pi = 4\pi, \\ \text{triangle} &= \tfrac{1}{2}(4)(4) = 8, \\ \text{segment} &= 4\pi - 8 \approx 12.56 - 8 = 4.56. \end{aligned}$$

### Finding the center and radius by completing the square

An equation like $x^2 + y^2 - 6x + 4y - 12 = 0$ is a circle in disguise. To find its center and radius, **complete the square** on the $x$-terms and the $y$-terms separately, adding the same amounts to *both* sides. Each time, take half the middle coefficient and square it.

### Completing the square

Put $x^2 + y^2 - 6x + 4y - 12 = 0$ into center–radius form. Group the $x$'s and $y$'s and move the constant over:

$$(x^2 - 6x) + (y^2 + 4y) = 12.$$

Half of $-6$ is $-3$, squared is 9; half of 4 is 2, squared is 4. Add 9 and 4 to both sides:

$$\begin{aligned} (x^2 - 6x + 9) + (y^2 + 4y + 4) &= 12 + 9 + 4, \\ (x-3)^2 + (y+2)^2 &= 25. \end{aligned}$$

So the center is $(3, -2)$ and the radius is $r = \sqrt{25} = 5$.

**One rule, many faces.** "Half the arc" powers inscribed angles, tangent–chord angles, and cyclic quadrilaterals. "Near $\times$ far" powers all three Power-of-a-Point setups. And "sector minus triangle" gives a segment. Spotting which familiar idea is hiding in a hard problem is most of the battle.

# 11 Area & Volume

*Flat shapes, solid shapes, and the formulas that measure them*

## 11.1 Perimeter & Area of Polygons

**Two rules to carry through everything:**

- **Perimeter** is a length, so its units are plain: cm, m, in. **Area** is a *covering*, so its units are *squared*: $\text{cm}^2$. **Volume** is a *filling*, so its units are *cubed*: $\text{cm}^3$.
- Whenever a formula uses $\pi$, we will round with $\pi \approx 3.14$ and *say so*. Always keep the correct unit on your final answer.

*A trapezoid has two parallel bases.*

*(diagram in the online version)*

**Formula Reference — Flat Shapes**

Let $b$ be a base and $h$ the matching *perpendicular* height.

- **Rectangle** (length $\ell$, width $w$): $P = 2\ell + 2w$, $A = \ell w$.
- **Triangle**: $P = a + b + c$ (add the three sides), $A = \frac{1}{2}bh$.
- **Parallelogram**: $P = 2a + 2b$, $A = bh$.
- **Trapezoid** (parallel sides $b_1$ and $b_2$): $A = \frac{1}{2}(b_1 + b_2)\,h$.

The height is always measured *straight across*, at a right angle to the base — never along a slanted side.

**Rectangle: perimeter and area**

For this rectangle, $\ell = 8$ cm and $w = 5$ cm:

$$P = 2(8) + 2(5) = 26 \text{ cm}, \qquad A = 8 \times 5 = 40 \text{ cm}^2.$$

**Triangle and parallelogram**

**Triangle** with base $b = 6$ and height $h = 4$:

$$A = \tfrac{1}{2}(6)(4) = 12 \text{ cm}^2.$$

**Parallelogram** with base $b = 10$ and height $h = 4$:

$$A = (10)(4) = 40 \text{ cm}^2.$$

**Trapezoid**

The two parallel sides are $b_1 = 6$ and $b_2 = 10$; the height between them is $h = 4$:

$$A = \tfrac{1}{2}(b_1 + b_2)\, h = \tfrac{1}{2}(6 + 10)(4) = \tfrac{1}{2}(16)(4) = 32 \text{ cm}^2.$$

**Tip:** The $\frac{1}{2}$ in the triangle and trapezoid formulas is not a coincidence — a triangle is half of a parallelogram, and a trapezoid is the *average* of its two bases times the height.

## 11.2 Regular Polygons & Circles

**Formula Reference — Regular Polygons and Circles**

- **Regular polygon** (all sides and angles equal): the *apothem* $a$ is the distance from the center straight out to the middle of a side. With perimeter $P$,

$$A = \tfrac{1}{2}\, a\, P.$$

- **Circle** with radius $r$ (and diameter $d = 2r$):

$$\text{Circumference } C = 2\pi r = \pi d, \qquad \text{Area } A = \pi r^2.$$

- **Sector** (a "pizza slice" with central angle $\theta$ degrees):

$$\text{Arc length} = \frac{\theta}{360} \cdot 2\pi r, \qquad \text{Sector area} = \frac{\theta}{360} \cdot \pi r^2.$$

**Regular hexagon using the apothem**

A regular hexagon has side 6, so $P = 6 \times 6 = 36$. Its apothem is about $a \approx 5.2$:

$$A = \tfrac{1}{2}(5.2)(36) = 93.6 \text{ cm}^2.$$

**Circle: circumference, area, and a sector (with $\pi \approx 3.14$)**

A circle has radius $r = 10$:

$$C = 2\pi r \approx 2(3.14)(10) = 62.8 \text{ cm}, \qquad A = \pi r^2 \approx 3.14(10)^2 = 314 \text{ cm}^2.$$

A **quarter** of a circle of radius 6 (so $\theta = 90°$) has area

$$\frac{90}{360} \cdot \pi r^2 \approx \tfrac{1}{4}(3.14)(36) = \tfrac{1}{4}(113.04) = 28.26 \text{ cm}^2.$$

**Remember:** $A = \frac{1}{2}aP$ works for *any* regular polygon. A circle is like a regular polygon with endlessly many tiny sides — its "apothem" is the radius and its "perimeter" is the circumference, and $\frac{1}{2}r \cdot 2\pi r = \pi r^2$. Neat!

### 11.3 Composite (Compound) Figures

**Break It Apart, Then Add or Subtract**

A **composite figure** is made of familiar shapes joined together. To find its area:

1. Cut the figure into shapes you know (rectangles, triangles, half-circles, ...).
2. Find each piece's area.
3. **Add** the pieces that are present; **subtract** any region that is cut out.

For perimeter, trace the *outside* edge only, and do not count any inner cut lines.

**A rectangle topped with a half-circle**

This shape is a $6 \times 4$ rectangle with a semicircle of radius 3 sitting on top. Add the two areas ($\pi \approx 3.14$):

$$A_{\text{rect}} = 6 \times 4 = 24, \qquad A_{\text{semi}} = \tfrac{1}{2}\pi r^2 \approx \tfrac{1}{2}(3.14)(3)^2 = 14.13.$$

$$A_{\text{total}} = 24 + 14.13 = 38.13 \text{ cm}^2.$$

**Tip:** When a piece is *removed* (a hole, a bite, a corner cut off), you *subtract* its area. When a piece is *added on*, you *add* it. Always ask: "Is this region part of my shape, or missing from it?"

### 11.4 Surface Area of Solids

### Formula Reference — Surface Area

Surface area (SA) is the total area of *all* the outside faces — the amount of wrapping paper needed. It is always in *square* units.

- **Prism** (base area $B$, base perimeter $P$, height $h$): $SA = 2B + Ph$.
- **Cube** (edge $s$): $SA = 6s^2$ (six equal square faces).
- **Cylinder** (radius $r$, height $h$): $SA = 2\pi r^2 + 2\pi rh$.
- **Pyramid** (base area $B$, base perimeter $P$, *slant height* $\ell$): $SA = B + \frac{1}{2}P\ell$.
- **Cone** (radius $r$, slant height $\ell$): $SA = \pi r^2 + \pi r\ell$.
- **Sphere** (radius $r$): $SA = 4\pi r^2$.

The **slant height** $\ell$ runs up the slanted *face* of a pyramid or cone — it is longer than the straight-up height.

*A cylinder.*

*(diagram in the online version)*

### Rectangular prism

A box measures $\ell = 4$, $w = 3$, $h = 2$. Add the areas of all six faces (opposite faces match):

$$SA = 2(\ell w + \ell h + wh) = 2(4 \cdot 3 + 4 \cdot 2 + 3 \cdot 2) = 2(12 + 8 + 6) = 52 \text{ cm}^2.$$

### Cylinder (with $\pi \approx 3.14$)

A can has radius $r = 3$ and height $h = 10$. The surface is two circular ends plus the wrapper:

$$SA = 2\pi r^2 + 2\pi rh \approx 2(3.14)(9) + 2(3.14)(3)(10) = 56.52 + 188.4 = 244.92 \text{ cm}^2.$$

### Pyramid and cone (using slant height $\ell$)

**Square pyramid**, base side 6 (so $B = 36$, $P = 24$), slant height $\ell = 5$:

$$SA = B + \frac{1}{2}P\ell = 36 + \frac{1}{2}(24)(5) = 36 + 60 = 96 \text{ cm}^2.$$

**Cone**, radius $r = 3$, slant height $\ell = 5$ ($\pi \approx 3.14$):

$$SA = \pi r^2 + \pi r\ell \approx 3.14(9) + 3.14(3)(5) = 28.26 + 47.1 = 75.36 \text{ cm}^2.$$

**Sphere (with $\pi \approx 3.14$)**

A ball has radius $r = 6$:

$$SA = 4\pi r^2 \approx 4(3.14)(6)^2 = 4(3.14)(36) = 452.16 \text{ cm}^2.$$

**Remember:** For a prism, $Ph$ is the "label around the can" (the sides), and $2B$ is the two ends. For a pyramid or cone, use the *slant* height $\ell$ for the slanted faces — save the straight-up height for *volume*.

### 11.5 Volume of Solids

**Formula Reference — Volume**

Volume is how much space fills the inside — always in *cubic* units.

- **Prism or cylinder**: $V = Bh$, where $B$ is the base area and $h$ the height.
  - Rectangular prism: $V = \ell wh$. Cube: $V = s^3$. Cylinder: $V = \pi r^2 h$.
- **Pyramid or cone**: $V = \frac{1}{3}Bh$ (exactly one-third of the matching prism or cylinder).
  - Cone: $V = \frac{1}{3}\pi r^2 h$.
- **Sphere**: $V = \frac{4}{3}\pi r^3$.

*A cylinder.*

*(diagram in the online version)*

**Prism, cylinder, cone, and sphere ($\pi \approx 3.14$)**

**Box** $4 \times 3 \times 2$: $V = \ell wh = 4 \cdot 3 \cdot 2 = 24 \text{ cm}^3$.

**Cylinder** $r = 3$, $h = 10$: $V = \pi r^2 h \approx 3.14(9)(10) = 282.6 \text{ cm}^3$.

**Cone** $r = 3$, $h = 4$: $V = \frac{1}{3}\pi r^2 h \approx \frac{1}{3}(3.14)(9)(4) = \frac{1}{3}(113.04) = 37.68 \text{ cm}^3$.

**Sphere** $r = 6$: $V = \frac{4}{3}\pi r^3 \approx \frac{4}{3}(3.14)(216) = \frac{4}{3}(678.24) = 904.32 \text{ cm}^3$.

**Pyramid: one-third of its box**

A square pyramid has base $6 \times 6$ (so $B = 36$) and height $h = 10$:

$$V = \frac{1}{3}Bh = \frac{1}{3}(36)(10) = 120 \text{ cm}^3.$$

A full prism with that base and height would hold 360 $cm^3$ — the pyramid is exactly one-third of it.

**Remember:** A pyramid or cone always holds $\frac{1}{3}$ of the prism or cylinder that shares its base and height. If you forget the $\frac{1}{3}$, your answer will be three times too big!

## 11.6 Scaling Dimensions & Real-World Problems

### What Happens When You Resize a Shape

If you multiply *every* length of a figure by the same factor $k$:

- Perimeter (a length) is multiplied by $k$.
- Area is multiplied by $k^2$.
- Volume is multiplied by $k^3$.

So doubling the sides ($k = 2$) makes area $4\times$ and volume $8\times$ bigger.

### Scaling a cube

A cube with edge 2 has volume $2^3 = 8$. Triple every edge to 6: the new volume is $6^3 = 216$. That is $216 \div 8 = 27$ times as large — and sure enough, $k^3 = 3^3 = 27$.

### A real-world cost problem

A rectangular floor is 5 m by 4 m. Carpet costs \$3 per square meter. The area is $5\times4 = 20$ $m^2$, so the cost is

$$20 \times \$3 = \$60.$$

**Tip:** Real-world word problems are just formula problems in a costume. Read carefully to spot which measurement you need — area for painting or carpeting, volume for filling or capacity — then round money sensibly and keep the units.

## 11.7 Going Deeper: Advanced Area & Volume

### Heron's Formula — Area from the Three Sides

Sometimes you know a triangle's three sides but *not* its height. **Heron's formula** finds the area directly. First compute the *semiperimeter* (half the perimeter):

$$s = \frac{a+b+c}{2}.$$

Then the area is

$$A = \sqrt{s\,(s-a)(s-b)(s-c)}\,.$$

This always works for a valid triangle, and it never needs a perpendicular height.

*Half of a rectangle with the same base and height.*

*(diagram in the online version)*

### Worked example: Heron's formula on a 13–14–15 triangle

A triangle has sides $a = 13$, $b = 14$, $c = 15$. First the semiperimeter:

$$s = \frac{13+14+15}{2} = \frac{42}{2} = 21.$$

Now the differences are $s-a = 8$, $s-b = 7$, $s-c = 6$, so

$$A = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{7056} = 84 \text{ cm}^2.$$

### From Coordinates: the Shoelace Formula & Pick's Theorem

When a polygon is given by the coordinates of its corners, two tools find its area without any height at all.

**Shoelace formula.** List the vertices in order (going around once) as $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, then loop back to the first. Multiply "down the diagonals" and subtract:

$$A = \tfrac{1}{2}\left|\sum_{i=1}^{n}\bigl(x_i\,y_{i+1} - x_{i+1}\,y_i\bigr)\right|.$$

It is called the *shoelace* because the cross-multiplications criss-cross like laces.

**Pick's theorem.** If *every* corner sits on a grid (lattice) point, count $I$, the interior grid points, and $B$, the grid points *on* the boundary. Then

$$A = I + \frac{B}{2} - 1.$$

Two very different counts, one exact area — and they must agree.

### Worked example: shoelace and Pick's theorem agree

Take the lattice triangle with corners $(0,0)$, $(4,0)$, and $(2,3)$.

**Shoelace** (list the corners, wrap back to the start):

$$A = \tfrac{1}{2}\left|(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\right|.$$

$$A = \tfrac{1}{2}\left|(0\cdot 0 - 4\cdot 0) + (4\cdot 3 - 2\cdot 0) + (2\cdot 0 - 0\cdot 3)\right| = \tfrac{1}{2}|0 + 12 + 0| = 6 \text{ units}^2.$$

**Pick's theorem.** The boundary points are the 3 corners plus the 3 extra grid points along the bottom edge from $(0,0)$ to $(4,0)$ — the slanted edges pass through no grid points — so $B = 6$. The interior points are $(1,1), (2,1), (3,1), (2,2)$, so $I = 4$:

$$A = I + \frac{B}{2} - 1 = 4 + \frac{6}{2} - 1 = 4 + 3 - 1 = 6 \text{ units}^2.$$

Both methods give 6.

### Combined Solids — Deriving SA and Volume

A **combined (composite) solid** is glued from familiar solids: a cylinder capped by a hemisphere, an ice-cream cone (cone + hemisphere), a house (prism + pyramid), and so on. Treat it like a composite figure, but in 3D:

- **Volume:** *add* the volumes of the separate pieces. Nothing is lost where they join.
- **Surface area:** add only the surfaces that are *exposed* on the outside. The circle (or face) where two pieces meet is *inside* the solid — do *not* count it.

So for a cone stacked on a hemisphere of the same radius, the exposed surface is the cone's *slant* side $\pi r\ell$ plus the hemisphere's curved half $2\pi r^2$ — neither flat base circle is counted, because they are hidden at the join.

### Worked example: cone on a hemisphere ($\pi \approx 3.14$)

A solid is a cone (radius $r = 3$, height $h = 4$, so slant height $\ell = 5$) sitting on a hemisphere of the same radius $r = 3$.

**Volume** = cone + hemisphere:

$$V = \tfrac{1}{3}\pi r^2 h + \tfrac{2}{3}\pi r^3 \approx \tfrac{1}{3}(3.14)(9)(4) + \tfrac{2}{3}(3.14)(27) = 37.68 + 56.52 = 94.2 \text{ cm}^3.$$

**Surface area** = cone's slant side + hemisphere's curved half (the shared circle is hidden):

$$SA = \pi r\ell + 2\pi r^2 \approx (3.14)(3)(5) + 2(3.14)(9) = 47.1 + 56.52 = 103.62 \text{ cm}^2.$$

### Cross-Sections, Nets, and the Scaling Laws

**Cross-section:** the flat shape you see when you slice straight through a solid. Slicing a cylinder horizontally gives a *circle*; slicing it vertically through the middle gives a *rectangle*.

Slicing a cube can give a square, a rectangle, or even a triangle or hexagon, depending on the angle.

**Net:** a solid unfolded flat, showing every face at once. A cube's net is 6 squares; a cylinder's net is two circles plus a rectangle whose *width equals the circumference* $2\pi r$. A net is the fastest way to *see* why a surface-area formula adds up — the total area of the net *is* the surface area.

**Scaling:** multiply every length by a factor $k$ and lengths grow by $k$, areas by $k^2$, and volumes by $k^3$. A subtle consequence: the surface-area-to-volume ratio scales like $k^2/k^3 = 1/k$, so *bigger objects have relatively less surface* — which is why large animals hold heat and small ones lose it fast.

**Worked example: an area optimization**

A farmer has 40 m of fencing for a rectangular pen. Which rectangle encloses the *most* area? With perimeter fixed at 40, the width $w$ and length $\ell$ satisfy $\ell + w = 20$, so try a few:

The area is largest when $\ell = w = 10$ — a *square*. Among all rectangles of a given perimeter, the square always wins. (The same pattern holds in 3D: of all rectangular boxes with a fixed surface area, the cube holds the most volume.)

**Big picture:** Every advanced tool here is a shortcut around a missing height. Heron uses the three sides; the shoelace and Pick's theorem use coordinates; combined solids reuse the basic formulas piece by piece. And whenever a length is scaled by $k$, remember the ladder: length $\to k$, area $\to k^2$, volume $\to k^3$.

# 12 Course Review

*Every key definition, theorem, and formula from all ten units*

## 12.1 Points, Lines, Planes & Segments

**How to use this sheet.** Each section is one unit. *Concept boxes* give the definitions and rules, *example boxes* show a worked calculation, and the red *tip box* at the end of each unit warns you about the most common mistake. Keep answers exact (leave $\pi$ and radicals) unless a decimal is requested.

**The undefined terms and basic figures**

- **Point:** a location, no size (named $A$). **Line:** extends forever in two directions ($\overleftrightarrow{AB}$). **Plane:** a flat surface extending forever.
- **Collinear** points lie on one line; **coplanar** points lie in one plane.

- **Segment** $\overline{AB}$: two endpoints and all points between (has length $AB$). **Ray** $\overrightarrow{AB}$: one endpoint, goes forever one way.
- **Congruent** segments have equal length: $\overline{AB} \cong \overline{CD}$ means $AB = CD$.

### Segment addition, midpoint, and distance

- **Segment Addition Postulate:** if $B$ is between $A$ and $C$, then $AB + BC = AC$.
- **Midpoint** $M$ of $\overline{AB}$: the point with $AM = MB$ (it *bisects* the segment).

On a coordinate plane, for $A(x_1, y_1)$ and $B(x_2, y_2)$:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\text{Distance } AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

### Midpoint and distance

For $A(1, 2)$ and $B(7, 10)$:

$$M = \left(\tfrac{1+7}{2}, \tfrac{2+10}{2}\right) = (4, 6), \qquad AB = \sqrt{(7-1)^2 + (10-2)^2} = \sqrt{36 + 64} = \sqrt{100} = 10.$$

**Midpoint adds, distance subtracts.** The midpoint formula *averages* (add the coordinates, divide by 2); the distance formula *subtracts* coordinates first, then squares. Squaring makes signs irrelevant, so order of subtraction does not matter.

## 12.2 Angles & Angle Relationships

### Types of angles

Measured in degrees (°): **acute** $< 90°$, **right** $= 90°$, **obtuse** between $90°$ and $180°$, **straight** $= 180°$. An **angle bisector** splits an angle into two equal halves.

### Angle pairs

- **Complementary:** two angles that add to $90°$.
- **Supplementary:** two angles that add to $180°$.
- **Linear pair:** adjacent angles on a straight line — they are supplementary ($= 180°$).
- **Vertical angles:** the opposite angles formed by two crossing lines — they are *congruent*

(equal).

- **Angle Addition Postulate:** if $D$ is inside $\angle ABC$, then $\angle ABD + \angle DBC = \angle ABC$.

**Vertical and linear pair**

Two lines cross. One angle is $70°$. Its *vertical* angle is also $70°$. Each angle next to it (linear pair) is $180° - 70° = 110°$.

**Vertical vs. linear.** Vertical angles are *equal*; a linear pair is *supplementary* (adds to $180°$). Do not confuse "complementary" ($90°$) with "supplementary" ($180°$) — $c$ comes before $s$, and 90 before 180.

### 12.3 Parallel & Perpendicular Lines

**Angles formed by a transversal**

When a **transversal** cuts two **parallel** lines, these pairs are formed:

- **Corresponding angles:** congruent (equal).
- **Alternate interior angles:** congruent.
- **Alternate exterior angles:** congruent.
- **Co-interior (same-side interior) angles:** supplementary ($= 180°$).

The converse also works: if any of these relationships holds, the lines *are* parallel.

**Slopes of parallel and perpendicular lines**

For slope $m$ (recall $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, "rise over run"):

- **Parallel** lines have *equal* slopes: $m_1 = m_2$.
- **Perpendicular** lines have *opposite-reciprocal* slopes: $m_1 \cdot m_2 = -1$, i.e. $m_2 = -\dfrac{1}{m_1}$.

Horizontal ($m = 0$) and vertical (undefined slope) lines are perpendicular to each other.

**Parallel and perpendicular slopes**

A line has slope $m = \frac{2}{3}$. A line *parallel* to it also has slope $\frac{2}{3}$. A line *perpendicular* to it has slope $-\frac{3}{2}$ (flip and negate).

**Flip and negate.** Perpendicular slope = reciprocal *with the sign changed.* Just flipping ($\frac{2}{3} \to \frac{3}{2}$) or just negating ($\frac{2}{3} \to -\frac{2}{3}$) is not enough — you must do *both.*

## 12.4 Triangles & Congruence

**Angle facts in triangles**

- **Triangle Angle Sum:** the three interior angles add to $180°$.
- **Exterior Angle Theorem:** an exterior angle equals the sum of the two *remote* (non-adjacent) interior angles.
- **Classify by sides:** scalene (none equal), isosceles (two equal), equilateral (all equal). **By angles:** acute, right, obtuse.

**Proving triangles congruent**

Two triangles are congruent if any one of these matches:

- **SSS** — three sides
- **SAS** — two sides + included angle
- **ASA** — two angles + included side
- **AAS** — two angles + a non-included side
- **HL** — (right triangles) hypotenuse + a leg

**SSA** and **AAA** do *not* prove congruence. After proving congruence, **CPCTC** lets you conclude the matching parts are equal.

**Isosceles triangle & the triangle inequality**

- **Isosceles Triangle Theorem:** the angles opposite the two equal sides (*base angles*) are equal — and the converse holds.
- **Triangle Inequality:** the sum of any two sides is greater than the third: $a + b > c$. The third side lies between $|a - b|$ and $a + b$.

**Missing angle and side range**

A triangle has angles $50°$ and $60°$; the third is $180° - 50° - 60° = 70°$. If two sides are 7 and 10, the third side $x$ satisfies $10 - 7 < x < 10 + 7$, i.e. $3 < x < 17$.

**"Included" is everything.** SAS needs the angle *between* the two sides; ASA needs the side *between* the two angles. If the parts are not in that position, the shortcut does not apply (that is why SSA fails).

## 12.5 Similarity

### Similar triangles

Similar figures ($\sim$) have **equal corresponding angles** and **proportional corresponding sides**. Prove triangles similar by:

- **AA** — two pairs of equal angles.
- **SSS$\sim$** — all three side ratios equal.
- **SAS$\sim$** — two side ratios equal with equal included angle.

The common ratio of the sides is the **scale factor** $k$.

### Proportions, midsegment, and scaling

- Solve proportions by **cross-multiplying**: $\frac{a}{b} = \frac{c}{d} \Rightarrow ad = bc$.
- **Triangle Midsegment:** the segment joining midpoints of two sides is *parallel* to the third side and *half* its length.
- **Side-Splitter:** a line parallel to one side cuts the other two sides proportionally.

For scale factor $k$: ratio of **perimeters** $= k$, ratio of **areas** $= k^2$, ratio of **volumes** $= k^3$.

### Scale factor and area

Two similar triangles have sides in ratio $k = \frac{3}{2}$. Their perimeters are in ratio $\frac{3}{2}$, and their areas are in ratio $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

**Square the scale factor for area.** If lengths scale by $k$, area scales by $k^2$ (and volume by $k^3$). Doubling every side does *not* double the area — it makes it 4 times bigger.

## 12.6 Right Triangles & Trigonometry

**Pythagorean Theorem and its converse**

In a right triangle with legs $a, b$ and hypotenuse $c$ (the side opposite the right angle):

$$a^2 + b^2 = c^2.$$

**Converse:** if $a^2 + b^2 = c^2$, the triangle is right. If $a^2 + b^2 > c^2$ it is acute; if $a^2 + b^2 < c^2$ it is obtuse. Common **triples:** 3-4-5, 5-12-13, 8-15-17, 7-24-25 (and their multiples).

**Special right triangles**

$$\begin{aligned} 45°\text{-}45°\text{-}90° &: \ \text{legs equal } x, \ \text{hypotenuse } = x\sqrt{2} \\ 30°\text{-}60°\text{-}90° &: \ \text{short leg } x, \ \text{long leg } = x\sqrt{3}, \ \text{hyp } = 2x \end{aligned}$$

(In 30-60-90 the short leg is opposite 30°; the long leg is opposite 60°.)

**Trigonometric ratios (SOH-CAH-TOA)**

For an acute angle $\theta$ in a right triangle:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

To find a missing *angle*, use **inverse trig**: $\theta = \sin^{-1}\left(\frac{\text{opp}}{\text{hyp}}\right)$, and likewise $\cos^{-1}$, $\tan^{-1}$.

**Finding a side and an angle**

A right triangle has an angle $\theta$ with opposite side 3 and hypotenuse 5. Then $\sin\theta = \frac{3}{5}$, so $\theta = \sin^{-1}(0.6) \approx 36.9°$. The adjacent leg is $\sqrt{5^2 - 3^2} = 4$.

**Ratio for sides, inverse for angles.** If you know the angle and want a side, use sin / cos / tan. If you know two sides and want the angle, use $\sin^{-1}$ / $\cos^{-1}$ / $\tan^{-1}$. Set your calculator to *degrees*.

## 12.7 Quadrilaterals & Polygons

**Angle sums in any polygon**

For a polygon with $n$ sides:

$$\text{sum of interior angles} = (n-2)\cdot 180°, \qquad \text{sum of exterior angles} = 360°.$$

For a **regular** polygon: each interior angle $= \dfrac{(n-2)180°}{n}$, each exterior angle $= \dfrac{360°}{n}$.

**Parallelogram properties**

In every **parallelogram**: opposite sides are parallel *and* congruent; opposite angles are congruent; consecutive angles are supplementary; and the *diagonals bisect each other*.

**The special quadrilaterals**

- **Rectangle:** parallelogram with 4 right angles; diagonals are *congruent*.
- **Rhombus:** parallelogram with 4 equal sides; diagonals are *perpendicular* and bisect the angles.
- **Square:** both a rectangle and a rhombus (all properties of each).
- **Trapezoid:** exactly one pair of parallel sides; *isosceles* trapezoid has congruent legs, congruent base angles, and congruent diagonals.
- **Kite:** two pairs of adjacent congruent sides; diagonals are perpendicular.

**Interior angle of a regular polygon**

A regular hexagon has $n = 6$: interior-angle sum $= (6-2)180° = 720°$, so each angle $= \frac{720°}{6} = 120°$. Each exterior angle $= \frac{360°}{6} = 60°$.

**Exterior angles always total 360°.** No matter how many sides, the exterior angles sum to 360°. That is the fast way to find a regular polygon's angle: exterior $= \frac{360°}{n}$, then interior $= 180° -$ exterior.

## 12.8 Circles

**Circumference, area, and arcs**

With radius $r$ and diameter $d = 2r$:

$$C = 2\pi r = \pi d, \qquad A = \pi r^2.$$

For a central angle of $n°$: arc length $= \dfrac{n}{360} \cdot 2\pi r$, sector area $= \dfrac{n}{360} \cdot \pi r^2$.

**Central, inscribed, and other angles**

- **Central angle** (vertex at center) = its intercepted arc.

- **Inscribed angle** (vertex on circle) $= \frac{1}{2}$ its intercepted arc. An angle inscribed in a semicircle is $90°$.
- **Tangent** $\perp$ radius at the point of tangency; two tangents from one outside point are congruent.

**Power of a Point**

From a point, the products of the two chord/secant/tangent pieces are equal:

$$\begin{aligned} \text{Two chords:} \quad & a \cdot b = c \cdot d \\ \text{Two secants:} \quad & (\text{whole}_1)(\text{outside}_1) = (\text{whole}_2)(\text{outside}_2) \\ \text{Tangent \& secant:} \quad & t^2 = (\text{whole})(\text{outside}) \end{aligned}$$

**Equation of a circle**

A circle with center $(h, k)$ and radius $r$:

$$(x-h)^2 + (y-k)^2 = r^2.$$

The center uses the *opposite* sign of what appears; the right side is $r^2$, so take a square root for $r$.

**Inscribed angle and a circle equation**

An inscribed angle intercepting a $100°$ arc measures $\frac{1}{2}(100°) = 50°$. The equation $(x-2)^2 + (y+3)^2 = 25$ has center $(2, -3)$ and radius $\sqrt{25} = 5$.

**Central equals, inscribed halves.** A central angle *equals* its arc; an inscribed angle is *half* its arc. And length answers (circumference, arc) use plain units, while area answers (circle, sector) use *square* units.

## 12.9 Perimeter, Area & Volume

**Area of 2D figures**

- Rectangle: $A = bh$
- Parallelogram: $A = bh$
- Triangle: $A = \frac{1}{2}bh$
- Trapezoid: $A = \frac{1}{2}(b_1 + b_2)h$
- Rhombus / kite: $A = \frac{1}{2}d_1 d_2$
- Circle: $A = \pi r^2$
- Regular polygon: $A = \frac{1}{2}(\text{apothem})(\text{perimeter})$

**Perimeter** is the distance around; the **circumference** $C = 2\pi r$ is a circle's perimeter.

**Surface area and volume of solids**

Let $B$ = area of the base, $P$ = base perimeter, $h$ = height, $\ell$ = slant height.

$$\begin{aligned} \text{Prism:} \quad & V = Bh, && SA = 2B + Ph \\ \text{Cylinder:} \quad & V = \pi r^2 h, && SA = 2\pi r^2 + 2\pi rh \\ \text{Pyramid:} \quad & V = \tfrac{1}{3}Bh, && SA = B + \tfrac{1}{2}P\ell \\ \text{Cone:} \quad & V = \tfrac{1}{3}\pi r^2 h, && SA = \pi r^2 + \pi r\ell \\ \text{Sphere:} \quad & V = \tfrac{4}{3}\pi r^3, && SA = 4\pi r^2 \end{aligned}$$

**Volume of a cylinder and a cone**

A cylinder with $r = 3$, $h = 10$: $V = \pi(3)^2(10) = 90\pi$. A cone with the same base and height holds one third as much: $V = \frac{1}{3}\pi(3)^2(10) = 30\pi$.

**The one-third family.** Pyramids and cones each get a $\frac{1}{3}$ compared to the prism or cylinder with the same base and height. Area uses *square* units; volume uses *cubic* units — check your units to catch errors.

## 12.10 Transformations & Symmetry

**Rigid motions (isometries)**

These preserve size and shape (image $\cong$ pre-image):

$$\begin{aligned} \text{Translation by } (a,b): \quad & (x,y) \to (x+a,\ y+b) \\ \text{Reflection over } x\text{-axis:} \quad & (x,y) \to (x,\ -y) \\ \text{Reflection over } y\text{-axis:} \quad & (x,y) \to (-x,\ y) \\ \text{Reflection over } y = x: \quad & (x,y) \to (y,\ x) \\ \text{Reflection over } y = -x: \quad & (x,y) \to (-y,\ -x) \end{aligned}$$

**Rotations and dilations**

Rotations about the origin (counterclockwise):

$$\begin{aligned} 90^\circ: \quad & (x,y) \to (-y,\ x) \\ 180^\circ: \quad & (x,y) \to (-x,\ -y) \\ 270^\circ: \quad & (x,y) \to (y,\ -x) \end{aligned}$$

**Dilation** (center origin, scale factor $k$): $(x, y) \to (kx,\ ky)$. This is *not* rigid — it changes size (lengths $\times k$, area $\times k^2$) but keeps shape, so figures are *similar*.

### Symmetry

- **Line (reflection) symmetry:** a fold line maps the figure onto itself.
- **Rotational symmetry:** a turn of less than 360° about a center maps the figure onto itself (a regular $n$-gon has order $n$, smallest angle $\frac{360°}{n}$).

### Applying a rule

Reflect $P(3, 5)$ over the $x$-axis: $(x, y) \to (x, -y)$ gives $P'(3, -5)$. Then rotate $P'$ by 90°: $(x, y) \to (-y, x)$ gives $(5, 3)$.

**Rigid motions keep congruence; dilations make similarity.** Translations, reflections, and rotations preserve length and angle (image $\cong$ pre-image). Only a dilation changes size — and it scales lengths by $k$, area by $k^2$.