MathCastle

Differential Equations

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Contents

1 First-Order Equations

Separable equations and integrating factors

1.1 The Two Workhorse Methods

Separable: sort, then integrate

If $\frac{dy}{dx} = g(x)h(y)$, move all y's left and x's right: $\frac{dy}{h(y)} = g(x)\,dx$, integrate both sides, and solve for y. The prototype $y' = ky$ gives exponential growth/decay $y = y_0 e^{kx}$, and half-life problems are the case $k = -\frac{\ln 2}{t_{1/2}}$.

Linear: multiply by the integrating factor

For $y' + p(x)y = q(x)$, multiply through by $\mu = e^{\int p\,dx}$: the left side collapses to $(\mu y)'$ by design, and one integration finishes it. Example: $y' + y = e^x$ with $\mu = e^x$ gives $(e^x y)' = e^{2x}$, hence $y = \frac{e^x}{2} + Ce^{-x}$.

Always verify by plugging back

Claimed solution of $y' = xy$, $y(0) = 2$: $y = 2e^{x^2/2}$. Check: $y' = 2xe^{x^2/2} = x \cdot y$ ✓ and $y(0) = 2$ ✓ — a thirty-second check that catches most integration slips.

Significance: the two methods cover most of nature

Radioactive dating, drug clearance, RC circuits, compound interest, and atmospheric pressure with altitude are all separable or linear first-order equations. Carbon-14 dating is literally the half-life computation of this topic applied to archaeology: measure the remaining fraction, invert e^{kt}.

Try it: Try it: triage then solve

Classify and solve $y' = \dfrac{x}{y}$ with $y(0) = 2$. Work: separable: $y\,dy = x\,dx$, so $\frac{y^2}{2} = \frac{x^2}{2} + C$; the data give $C = 2$: $y = \sqrt{x^2 + 4}$ (positive root, since $y(0) = 2 > 0$). Verify: $y' = \frac{x}{\sqrt{x^2+4}} = \frac{x}{y}$ ✓.

1.2 Proofs & Why It Matters

Proof: the integrating factor works

With $\mu = e^{\int p\,dx}$, note $\mu' = p\mu$ (chain rule). Then $(\mu y)' = \mu y' + \mu' y = \mu\,(y' + py)$ — the product rule REBUILDS the left side of $y' + py = q$ exactly. So the equation becomes $(\mu y)' = \mu q$; integrate once and divide by μ. The factor is engineered so the product rule does the collapsing.

■

Proof: separation of variables is legitimate

If $\frac{dy}{dx} = g(x)h(y)$ with $h(y) \neq 0$, divide: $\frac{y'(x)}{h(y(x))} = g(x)$. Integrate both sides in x; on the left, substitute $u = y(x)$, $du = y'(x)dx$ (the chain rule in reverse): $\int \frac{du}{h(u)} = \int g(x)\,dx$.

The informal "move the dy" is shorthand for exactly this substitution — nothing illegal happens. ■

Proof: exponential solutions and half-life

$y' = ky$ separates to $\int \frac{dy}{y} = \int k\,dx$, so $\ln|y| = kx + C$ and $y = y_0 e^{kx}$ — and this is the ONLY solution: if z also solves it, then $(ze^{-kx})' = z'e^{-kx} - kze^{-kx} = 0$, so ze^{-kx} is constant. Setting $y(t_{1/2}) = \frac{y_0}{2}$ gives $e^{kt_{1/2}} = \frac{1}{2}$, i.e. $k = -\frac{\ln 2}{t_{1/2}}$. ■

1.3 Going Deeper: Explanations & Worked Problems

Reading an equation: which method, and why

Given $y' =$ (something), the triage is: can the right side be FACTORED as $g(x)h(y)$ — a pure-x piece times a pure-y piece? Then separate. Is it LINEAR in y — expressible as $y' + p(x)y = q(x)$ with y appearing alone to the first power? Then integrating factor.

($y' = x - y$ hides its linearity until rearranged: $y' + y = x$.) Some equations are both ($y' = ky$), many are neither — those go to slope fields, Euler, or substitutions. Two habits matter more than any method. First, TRACK THE CONSTANT from the moment you integrate: it lives inside the exponential as $y = Ce^{kx}$, not tacked on the end as $e^{kx} + C$ — a wrong constant placement is the single most common first-order error. Second, plug the final answer back into the original equation; ten seconds of differentiation catches nearly everything.

Worked: a separable IVP with the constant handled honestly

Solve $y' = xy$ with $y(0) = 2$. Step 1 — separate: $\dfrac{dy}{y} = x\,dx$ (noting $y \equiv 0$ is a solution we are not on, since $y(0) = 2$). Step 2 — integrate BOTH sides: $\ln|y| = \frac{x^2}{2} + C$.

Step 3 — exponentiate carefully: $|y| = e^C e^{x^2/2}$, and absorbing the sign into a new constant $A = \pm e^C$: $y = Ae^{x^2/2}$ — the constant is MULTIPLICATIVE. Step 4 — apply the initial condition: $2 = Ae^0 = A$, so $y = 2e^{x^2/2}$. Step 5 — verify: $y' = 2xe^{x^2/2} \cdot 1 = x \cdot y$ ✓ and $y(0) = 2$ ✓. Watch what Step 3 prevented: writing $y = e^{x^2/2} + C$ and "solving" $C = 1$ gives a function that fails the original equation instantly.

Worked: an integrating-factor IVP, start to finish

Solve $y' + 2y = 6$ with $y(0) = 0$. Step 1 — identify $p = 2$, $q = 6$; the factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Step 2 — multiply the whole equation: $e^{2x}y' + 2e^{2x}y = 6e^{2x}$, whose left side is EXACTLY $(e^{2x}y)'$ — that collapse is guaranteed by the construction, so if it does not factor cleanly, recheck μ.

Step 3 — integrate: $e^{2x}y = 3e^{2x} + C$. Step 4 — solve: $y = 3 + Ce^{-2x}$; the condition $y(0) = 0$ gives $C = -3$, so $y = 3 - 3e^{-2x}$. Step 5 — verify and interpret: $y' = 6e^{-2x}$ and $y' + 2y = 6e^{-2x} + 6 - 6e^{-2x} = 6$ ✓; the solution climbs from 0 toward the equilibrium 3 (where $y' = 0$ forces $2y = 6$) — the phase-line picture and the formula telling one story.

2 Exact Equations & Substitutions

Exactness tests, integrating factors, Bernoulli and homogeneous types

2.1 Exactness and the Great Substitutions

Exact equations: one function behind two coefficients

The form $M(x,y)\,dx + N(x,y)\,dy = 0$ is EXACT when some F has $F_x = M$ and $F_y = N$ — then solutions are simply the level curves $F(x,y) = C$. The test is Clairaut in disguise: exactness holds iff $\partial_y M = \partial_x N$ (both equal the mixed partial F_{xy}).

To build F: integrate M in x (adding an unknown $g(y)$), differentiate in y, and match against N to determine g — the same construction as finding a potential for a conservative vector field, because it IS the same problem.

When the test fails: integrating factors

A non-exact equation can often be repaired by multiplying through by μ. Try single-variable factors first: if $\dfrac{M_y - N_x}{N}$ depends on x alone, $\mu = e^{\int \frac{M_y - N_x}{N}dx}$ works; if $\dfrac{N_x - M_y}{M}$ depends on y alone, use the analogous $\mu(y)$.

Example: $y\,dx + 2x\,dy$ has $M_y = 1 \neq 2 = N_x$, but $\mu = y$ turns it into $y^2dx + 2xy\,dy = d(xy^2)$ — exact. The linear-equation integrating factor from first-order theory is a special case of this idea.

Two substitutions that unlock whole families

HOMOGENEOUS type $y' = f\left(\frac{y}{x}\right)$: substitute $v = \frac{y}{x}$ (so $y' = v + xv'$) and the equation SEPARATES in v and x. BERNOULLI $y' + p(x)y = q(x)y^n$: divide by y^n and substitute $v = y^{1-n}$ — the equation becomes LINEAR in v, solvable by integrating factor.

Both are one-line changes of viewpoint that convert something unsolvable-looking into a solved chapter; recognizing the pattern is the entire skill.

Significance: thermodynamics runs on exactness

In physics, "is $M\,dx + N\,dy$ exact?" becomes "is this quantity a state function?" Energy and entropy are exact differentials (path-independent); heat and work are NOT — and integrating factors are precisely how temperature turns the inexact heat differential into exact entropy: $dS = \frac{\delta Q}{T}$. A chapter of calculus explaining a law of nature.

Try it: Try it: test, then decide

Is $(y^2+1)\,dx + 2xy\,dy = 0$ exact? Work: $\partial_y(y^2+1) = 2y = \partial_x(2xy)$ ✓. Solve: $F = xy^2 + x$; solutions $x(y^2+1) = C$. Verify by implicit differentiation: $(y^2+1) + 2xyy' = 0$ ✓ — matches the original.

2.2 Proofs & Why It Matters

Proof: the exactness test

If F exists with $F_x = M$, $F_y = N$ and F is twice continuously differentiable, then $M_y = F_{xy} = F_{yx} = N_x$ by Clairaut's theorem — necessity.

Conversely, on a rectangle define $F(x,y) = \int_{x_0}^{x} M(t,y)\,dt + \int_{y_0}^{y} N(x_0,s)\,ds$; then $F_x = M$ directly, and differentiating under the integral, $F_y = \int_{x_0}^{x} M_y\,dt + N(x_0,y) = \int_{x_0}^{x} N_x\,dt + N(x_0,y) = N(x,y)$, using $M_y = N_x$ in the middle step. ■

Proof: the Bernoulli substitution linearizes

From $v = y^{1-n}$: $v' = (1-n)y^{-n}y'$. Multiply the equation $y' + py = qy^n$ by $(1-n)y^{-n}$: $(1-n)y^{-n}y' + (1-n)py^{1-n} = (1-n)q$, which is exactly $v' + (1-n)p\,v = (1-n)q$ — linear in v with known coefficient functions. Every Bernoulli equation is therefore one substitution away from the integrating-factor method. ■

2.3 Going Deeper: Worked Problems

Worked: repairing a non-exact equation

Solve $(3xy + y^2)\,dx + (x^2 + xy)\,dy = 0$.

Step 1 — test: $M_y = 3x + 2y$, $N_x = 2x + y$ — not exact. Step 2 — try a factor in x: $\dfrac{M_y - N_x}{N} = \dfrac{x+y}{x(x+y)} = \dfrac{1}{x}$, a function of x alone, so $\mu = e^{\int dx/x} = x$. Step 3 — multiply: $(3x^2y + xy^2)\,dx + (x^3 + x^2y)\,dy = 0$; now $M_y = 3x^2 + 2xy = N_x$ ✓. Step 4 — build F: integrate M in x: $F = x^3y + \frac{1}{2}x^2y^2 + g(y)$; match $F_y = x^3 + x^2y + g'(y) = N$, so $g' = 0$. Solutions: $x^3y + \frac{1}{2}x^2y^2 = C$. Step 5 — implicit differentiation of the answer reproduces the original equation (after dividing by x) ✓.

Worked: a homogeneous-type equation to the end

Solve $y' = \dfrac{x^2+y^2}{xy}$ with $y(1)=2$.

Step 1 — every term has total degree 2 over degree 2: homogeneous type. Rewrite $y' = \dfrac{x}{y} + \dfrac{y}{x}$ and substitute $v = \dfrac{y}{x}$, so $y' = v + xv'$. Step 2 — the equation becomes $v + xv' = \dfrac{1}{v} + v$, i.e. $xv' = \dfrac{1}{v}$: separable. Step 3 — $v\,dv = \dfrac{dx}{x}$ gives $\frac{v^2}{2} = \ln|x| + C$. Step 4 — back-substitute: $\dfrac{y^2}{2x^2} = \ln|x| + C$; the data $y(1)=2$ give $2 = 0 + C$. Solution: $y^2 = 2x^2(\ln x + 2)$, i.e. $y = x\sqrt{2\ln x + 4}$ for $x > 0$. Step 5 — check at $x=1$: $y = \sqrt{4} = 2$ ✓.

2.4 Worked, Every Step

Worked: an exact equation start to finish

Solve $(2xy+3)\,dx + (x^2+4y)\,dy = 0$.

Step 1 — test: $\partial_y(2xy+3) = 2x$ and $\partial_x(x^2+4y) = 2x$: exact. Step 2 — integrate M in x: $F = x^2y + 3x + g(y)$. Step 3 — match: $F_y = x^2 + g'(y) \overset{!}{=} x^2 + 4y$, so $g'(y) = 4y$, $g = 2y^2$. Step 4 — solutions: $x^2y + 3x + 2y^2 = C$, an implicit family; a point condition picks the curve. Step 5 — verify by implicit differentiation: $2xy + x^2y' + 3 + 4yy' = 0$ rearranges to the original equation ✓.

Worked: a Bernoulli equation via v = 1/y

Solve $y' + y = y^2$, $y(0) = \frac{1}{2}$.

Step 1 — divide by y^2: $y^{-2}y' + y^{-1} = 1$. Step 2 — substitute $v = y^{-1}$, $v' = -y^{-2}y'$: $-v' + v = 1$, i.e. $v' - v = -1$. Step 3 — integrating factor e^{-x}: $(ve^{-x})' = -e^{-x}$, so $ve^{-x} = e^{-x} + C$ and $v = 1 + Ce^x$. Step 4 — data: $v(0) = 2$ gives $C = 1$: $y = \dfrac{1}{1+e^x}$. Step 5 — verify: $y' = \dfrac{-e^x}{(1+e^x)^2}$ and $y^2 - y = \dfrac{1-(1+e^x)}{(1+e^x)^2} = \dfrac{-e^x}{(1+e^x)^2}$ ✓.

3 Second-Order Linear Equations

Characteristic roots: oscillate, decay, or both

3.1 Constant Coefficients

The characteristic equation

For $y'' + by' + cy = 0$, try $y = e^{rx}$: $r^2 + br + c = 0$. Distinct real roots give $Ae^{r_1x} + Be^{r_2x}$; a DOUBLE root r gives $(A+Bx)e^{rx}$ (the extra x, exactly as in repeated-root recurrences);

complex roots $\alpha \pm \beta i$ give decaying oscillations $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.

Damping in one glance

For $y''+by'+\omega^2 y = 0$, the discriminant $b^2-4\omega^2$ sorts the physics: negative = underdamped (oscillates while decaying), zero = CRITICALLY damped (fastest settle, no overshoot), positive = overdamped. Pure $y'' + \omega^2 y = 0$ oscillates forever with period $\frac{2\pi}{\omega}$.

Reading off a solution

$y'' - 3y' + 2y = 0$: roots $1, 2$, so $y = Ae^x + Be^{2x}$; the initial data $y(0) = 3$, $y'(0) = 5$ pin $A = 1$, $B = 2$ via a tiny linear system — second-order ODEs always hand you two conditions for the two constants.

Significance: everything that vibrates

Car suspensions (critically damped on purpose), earthquake-resistant buildings, guitar strings, and RLC radio tuners all obey second-order linear equations. The discriminant taxonomy — underdamped, critical, overdamped — is a design menu: engineers CHOOSE b to put a system in the regime they want.

Try it: Try it: read the physics from the roots

For $y''+4y'+13y = 0$, describe the motion. Work: $r = \frac{-4\pm\sqrt{16-52}}{2} = -2\pm 3i$: underdamped — oscillates at frequency 3 inside a decaying envelope e^{-2t}. Half-life of the envelope: $\frac{\ln 2}{2} \approx 0.35$ time units. No solving needed to know all this.

3.2 Proofs & Why It Matters

Proof: why e^{rx} and the characteristic equation

Substitute $y = e^{rx}$ into $y'' + by' + cy = 0$: $e^{rx}(r^2 + br + c) = 0$, and since $e^{rx} \neq 0$, the exponent must satisfy $r^2 + br + c = 0$.

Two independent solutions of a second-order linear equation span ALL solutions (the solution space is 2-dimensional: a solution is pinned by $y(0), y'(0)$), so distinct roots finish the problem. ■

Proof: the double root needs xe^{rx}

If r is a double root then $r^2 + br + c = (\lambda - r)^2$ expanded gives $b = -2r$, $c = r^2$. Try $y = xe^{rx}$: $y' = (1 + rx)e^{rx}$, $y'' = (2r + r^2 x)e^{rx}$.

Substitute: $e^{rx}\left[(2r + r^2 x) + b(1 + rx) + cx\right] = e^{rx}\left[(2r + b) + x(r^2 + br + c)\right] = 0$ — both brackets vanish precisely because r is a double root. ■

Proof: complex roots give cosine and sine (Euler)

Euler's formula $e^{i\beta x} = \cos\beta x + i\sin\beta x$ (visible by comparing the three power series) turns the complex solutions $e^{(\alpha\pm i\beta)x}$ into $e^{\alpha x}(\cos\beta x \pm i\sin\beta x)$.

Real linear combinations — half the sum, and the difference over $2i$ — extract $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$, two REAL independent solutions. Decay times oscillation, straight from algebra. ■

3.3 Going Deeper: Explanations & Worked Problems

Why two constants, and how initial data pins them

A second-order equation reaches back two derivatives, so a solution is not determined until you specify both a starting VALUE $y(0)$ and a starting SLOPE $y'(0)$ — think of a mass on a spring: where it starts and how fast it is moving are independent facts.

Correspondingly the general solution carries exactly two free constants, and the solution space of the homogeneous equation is two-dimensional: find two independent solutions and every solution is a combination of them. The characteristic equation delivers those two: distinct real roots give two exponentials; a double root gives e^{rx} and xe^{rx}; complex roots give a cosine and a sine dressed in an exponential envelope. Applying initial data always ends the same way — a small linear system in the two constants, one equation from $y(0)$, one from $y'(0)$.

Worked: a full IVP, characteristic roots to final value

Solve $y'' - 3y' + 2y = 0$, $y(0) = 3$, $y'(0) = 5$, and evaluate at $x = \ln 2$.

Step 1 — characteristic equation: $r^2 - 3r + 2 = (r-1)(r-2) = 0$: roots $1, 2$. Step 2 — general solution: $y = Ae^x + Be^{2x}$. Step 3 — apply the data: $y(0) = A + B = 3$; since $y' = Ae^x + 2Be^{2x}$, $y'(0) = A + 2B = 5$. Step 4 — solve the little system: subtracting, $B = 2$, then $A = 1$: $y = e^x + 2e^{2x}$. Step 5 — evaluate: $e^{\ln 2} = 2$ and $e^{2\ln 2} = (e^{\ln 2})^2 = 4$, so $y(\ln 2) = 2 + 2 \cdot 4 = 10$. Step 6 — verify the ODE once: $y'' = e^x + 8e^{2x}$, $3y' = 3e^x + 12e^{2x}$, $2y = 2e^x + 4e^{2x}$; then $y'' - 3y' + 2y = (1 - 3 + 2)e^x + (8 - 12 + 4)e^{2x} = 0$ ✓.

Worked: complex roots, read as physics

Solve $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = -1$.

Step 1 — characteristic: $r^2 + 2r + 5 = 0$ gives $r = \dfrac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$. Step 2 — real solution form ($\alpha = -1$, $\beta = 2$): $y = e^{-x}(A\cos 2x + B\sin 2x)$. Step 3 — data: $y(0) = A = 1$; the product rule gives $y'(0) = -A + 2B = -1$, so $B = 0$: $y = e^{-x}\cos 2x$. Step 4 — read it: an oscillation of angular frequency 2 (period π) inside a decaying envelope e^{-x} — underdamped motion, crossing zero every $\frac{\pi}{2}$, each swing about $e^{-\pi} \approx 4\%$ the height of the one two crossings before. The root's real part is the decay rate; its imaginary part is the frequency: the algebra IS the physics.

4 The Wronskian & Variation of Parameters

Independence certificates, and the method that handles any forcing

4.1 The General Machinery

The Wronskian certifies independence

For solutions y_1, y_2 of the same second-order linear homogeneous equation, the Wronskian $W = y_1 y_2' - y_1' y_2$ is either NEVER zero or identically zero (Abel's theorem: $W = Ce^{-\int p\,dx}$).

Nonzero W means $\{y_1, y_2\}$ is a fundamental set — every solution is a combination of them, and every initial-value problem is solvable within their span. Computing W is a 2×2 determinant: for $\cos\beta x, \sin\beta x$ it is the constant β.

Variation of parameters: the universal particular solution

For $y'' + py' + qy = f$ with fundamental pair y_1, y_2: seek $y_p = u_1 y_1 + u_2 y_2$ with FUNCTIONS u_i. Imposing the standard side condition $u_1' y_1 + u_2' y_2 = 0$ leads to $u_1' = -\dfrac{y_2 f}{W}$, $u_2' = \dfrac{y_1 f}{W}$ — two integrations and done.

Unlike undetermined coefficients this needs NO guess and works for any continuous forcing: $\sec x$, $\frac{e^x}{x}$, $\ln x$. The cost is two possibly-ugly integrals; the gain is universality.

Worked: y + y = sec x, which no guess can touch

Step 1 — fundamental pair: $y_1 = \cos x$, $y_2 = \sin x$, $W = 1$. Step 2 — the formulas: $u_1' = -\sin x \sec x = -\tan x$ and $u_2' = \cos x \sec x = 1$. Step 3 — integrate: $u_1 = \ln(\cos x)$, $u_2 = x$. Step 4 — assemble: $y_p = \cos x\ \ln(\cos x) + x \sin x$.

Step 5 — sanity: the $x \sin x$ piece echoes resonance ($\sec x$ contains the natural frequency), and differentiating twice confirms $y_p'' + y_p = \sec x$ ✓.

Significance: the Wronskian outlives the course

Wronskians appear wherever independence of functions matters: quantum mechanics uses them to normalize scattering states, and numerical ODE solvers monitor them to detect degenerating solution bases. Variation of parameters, meanwhile, is the finite-dimensional ancestor of the "Green's function" method that solves PDEs by superposing impulse responses.

Try it: Try it: a Wronskian in one line

Compute $W(e^{2x}, xe^{2x})$. Work: $W = e^{2x}(e^{2x} + 2xe^{2x}) - 2e^{2x} \cdot xe^{2x} = e^{4x}(1 + 2x - 2x) = e^{4x}$ —

never zero, so the repeated-root pair really is independent, justifying the $(A + Bx)e^{2x}$ general solution.

4.2 Proofs & Why It Matters

Proof: the variation-of-parameters formulas

With $y_p = u_1y_1 + u_2y_2$ and side condition $u_1'y_1 + u_2'y_2 = 0$: then $y_p' = u_1y_1' + u_2y_2'$, and $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$. Substitute into the ODE; the u_i-times-homogeneous-solution terms vanish (each y_i solves the homogeneous equation), leaving $u_1'y_1' + u_2'y_2' = f$.

Together with the side condition this is a 2×2 linear system in u_1', u_2' whose determinant is exactly W; Cramer's rule gives $u_1' = -y_2f/W$, $u_2' = y_1f/W$. ■

Proof: Abel's theorem

Differentiate $W = y_1y_2' - y_1'y_2$: $W' = y_1y_2'' - y_1''y_2$ (the cross terms cancel). Substitute $y_i'' = -py_i' - qy_i$: $W' = y_1(-py_2' - qy_2) - (-py_1' - qy_1)y_2 = -p(y_1y_2' - y_1'y_2) = -pW$. So $W = Ce^{-\int p}$ — never zero unless $C = 0$, in which case it vanishes identically. Independence is an all-or-nothing affair. ■

4.3 Going Deeper: Worked Problems

Worked: variation of parameters with a non-standard forcing

Solve $y'' - 2y' + y = \dfrac{e^x}{x}$ for $x > 0$.

Step 1 — homogeneous: $(r - 1)^2 = 0$, double root: $y_1 = e^x$, $y_2 = xe^x$. Step 2 — Wronskian: $W = e^x(e^x + xe^x) - e^x \cdot xe^x = e^{2x}$. Step 3 — the formulas with $f = \frac{e^x}{x}$: $u_1' = -\dfrac{y_2f}{W} = -\dfrac{xe^x \cdot e^x/x}{e^{2x}} = -1$, and $u_2' = \dfrac{y_1f}{W} = \dfrac{e^x \cdot e^x/x}{e^{2x}} = \dfrac{1}{x}$. Step 4 — integrate: $u_1 = -x$, $u_2 = \ln x$. Step 5 — $y_p = -xe^x + xe^x \ln x$; the $-xe^x$ is a homogeneous solution and can be dropped: $y_p = xe^x \ln x$. General solution: $y = (c_1 + c_2x)e^x + xe^x \ln x$. Undetermined coefficients has no guess for $\frac{e^x}{x}$ — this method never needed one.

Worked: using the Wronskian to test a claimed fundamental set

Are $y_1 = x$ and $y_2 = x \ln x$ a fundamental set of solutions of $x^2y'' - xy' + y = 0$ on $x > 0$?

Step 1 — check each solves the equation: for $y = x$: $0 - x + x = 0$ ✓; for $y = x \ln x$: $y' = \ln x + 1$, $y'' = \frac{1}{x}$, so $x^2 \cdot \frac{1}{x} - x(\ln x + 1) + x \ln x = x - x \ln x - x + x \ln x = 0$ ✓. Step 2 — Wronskian: $W = x(\ln x + 1) - 1 \cdot x \ln x = x$, nonzero for $x > 0$. Step 3 — conclusion: yes — independent solutions, so every solution is $c_1x + c_2x \ln x$. Step 4 — connect: this is a Cauchy–Euler equation with double indicial root $r = 1$, and the $\ln x$ companion is the $x = e^t$ shadow of the te^{rt} rule. The Wronskian being exactly x matches Abel's formula $W = Ce^{-\int p}$ with $p = -\frac{1}{x}$.

5 Nonhomogeneous Equations & Resonance

Undetermined coefficients, and what forcing does

5.1 Undetermined Coefficients

Guess the shape of the forcing

For $y''+by'+cy = f(x)$: general solution = homogeneous + ONE particular solution. Guess y_p shaped like f: polynomials for polynomials ($4x$ gets $ax+b$), Ae^{kx} for exponentials, $A\cos + B\sin$ for sinusoids — substitute and match coefficients.

Resonance: multiply by x

When the forcing already SOLVES the homogeneous equation, the naive guess collapses — multiply it by x. For $y'' + 4y = \cos 2x$: $y_p = \frac{x}{4}\sin 2x$, whose amplitude GROWS without bound. This is resonance — the mathematics behind pushed swings and shattered bridges.

A full match, worked

$y'' + 3y' + 2y = 4x$ with $y_p = ax + b$: substituting gives $3a + 2(ax + b) = 4x$, so $a = 2$ and $b = -3$: $y_p = 2x - 3$. Matching term-by-term is bookkeeping, not cleverness — write the shape, then compare.

Significance: resonance is not a metaphor

The Broughton suspension bridge (1831) collapsed under soldiers marching in step; the Millennium Bridge in London wobbled dangerously on opening day from synchronized footfalls — both are the $x\sin\omega x$ growth of this topic in steel. Every tuned structure is designed to keep its natural frequencies away from expected forcing.

Try it: Try it: spot resonance before solving

Which forcing causes resonance in $y'' + 9y = f(x)$: $\cos 2x$, $\cos 3x$, or e^{3x}? Work: homogeneous solutions oscillate at frequency 3, so $\cos 3x$ resonates (guess needs the extra x); $\cos 2x$ and e^{3x} do not — e^{3x} is not a homogeneous solution here since roots are $\pm 3i$, not 3.

5.2 Proofs & Why It Matters

Proof: general = homogeneous + particular

Let y_p solve $Ly = f$ (writing L for the whole left side). If y is ANY solution of $Ly = f$, then

$L(y-y_p) = f - f = 0$ by linearity, so $y - y_p$ is a homogeneous solution: $y = y_h + y_p$. Conversely every $y_h + y_p$ solves $Ly = f$.

One particular solution plus the whole homogeneous family is therefore the complete solution set. ■

Proof: the resonant particular solution

For $y'' + \omega^2 y = \cos\omega x$, try $y = kx\sin\omega x$: $y' = k\sin\omega x + k\omega x\cos\omega x$ and $y'' = 2k\omega\cos\omega x - k\omega^2 x\sin\omega x$. Then $y'' + \omega^2 y = 2k\omega\cos\omega x$ — the x-terms cancel — so $k = \frac{1}{2\omega}$ ($= \frac{1}{4}$ when $\omega = 2$).

The linear-in-x amplitude is forced: bounded guesses die because they already solve the homogeneous equation. ■

5.3 Going Deeper: Explanations & Worked Problems

The guessing table, and the logic behind it

Undetermined coefficients works because differentiation maps each function family into itself: polynomials to polynomials, e^{kx} to itself, $\{\cos\omega x, \sin\omega x\}$ into each other.

So guess a general member of the forcing's family: for $4x$, guess $ax + b$ (BOTH terms — the derivative of ax makes constants, so a bare ax cannot balance); for e^{2x}, guess Ae^{2x}; for $\cos 2x$, guess $A\cos 2x + B\sin 2x$ (both, since y' turns cosine into sine); for products, multiply the guesses. THE EXCEPTION: if the guess already solves the homogeneous equation, substituting it gives zero on the left and the coefficient equations become unsolvable — multiply the guess by x (twice for a double root). That is resonance, and physically it is why forcing a system AT its natural frequency produces growing oscillations rather than a steady response.

Worked: polynomial forcing, all coefficients matched

Find the general solution of $y'' + 3y' + 2y = 4x$.

Step 1 — homogeneous part: $r^2 + 3r + 2 = (r+1)(r+2) = 0$, so $y_h = c_1e^{-x} + c_2e^{-2x}$. Step 2 — guess $y_p = ax + b$ (not a homogeneous solution — no x-multiplication needed): $y_p' = a$, $y_p'' = 0$. Step 3 — substitute: $0 + 3a + 2(ax + b) = 4x$, i.e. $2a\,x + (3a + 2b) = 4x + 0$. Step 4 — match coefficient BY coefficient: x: $2a = 4 \Rightarrow a = 2$; constants: $3(2) + 2b = 0 \Rightarrow b = -3$. Step 5 — assemble: $y = c_1e^{-x} + c_2e^{-2x} + 2x - 3$. Step 6 — long-run reading: both homogeneous pieces decay, so EVERY solution approaches the line $2x - 3$ — the particular solution is the system's steady response, the homogeneous part its fading memory of initial conditions.

Worked: resonance versus near-resonance, side by side

Compare $y'' + 4y = \cos 3x$ against $y'' + 4y = \cos 2x$ (natural frequency 2). NON-RESONANT:

guess $y_p = A\cos 3x$; then $y_p'' = -9A\cos 3x$, so $(-9A+4A)\cos 3x = \cos 3x$ gives $A = -\frac{1}{5}$ — a bounded steady oscillation, done.

RESONANT: the naive guess $A\cos 2x$ gives $(-4A+4A)\cos 2x = 0 \neq \cos 2x$ — unsolvable, exactly because $\cos 2x$ solves the homogeneous equation. Multiply by x: $y_p = x(A\cos 2x + B\sin 2x)$. Differentiate twice (product rule, carefully): $y_p'' = -4x(A\cos 2x + B\sin 2x) + 2(-2A\sin 2x + 2B\cos 2x)$; adding $4y_p$ cancels the x-terms, leaving $4B\cos 2x - 4A\sin 2x = \cos 2x$: $B = \frac{1}{4}$, $A = 0$, so $y_p = \frac{x}{4}\sin 2x$. The amplitude GROWS linearly forever — same equation shape, one frequency changed, qualitatively different fate.

6 Systems of Differential Equations

Eigenvalues run the phase portrait

6.1 Linear Systems

Solve along eigenvectors

For $\mathbf{u}' = A\mathbf{u}$, each eigenpair gives a straight-line solution $e^{\lambda t}\mathbf{v}$, and the general solution mixes them. The eigenvalue SIGNS shape the picture: both negative — everything decays to the origin (sink); both positive — source; opposite signs ($\det A < 0$) — a SADDLE.

Complex eigenvalues spiral

Eigenvalues $\alpha \pm \beta i$ make trajectories rotate with rate β while $e^{\alpha t}$ scales them: spirals in or out, or pure circles when $\alpha = 0$. The system $x' = y$, $y' = -x$ is the cleanest case: $x(t) = \cos t$ from $(1, 0)$ — a rotation, forever.

Decoupled warm-up

$x' = 2x$, $y' = 3y$ solves coordinate-wise: (e^{2t}, e^{3t}). Diagonalization is the art of choosing coordinates that make EVERY system look like this one.

Significance: coupled everything

Predator-prey cycles, coupled pendulums, chemical reaction networks, epidemic compartments (S-I-R), and multi-loop circuits are systems of ODEs. The eigenvalue portrait — sink, source, saddle, spiral — is the universal vocabulary for ”what happens in the long run” across all of them.

Try it: Try it: classify at a glance

Classify $\mathbf{u}' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\mathbf{u}$. Work: trace $-6 < 0$, det $= 9 - 1 = 8 > 0$, discriminant $36 - 32 = 4 > 0$: two real negative eigenvalues (-2 and -4) — a stable NODE; everything decays to the origin, fastest along the $\lambda = -4$ eigendirection $(1, -1)$.

6.2 Proofs & Why It Matters

Proof: $e^t v$ solves u = Au

Differentiate $\mathbf{u}(t) = e^{\lambda t}\mathbf{v}$: $\mathbf{u}' = \lambda e^{\lambda t}\mathbf{v}$. Apply the matrix instead: $A\mathbf{u} = e^{\lambda t}A\mathbf{v} = e^{\lambda t}\lambda\mathbf{v}$ using $A\mathbf{v} = \lambda\mathbf{v}$.

The two agree, so every eigenpair yields a straight-line solution; independence of eigenvectors makes their combinations the general solution. ■

Proof: x = y, y = x moves in circles

Let $E(t) = x(t)^2 + y(t)^2$. Then $E' = 2xx' + 2yy' = 2xy - 2yx = 0$: E is CONSTANT, so trajectories stay on circles about the origin. Differentiating $x' = y$ gives $x'' = y' = -x$, simple harmonic motion, so from $(1, 0)$: $x = \cos t$, $y = -\sin t$ — uniform rotation, confirming the circle. ■

6.3 Going Deeper: Explanations & Worked Problems

From eigenpairs to the phase portrait, step by step

To solve $\mathbf{u}' = A\mathbf{u}$: find the eigenpairs of A; each gives a straight-line solution $e^{\lambda t}\mathbf{v}$; the general solution is $c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2$; initial data fixes c_1, c_2 by writing $\mathbf{u}(0)$ in the eigenbasis.

The PICTURE follows from the signs alone: both $\lambda < 0$ — every trajectory slides to the origin (stable node); both > 0 — everything escapes (source); opposite signs — a SADDLE: trajectories sweep in along the negative eigenline and out along the positive one, and $\det A = \lambda_1\lambda_2 < 0$ detects this without computing anything else. Complex $\alpha \pm \beta i$: spirals, inward if $\alpha < 0$, outward if $\alpha > 0$, closed ellipses if $\alpha = 0$. Trace and determinant of a 2×2 thus classify the whole portrait before any solving happens.

Worked: a full system solve with initial data

Solve $\mathbf{u}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = (3, 1)$.

Step 1 — eigenpairs (computed in the eigenvalue topic): $\lambda = 3$ with $(1, 1)$; $\lambda = 1$ with $(1, -1)$. Step 2 — general solution: $\mathbf{u} = c_1e^{3t}(1, 1) + c_2e^t(1, -1)$. Step 3 — initial data: $(3, 1) = c_1(1, 1) + c_2(1, -1)$ means $c_1 + c_2 = 3$, $c_1 - c_2 = 1$: $c_1 = 2$, $c_2 = 1$. Step 4 — the answer: $\mathbf{u}(t) = 2e^{3t}(1, 1) + e^t(1, -1)$, i.e. $x = 2e^{3t} + e^t$, $y = 2e^{3t} - e^t$. Step 5 — behavior: both eigenvalues are positive, so the origin is a source; as t grows the e^{3t} term dominates and the trajectory hugs the line $y = x$ — the fast eigendirection — while for $t \to -\infty$ it aligned with $(1, -1)$. Eigenvectors are the skeleton every trajectory drapes itself on.

Worked: classify without solving

Classify the equilibrium of $\mathbf{u}' = A\mathbf{u}$ for $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, then for $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, using only trace and determinant.

For A: $\det = 1 - 4 = -3 < 0$ — eigenvalues have opposite signs (their product is negative): a SADDLE, unstable, no further computation required; indeed $\lambda = 3, -1$ with eigenlines $y = x$ (outflow) and $y = -x$ (inflow). For B: $\mathrm{tr} = 0$, $\det = 1$: eigenvalues satisfy $\lambda^2 + 1 = 0$, so $\lambda = \pm i$ — pure imaginary: closed orbits, a CENTER; and indeed $x' = y, y' = -x$ conserves $x^2 + y^2$, so trajectories are literal circles. Two matrices, four numbers total examined, both portraits fully determined.

7 Nonlinear Systems & Linearization

Jacobians, predator–prey, competition, and the pendulum

7.1 Local Pictures of Nonlinear Worlds

Linearize at each equilibrium

A nonlinear system $x' = f(x, y)$, $y' = g(x, y)$ is understood equilibrium by equilibrium: find all points where $f = g = 0$, compute the JACOBIAN $\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$, and evaluate it at each.

Near the equilibrium the system behaves like $\mathbf{u}' = J\mathbf{u}$, so the linear classification (saddle, node, spiral — by trace and determinant) applies locally. The Hartman–Grobman principle: the linearization tells the truth except in the borderline center case.

Predator–prey and competition, read from Jacobians

Lotka–Volterra $x' = x - xy$, $y' = -y + xy$: at the coexistence point $(1, 1)$ the Jacobian is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ — trace 0, eigenvalues $\pm i$: closed orbits, populations cycling forever out of phase.

Competing species $x' = x(3 - x - 2y)$, $y' = y(2 - x - y)$: the interior equilibrium $(1, 1)$ has Jacobian determinant $-1 < 0$ — a SADDLE, so coexistence is unstable and one species excludes the other. Ecology, decided by a determinant.

Worked: the pendulum, honestly then linearized

The pendulum $\theta'' + \sin\theta = 0$ has equilibria at $\theta = 0$ (hanging) and $\theta = \pi$ (balanced upright).

Step 1 — linearize at 0: $\sin\theta \approx \theta$ gives $\theta'' + \theta = 0$, a center: small swings oscillate with period 2π. Step 2 — linearize at π: $\sin\theta \approx -(\theta - \pi)$ gives $u'' - u = 0$, eigenvalues ± 1: a SADDLE — the inverted pendulum falls. Step 3 — what linearization misses: the true period GROWS with amplitude, and the special orbits into the saddle separate swinging from over-the-top spinning. The phase portrait stitches all of this together.

Significance: where chaos begins

Linearization explains the local picture, but genuinely nonlinear systems can do what linear ones never can: multiple equilibria, limit cycles (heartbeats are one), and chaos (weather). The Jacobian toolkit of this topic is the entry door; the Lorenz attractor is what lives three doors down.

Try it: Try it: equilibria of a damped pendulum

For $x' = y$, $y' = -\sin x - y$, find the equilibria and classify $(0,0)$. Work: equilibria at $(k\pi, 0)$. Jacobian $\begin{pmatrix} 0 & 1 \\ -\cos x & -1 \end{pmatrix}$; at origin: trace -1, det 1, discriminant $1 - 4 < 0$: a stable SPIRAL — the pendulum rings down, exactly as friction predicts.

7.2 Proofs & Why It Matters

Proof: why the Jacobian is the right linearization

Write $\mathbf{x} = \mathbf{x}^* + \mathbf{u}$ for a small displacement from the equilibrium $\mathbf{x}^*$. Taylor: $f(\mathbf{x}^* + \mathbf{u}) = f(\mathbf{x}^*) + f_x u_1 + f_y u_2 + O(|\mathbf{u}|^2)$, and $f(\mathbf{x}^*) = 0$ by definition of equilibrium.

So $\mathbf{u}' = J\mathbf{u} + O(|\mathbf{u}|^2)$: to first order the displacement obeys the linear system with matrix J, and when J's eigenvalues have nonzero real part the quadratic remainder is too weak to change the picture. ■

Proof: Lotka–Volterra orbits really close up

The quantity $E(x,y) = x - \ln x + y - \ln y$ is conserved: $\frac{dE}{dt} = \left(1 - \frac{1}{x}\right) x' + \left(1 - \frac{1}{y}\right) y' = (x-1)(1-y) + (y-1)(x-1)\cdot(-1)\cdot(-1)\ldots$ — expanding, $(1-\frac{1}{x})(x - xy) + (1 - \frac{1}{y})(-y + xy) = (x-1)(1-y) + (y-1)(x-1) = 0$.

Trajectories live on level curves of E, which are closed loops around $(1,1)$ — so the center predicted by the linearization is genuine, not an artifact. ■

7.3 Going Deeper: Worked Problems

Worked: a full phase-portrait classification

Classify every equilibrium of $x' = x(2-x-y)$, $y' = y(3-2x-y)$.

Step 1 — equilibria: each equation vanishes on two curves; intersections: $(0,0)$, $(2,0)$, $(0,3)$, and the interior solution of $x+y=2$, $2x+y=3$: $(1,1)$. Step 2 — Jacobian: $\begin{pmatrix} 2-2x-y & -x \\ -2y & 3-2x-2y \end{pmatrix}$. Step 3 — evaluate and classify: at $(0,0)$: $\mathrm{diag}(2,3)$, both positive — source. At $(2,0)$: $\begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}$, eigenvalues $-2,-1$ — stable node. At $(0,3)$: $\begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}$, eigenvalues $-1,-3$ — stable node. At $(1,1)$: $\begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}$, $\det = 1-2 = -1 < 0$ — saddle. Step 4 — the story: two stable "one species wins" states, an unstable coexistence saddle whose stable manifold is the knife-edge dividing the winners. Competitive exclusion, read off four small matrices.

Worked: linearization that lies, and what settles it

The system $x' = -y + x(x^2+y^2)$, $y' = x + y(x^2+y^2)$ has Jacobian $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ at the origin — a center. Is the origin actually stable?

Step 1 — the linearization says "center" (eigenvalues $\pm i$), the one case Hartman–Grobman does NOT cover. Step 2 — switch to polar coordinates: with $r^2 = x^2+y^2$, compute $rr' = xx' + yy' = x(-y+xr^2) + y(x+yr^2) = r^4$, so $r' = r^3 > 0$. Step 3 — the radius GROWS: trajectories spiral outward, and the origin is unstable — the cubic terms the linearization discarded decide everything. Step 4 — the lesson: when the linear part gives a center, look for a conserved quantity (as with Lotka–Volterra) or a radial equation; only nonlinear information can break the tie.

8 Laplace Transforms

Turning calculus into algebra

8.1 The Transform Method

The transform and the short table

$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}dt$. The working table: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, $\mathcal{L}\{\sin bt\} = \frac{b}{s^2+b^2}$, $\mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$.

Derivatives become multiplication

$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0)$: differentiation turns into multiplying by s, with the initial condition built in automatically. An initial-value problem becomes ALGEBRA in s — solve for $\mathcal{L}\{y\}$, split by partial fractions, read the table backward.

Reading the table backward

$F(s) = \frac{6}{s^4}$: since $\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$, the inverse transform is $f(t) = t^3$. Recognizing table entries inside partial-fraction pieces is the entire skill.

Significance: the engineer's transform

Control theory runs on Laplace: a system's transfer function $H(s)$ IS the transform of its impulse response, and stability is "poles in the left half-plane" — statements about s, not t. Circuit analysis, signal processing, and PID controller design all speak this language natively.

Try it: Try it: one transform, no tables open

Compute $\mathcal{L}\{5e^{-2t} + 3t\}$. Work: linearity: $\frac{5}{s+2} + \frac{3}{s^2}$. Inverse practice: what has transform $\frac{4}{s^2+4}$? Work: $\frac{b}{s^2+b^2}$ with $b = 2$, scaled by 2: $2\sin 2t$.

8.2 Proofs & Why It Matters

Proof: Le^{at} = 1/(sa)

Directly: $\int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}$ for $s > a$ (the boundary term at ∞ dies because $s - a > 0$). Setting $a = 0$ gives $\mathcal{L}\{1\} = \frac{1}{s}$ as a corollary. ■

Proof: Ly = sLy y(0)

Integrate by parts: $\int_0^\infty y'(t)e^{-st}dt = \left[y(t)e^{-st}\right]_0^\infty + s\int_0^\infty y(t)e^{-st}dt = -y(0) + s\,\mathcal{L}\{y\}$, assuming y grows slower than e^{st} so the boundary term at infinity vanishes.

Iterating gives $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$ — initial conditions enter automatically, which is the method's superpower. ■

Proof: Lt = $n!/s^{n+1}$

Induction with parts: $\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt = \frac{n}{s}\,\mathcal{L}\{t^{n-1}\}$ (boundary terms vanish). Unwinding from $\mathcal{L}\{1\} = \frac{1}{s}$: $\mathcal{L}\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdots\frac{1}{s}\cdot\frac{1}{s} = \frac{n!}{s^{n+1}}$. ■

8.3 Going Deeper: Explanations & Worked Problems

The transform pipeline, and where each rule enters

Solving an IVP by Laplace is a fixed pipeline. (1) TRANSFORM the equation term by term: $\mathcal{L}\{y'\} = sY - y(0)$ and $\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0)$ fold the initial conditions in at the start — no constants to find later.

(2) SOLVE the resulting ALGEBRAIC equation for $Y(s)$ — always possible, it is linear in Y. (3) PARTIAL-FRACTION Y into table-sized pieces: $\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right)$, quadratic pieces aimed at the sine/cosine entries. (4) INVERT with the table. The bookkeeping wins where classical methods strain: piecewise forcing (step functions), impulses, and long chains of initial conditions all ride through the algebra unchanged. The cost is fluency with a small table — and partial fractions, which do the heavy lifting in every problem.

Worked: an IVP through the full pipeline

Solve $y' + y = 1$, $y(0) = 0$. Step 1 — transform both sides: $(sY - y(0)) + Y = \frac{1}{s}$, so $(s+1)Y = \frac{1}{s}$. Step 2 — algebra: $Y = \dfrac{1}{s(s+1)}$. Step 3 — partial fractions: write $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$; multiplying out, $1 = A(s+1) + Bs$; set $s = 0$: $A = 1$; set $s = -1$: $B = -1$.

So $Y = \frac{1}{s} - \frac{1}{s+1}$. Step 4 — invert with the table: $y(t) = 1 - e^{-t}$. Step 5 — verify: $y' = e^{-t}$ and $y' + y = e^{-t} + 1 - e^{-t} = 1$ ✓, $y(0) = 0$ ✓. Compare with the integrating-factor route: same answer, but here the initial condition entered at Step 1 and no constant C ever appeared.

Worked: a second-order IVP with a sine, by table and partial fractions

Solve $y'' + y = 0$, $y(0) = 0$, $y'(0) = 3$ — deliberately simple, to watch the machinery.

Step 1 — transform: $s^2Y - s \cdot 0 - 3 + Y = 0$. Step 2 — solve: $Y = \dfrac{3}{s^2+1}$. Step 3 — no partial fractions needed: this is 3 times the table entry $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$. Step 4 — invert: $y = 3\sin t$. Step 5 — verify: $y'' = -3\sin t = -y$ ✓, $y(0) = 0$ ✓, $y'(0) = 3\cos 0 = 3$ ✓. Now the same skeleton with forcing, damping, or a step function only lengthens Step 3 — the pipeline itself never changes shape, which is exactly its virtue.

9 Laplace II: Steps, Impulses & Convolution

Piecewise forcing, hammer blows, and the convolution theorem

9.1 Discontinuous and Impulsive Forcing

Unit steps carry delays

The Heaviside step $u(t-a)$ switches on at $t = a$; any piecewise forcing is a sum of stepped pieces. The shift rule both ways: $\mathcal{L}\{g(t-a)\,u(t-a)\} = e^{-as}G(s)$, so a DELAY in time is an EXPONENTIAL factor in s — and when inverting, every e^{-as} in $Y(s)$ announces "the following response, but starting at $t = a$." This is how one algebraic computation handles a force that turns on at $t = 3$ and off at $t = 7$.

The impulse: and what it does

The Dirac impulse $\delta(t-a)$ models a hammer blow: zero except at one instant, total area 1, transform $\mathcal{L}\{\delta(t-a)\} = e^{-as}$.

Forcing an equation with δ produces an instantaneous JUMP in the highest-derivative-minus-one: $y' = \delta(t-1)$, $y(0) = 0$ gives the step $y = u(t-1)$ — flat, jump by 1, flat. The solution to δ-forcing is the system's IMPULSE RESPONSE, its fingerprint.

Convolution: multiplication in disguise

The convolution $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$ satisfies $\mathcal{L}\{f*g\} = F(s)G(s)$ — products of transforms are convolutions of functions. Consequence: the response to ANY forcing f is $h*f$ where h is the impulse response, because $Y = H(s)F(s)$.

One integral formula expresses every solution — the superposition of infinitely many scaled, delayed hammer blows.

Significance: real inputs are not smooth

Real systems get switched on, struck, and pulsed: a thermostat clicking, a hammer test on a bridge, a defibrillator pulse. Steps and deltas model these honestly, and the convolution integral is how audio engineers apply reverb: recorded impulse response of a cathedral, convolved with your dry vocal track.

Try it: Try it: read a delayed inverse

Invert $Y(s) = \dfrac{e^{-3s}}{s+2}$. Work: without the exponential, $\frac{1}{s+2}$ inverts to e^{-2t}; the e^{-3s} delays it: $y(t) = e^{-2(t-3)}u(t-3)$ — silence until $t = 3$, then the decay begins, shifted intact.

9.2 Proofs & Why It Matters

Proof: the t-shift rule

Directly: $\mathcal{L}\{g(t-a)u(t-a)\} = \int_a^\infty g(t-a)e^{-st}dt$ (the step kills $t < a$). Substitute $\tau = t - a$: $\int_0^\infty g(\tau)e^{-s(\tau+a)}d\tau = e^{-as}\int_0^\infty g(\tau)e^{-s\tau}d\tau = e^{-as}G(s)$. ■

Proof: the convolution theorem

$F(s)G(s) = \int_0^\infty \int_0^\infty f(\tau)g(\sigma)e^{-s(\tau+\sigma)}\,d\sigma\,d\tau$ — a double integral over the quarter-plane.

Change variables to $t = \tau + \sigma$ (total time) and τ: for each fixed t, τ ranges over $[0, t]$, giving $\int_0^\infty e^{-st}\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right]dt = \mathcal{L}\{f * g\}$. A Fubini swap turns a product into a convolution. ■

9.3 Going Deeper: Worked Problems

Worked: a force that switches on, then off

Solve $y'' + y = f(t)$, $y(0) = y'(0) = 0$, where $f(t) = 1$ for $0 \le t < \pi$ and 0 afterward.

Step 1 — write the forcing with steps: $f = 1 - u(t-\pi)$, so $\mathcal{L}\{f\} = \dfrac{1}{s} - \dfrac{e^{-\pi s}}{s}$. Step 2 — transform: $(s^2+1)Y = \dfrac{1-e^{-\pi s}}{s}$, so $Y = \dfrac{1-e^{-\pi s}}{s(s^2+1)}$. Step 3 — partial fractions on $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$, which inverts to $g(t) = 1 - \cos t$. Step 4 — the $e^{-\pi s}$ copy is the same response delayed: $y = g(t) - g(t-\pi)u(t-\pi)$. Step 5 — read it: for $t < \pi$, $y = 1 - \cos t$ (pushed from rest); for $t \ge \pi$, $y = (1-\cos t) - (1 - \cos(t-\pi)) = -2\cos t$ — the force released exactly when the oscillator was at its far point, leaving it swinging with amplitude 2 forever. Piecewise forcing, zero case-splitting.

Worked: an impulse response and a convolution

Find the impulse response of $y'' + 3y' + 2y = \delta(t)$ (zero initial conditions), then use convolution to solve the same equation with forcing $f(t) = e^{-t}$.

Step 1 — transform: $(s^2+3s+2)H = 1$, so $H(s) = \dfrac{1}{(s+1)(s+2)} = \dfrac{1}{s+1} - \dfrac{1}{s+2}$: impulse response $h(t) = e^{-t} - e^{-2t}$. Step 2 — for general forcing, $y = h * f$: $y(t) = \displaystyle\int_0^t \left(e^{-\tau} - e^{-2\tau}\right) e^{-(t-\tau)}\,d\tau$. Step 3 — simplify: $e^{-t}\displaystyle\int_0^t \left(1 - e^{-\tau}\right) d\tau = e^{-t}\left(t - 1 + e^{-t}\right)$. Step 4 — so $y = te^{-t} - e^{-t} + e^{-2t}$; check: $y(0) = 0$ ✓, and the te^{-t} term is exactly what resonance-style forcing at a homogeneous rate produces. One impulse response, every forcing solved by an integral.

10 Series Solutions

Power series where formulas fail

10.1 The Power Series Method

Substitute, shift, match

Write $y = \sum a_n x^n$, substitute into the equation, shift indices so all powers align, and set each coefficient to zero — a RECURRENCE for the a_n. For $y' = y$: $(n+1)a_{n+1} = a_n$, giving

$a_n = \frac{1}{n!}$ — the series discovers e^x on its own.

Equations beyond elementary functions

The Airy-type equation $y'' = xy$ has no closed-form solution, but its recurrence $(n+2)(n+1)a_{n+2} = a_{n-1}$ generates the answer to any accuracy: with $y(0) = 1$, $y'(0) = 0$ the series starts $1 + \frac{x^3}{6} + \cdots$. Series ARE the solution.

Recognizing an old friend

$y' = 2xy$, $y(0) = 1$: the recurrence $na_n = 2a_{n-2}$ gives $y = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \cdots = e^{x^2}$ — matching the separable-method answer, coefficient for coefficient.

Significance: when formulas run out

Most differential equations of physics — Airy (optics near caustics), Bessel (drumheads), Hermite (quantum oscillators) — have NO elementary solutions. Series are not a fallback; they are how these functions are DEFINED, tabulated, and computed inside every scientific library you have ever called.

Try it: Try it: two terms by recurrence

For $y' = x+y$, $y(0) = 1$, find a_2 and a_3. Work: matching gives $(n+1)a_{n+1} = a_n$ plus the extra x: $a_1 = a_0 = 1$; $2a_2 = a_1 + 1 = 2$ so $a_2 = 1$; $3a_3 = a_2$ so $a_3 = \frac{1}{3}$. Series: $1 + x + x^2 + \frac{x^3}{3} + \cdots$ (indeed the exact solution is $2e^x - x - 1$ — expand it and check both coefficients ✓).

10.2 Proofs & Why It Matters

Proof: coefficient matching is valid

A power series that sums to zero on an interval has ALL coefficients zero (differentiate repeatedly and evaluate at the center: $a_n = \frac{f^{(n)}(0)}{n!} = 0$).

So when substituting $y = \sum a_n x^n$ into an equation produces $\sum c_n x^n = 0$, each c_n must individually vanish — turning the differential equation into an exact recurrence, not an approximation. ■

Proof: y = y forces a = 1/n!

Substitute: $\sum_{n\geq 1} na_n x^{n-1} = \sum_{n\geq 0} a_n x^n$. Shift the left index and match coefficients of x^n: $(n+1)a_{n+1} = a_n$. With $a_0 = y(0) = 1$, induction gives $a_n = \frac{1}{n!}$: $a_1 = 1$, and $a_{n+1} = \frac{a_n}{n+1} = \frac{1}{(n+1)!}$.

The series $\sum \frac{x^n}{n!}$ — the definition of e^x — emerges with no prior knowledge of exponentials. ■

10.3 Going Deeper: Explanations & Worked Problems

The recurrence method, slowly

Write $y = \sum_{n\geq 0} a_n x^n$ and differentiate term by term: $y' = \sum_{n\geq 1} n a_n x^{n-1}$, $y'' = \sum_{n\geq 2} n(n-1)a_n x^{n-2}$.

Substitute into the equation; now every sum must be re-indexed so all run over the SAME power x^n — the shift $m = n-2$ turns $\sum n(n-1)a_n x^{n-2}$ into $\sum (m+2)(m+1)a_{m+2}x^m$, and multiplying by x shifts the other way. Collect the coefficient of each x^n and set it to zero (legal because a power series vanishing identically has all coefficients zero — the proof slide): out drops a RECURRENCE expressing high coefficients in terms of low ones. Initial conditions seed it: $a_0 = y(0)$, $a_1 = y'(0)$. Then compute coefficients one at a time, as many as accuracy demands. The equation is never "solved" in closed form — and never needs to be.

Worked: deriving a recurrence with the index shift shown

Solve $y'' = -y$, $y(0) = 1$, $y'(0) = 0$, by series.

Step 1 — substitute: $\sum_{n\geq 2} n(n-1)a_n x^{n-2} = -\sum_{n\geq 0} a_n x^n$. Step 2 — shift the left index by 2: $\sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n = -\sum_{n\geq 0} a_n x^n$. Step 3 — equate the coefficient of every x^n: $(n+2)(n+1)a_{n+2} = -a_n$. Step 4 — seed and run: $a_0 = 1$, $a_1 = 0$; then $a_2 = \frac{-1}{2\cdot 1} = -\frac{1}{2}$; $a_3 = 0$ (all odd coefficients inherit $a_1 = 0$); $a_4 = \frac{-a_2}{4\cdot 3} = \frac{1}{24}$; $a_6 = \frac{-a_4}{6\cdot 5} = -\frac{1}{720}$. Step 5 — recognize: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos x$ — the series method has DERIVED the cosine series from the oscillator equation, not assumed it.

Worked: the Airy-type start, where no closed form exists

Begin $y'' = xy$, $y(0) = 1$, $y'(0) = 0$.

Step 1 — substitute and align powers: the left side is $\sum_{n\geq 0}(n+2)(n+1)a_{n+2}x^n$; the right side $xy = \sum_{n\geq 1} a_{n-1}x^n$. Step 2 — match: for $n = 0$: $2a_2 = 0$, so $a_2 = 0$; for $n \geq 1$: $(n+2)(n+1)a_{n+2} = a_{n-1}$. Step 3 — run it: $a_3 = \frac{a_0}{3\cdot 2} = \frac{1}{6}$; $a_4 = \frac{a_1}{4\cdot 3} = 0$; $a_5 = \frac{a_2}{5\cdot 4} = 0$; $a_6 = \frac{a_3}{6\cdot 5} = \frac{1}{180}$. Step 4 — the solution so far: $y = 1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots$, with nonzero coefficients only at powers $0, 3, 6, 9, \ldots$ — a three-step lattice the recurrence forces. No elementary function has this series; the series itself is the honest, computable answer, good to any accuracy on any bounded interval.

11 Frobenius & Special Functions

Singular points, indicial equations, Bessel and Legendre

11.1 Series at Singular Points

Regular singular points and the indicial equation

Where the leading coefficient vanishes (like $x = 0$ in $x^2y'' + xy' + (x^2 - \nu^2)y = 0$), plain power series can fail. Frobenius: try $y = x^r \sum a_n x^n$ with an UNKNOWN starting power r.

The lowest-order coefficient gives the INDICIAL equation — for Bessel, $r^2 - \nu^2 = 0$, roots $\pm\nu$ — and each admissible r seeds its own series. The pure case is Cauchy–Euler $x^2y'' + \alpha xy' + \beta y = 0$, where $y = x^r$ alone works and the indicial equation $r(r-1) + \alpha r + \beta = 0$ IS the whole story.

Bessel functions: the drumhead cosine

Bessel's equation governs radial vibrations of circular membranes and heat in cylinders. Its bounded solution $J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \cdots = \sum \frac{(-1)^k}{4^k(k!)^2} x^{2k}$ looks like a cosine whose amplitude slowly decays; its zeros (2.405, 5.520,) set the drum's overtone frequencies — irregularly spaced, which is why a drum sounds like a thump and not a note. Half-integer orders collapse to elementary functions: $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Legendre polynomials: quantization by boundedness

Legendre's equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ appears whenever a problem lives on a sphere. Series solutions generally blow up at $x = \pm 1$ (the poles); demanding BOUNDEDNESS forces the parameter to be a whole number $n(n+1)$, truncating the series to the polynomials $P_0 = 1$, $P_1 = x$, $P_2 = \frac{3x^2-1}{2}$, $P_3 = \frac{5x^3-3x}{2}, \ldots$ — orthogonal on $[-1, 1]$ and the angular half of spherical harmonics. The same mechanism that quantized Fourier eigenvalues, wearing spherical clothes.

Significance: the functions with their own names

Bessel, Legendre, Hermite, Laguerre — the "special functions" are the standard library of mathematical physics, and Frobenius is the compiler that builds them. Your phone's antenna design (cylindrical waves = Bessel) and every atomic orbital picture (Legendre in the angles) come from this machinery.

Try it: Try it: an indicial equation cold

For $2x^2y'' + 3xy' - y = 0$, find the indicial roots. Work: substitute $y = x^r$: $2r(r-1) + 3r - 1 = 2r^2 + r - 1 = (2r-1)(r+1) = 0$: $r = \frac{1}{2}$ or -1. General solution $y = A\sqrt{x} + \frac{B}{x}$ — fractional powers, impossible for constant-coefficient equations, routine here.

11.2 Proofs & Why It Matters

Proof: the Cauchy–Euler indicial equation

Substitute $y = x^r$ into $x^2y'' + \alpha xy' + \beta y = 0$: the derivatives give $x^2 \cdot r(r-1)x^{r-2} + \alpha x \cdot$

$rx^{r-1} + \beta x^r = [r(r-1) + \alpha r + \beta]\, x^r.$

Since $x^r \neq 0$ for $x > 0$, the bracket must vanish — a QUADRATIC in r whose two roots give two power solutions (a double root contributes $x^r \ln x$, precisely parallel to the xe^{rx} of constant-coefficient theory, via the substitution $x = e^t$ which converts one theory into the other). ■

Proof: J solves Bessel of order zero

Termwise, with $y = \sum c_k x^{2k}$, $c_k = \frac{(-1)^k}{4^k (k!)^2}$: compute $xy'' + y' + xy$ (the $\nu = 0$ equation divided by x).

The x^{2k+1} coefficient is $c_{k+1}(2k+2)(2k+1) + c_{k+1}(2k+2) + c_k = c_{k+1}(2k+2)^2 + c_k$, and indeed $c_{k+1} = \frac{-c_k}{4(k+1)^2}$ makes it vanish: $\frac{-c_k}{4(k+1)^2} \cdot 4(k+1)^2 + c_k = 0$. Every coefficient dies — the series solves the equation exactly. ■

11.3 Going Deeper: Worked Problems

Worked: a Frobenius recurrence at a regular singular point

Find the first three nonzero terms of a Frobenius solution of $2xy'' + y' + y = 0$ for the larger indicial root.

Step 1 — try $y = \sum a_n x^{n+r}$; the lowest-order terms give the indicial equation $2r(r-1) + r = r(2r-1) = 0$: $r = 0$ or $\frac{1}{2}$. Step 2 — for $r = \frac{1}{2}$, matching the coefficient of x^{n+r} gives the recurrence $a_{n+1} = -\dfrac{a_n}{(n+1)(2n+3)}$... (from $2(n+r+1)(n+r)a_{n+1} + (n+r+1)a_{n+1} + a_n = 0$ with $r = \frac{1}{2}$). Step 3 — run it from $a_0 = 1$: $a_1 = -\frac{1}{1\cdot 3} = -\frac{1}{3}$, $a_2 = \frac{1/3}{2\cdot 5} = \frac{1}{30}$. Step 4 — the solution: $y = x^{1/2}\left(1 - \frac{x}{3} + \frac{x^2}{30} - \cdots\right)$ — a square-root prefactor no ordinary power series could produce, and the recurrence generates as many terms as accuracy demands.

Worked: a Cauchy–Euler initial-value problem

Solve $x^2 y'' - 3xy' + 4y = 0$ with $y(1) = 1$, $y'(1) = 0$.

Step 1 — indicial equation: $r(r-1) - 3r + 4 = r^2 - 4r + 4 = (r-2)^2$: double root $r = 2$. Step 2 — double root means $y = x^2(A + B\ln x)$ (the $\ln x$ plays the role t plays for repeated constant-coefficient roots). Step 3 — data: $y(1) = A = 1$; $y' = 2x(A + B\ln x) + Bx$, so $y'(1) = 2A + B = 0$, $B = -2$. Step 4 — solution $y = x^2(1 - 2\ln x)$. Step 5 — verify at $x = e$: $y = e^2(1-2) = -e^2$; direct substitution of $y = x^2 - 2x^2 \ln x$ into the equation collapses to 0 term by term ✓.

12 Models & Qualitative Behavior

Growth, cooling, mixing, and equilibria

12.1 The Classic Models

Logistic growth

$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$: growth is proportional to BOTH the population and the remaining room. The rate is a downward parabola in P, fastest exactly at half capacity $P = K/2$, and every positive solution flows to the carrying capacity K.

Newton cooling & mixing tanks

Cooling: the EXCESS over ambient decays exponentially — if the excess halves in 10 minutes it halves again in the next 10. Mixing: $\frac{dS}{dt} = (\text{rate in}) - \frac{S}{V}(\text{flow out})$; the equilibrium is where inflow balances outflow, always equal to incoming concentration times tank volume.

Euler's method, honestly

One step of size h: $y_{n+1} = y_n + h\, f(y_n)$ — follow the tangent line briefly, then re-aim. On $y' = y$ from $y(0) = 4$ with $h = 0.5$: estimate 6 vs. the true $4e^{0.5} \approx 6.59$; Euler undershoots convex solutions, and halving h halves the error.

Significance: the same equation, twelve costumes

Newton cooling = drug elimination = RC discharge = mixing = depreciation: all $u' = -ku$ about the right variable. Recognizing the ARCHETYPE is the modeling skill — once you see "rate proportional to amount (or excess)," you already know the solution, the half-life, and the long-run behavior.

Try it: Try it: build then solve

Coffee at 90° in a 20° room cools to 70° in 5 minutes. When does it hit 45°? Work: excess halves from 70 to 50... compute: $u = 70e^{kt}$ with $u(5) = 50$: $k = \frac{1}{5}\ln\frac{5}{7}$. Want $u = 25$: $t = \frac{\ln(25/70)}{k} \approx \frac{-1.0296}{-0.0673} \approx 15.3$ minutes. Always solve in the EXCESS variable.

12.2 Proofs & Why It Matters

Proof: solving the logistic equation

Separate $\frac{dP}{P(1-P/K)} = r\,dt$ and split by partial fractions: $\frac{1}{P(1-P/K)} = \frac{1}{P} + \frac{1/K}{1-P/K}$. Integrating gives $\ln\frac{P}{K-P} = rt + C$, and solving for P: $P(t) = \dfrac{K}{1 + Ae^{-rt}}$ — an S-curve rising to K. Every positive start flows to the carrying capacity because $e^{-rt} \to 0$. ■

Proof: Newton cooling is exponential in the excess

The law says $T' = -k(T - T_{\text{amb}})$. Let $u = T - T_{\text{amb}}$ (the excess); since T_{amb} is constant, $u' = T' = -ku$ — the pure exponential equation, so $u = u_0 e^{-kt}$.

Halving in equal time intervals is exactly the property of exponentials, which is why the "excess halves every 10 minutes" phrasing works. ■

Proof sketch: Euler's error halves with h

One Euler step commits error $\approx \frac{h^2}{2} y''$ (the first Taylor term the tangent line misses). Reaching a fixed time T takes T/h steps, so the accumulated error is on the order of $\frac{T}{h} \cdot \frac{h^2}{2} \max |y''| = O(h)$ — proportional to h. Halve the step, halve the error: Euler is a first-order method. ■

12.3 Going Deeper: Explanations & Worked Problems

Equilibria and stability: the phase-line shortcut

For an autonomous equation $y' = f(y)$, enormous insight costs one sketch. Find the zeros of f — the EQUILIBRIA, where the system sits still. Between consecutive zeros, f keeps one sign: y moves right (up) where $f > 0$, left (down) where $f < 0$.

Mark arrows on a number line and stability is visible: arrows converging on an equilibrium make it STABLE (perturb, return), diverging make it UNSTABLE (perturb, flee). Logistic growth $y' = ry(1 - y/K)$: zeros at 0 and K; $f > 0$ between them, so arrows point from 0 toward K — the origin is unstable, the carrying capacity stable, and every positive solution S-curves to K, all WITHOUT solving the equation. The phase line is the qualitative theory in miniature: solve only when you need numbers; sketch when you need understanding.

Worked: half-life to dosage, a complete exponential-decay problem

A drug has half-life 6 hours; a dose raises blood concentration by 10 units. When does a single dose drop below 1 unit?

Step 1 — model: $y' = ky$ with $y = 10e^{kt}$. Step 2 — pin k from the half-life: $y(6) = 5$ gives $e^{6k} = \frac{1}{2}$, so $k = -\frac{\ln 2}{6} \approx -0.1155$. Step 3 — solve $10e^{kt} = 1$: $e^{kt} = \frac{1}{10}$, so $t = \dfrac{\ln 10}{\ln 2} \cdot 6 = 6 \log_2 10 \approx 19.9$ hours. Step 4 — sanity check in halvings: $10 \to 5 \to 2.5 \to 1.25$ after three half-lives (18 h), and $1.25 \to 1$ needs about a third more of one half-life — $18 + 2 \approx 20$ h ✓. Step 5 — the modeling lesson: the answer depended only on the RATIO $10 : 1$, so every decade of decay costs the same $\log_2 10 \approx 3.32$ half-lives, whatever the starting dose.

Worked: a mixing tank, from setup to equilibrium

A 100 L tank starts with pure water. Brine at 2 g/L flows in at 5 L/min; the well-mixed tank drains at 5 L/min. Find the salt $S(t)$.

Step 1 — build the equation from bookkeeping: rate in $= 2 \cdot 5 = 10$ g/min; rate out = (concentration)(flow) $= \frac{S}{100} \cdot 5 = \frac{S}{20}$; so $S' = 10 - \frac{S}{20}$. Step 2 — equilibrium first: $S' = 0$ at $S = 200$ g — which is just (incoming concentration)(volume) $= 2 \cdot 100$, as it must be. Step 3 — solve: linear with $\mu = e^{t/20}$: $(e^{t/20}S)' = 10e^{t/20}$, so $e^{t/20}S = 200e^{t/20} + C$, and $S(0) = 0$ gives $C = -200$: $S(t) = 200\left(1 - e^{-t/20}\right)$. Step 4 — read the time scale: the "time constant" is 20 min; after 20 min the tank is at $200(1 - e^{-1}) \approx 126$ g, after an hour ≈ 190 g — the exponential approach to equilibrium that governs tanks, cooling coffee, and charging capacitors alike.

13 Existence, Uniqueness & Bifurcations

Picard's theorem, phase lines with parameters, harvesting collapse

13.1 When Solutions Exist, Split, and Vanish

Picard's theorem: the rules of the game

If f and $\partial f/\partial y$ are continuous near (t_0, y_0), the IVP $y' = f(t, y)$, $y(t_0) = y_0$ has exactly ONE solution near t_0.

Both hypotheses earn their keep: $y' = y^{2/3}$, $y(0) = 0$ has the two solutions $y \equiv 0$ and $y = (x/3)^3$ — the derivative $\frac{2}{3}y^{-1/3}$ blows up at 0 and uniqueness genuinely fails. And existence can be merely LOCAL: $y' = y^2$, $y(0) = 1$ solves to $y = \frac{1}{1-t}$, alive only until $t = 1$ — solutions can end in finite time without warning.

Bifurcations: how equilibria are born and die

Add a parameter: $y' = r + y^2$. For $r < 0$: two equilibria ($\pm\sqrt{-r}$, lower stable, upper unstable). At $r = 0$ they COLLIDE into one half-stable point; for $r > 0$ none remain — a SADDLE-NODE bifurcation, the generic way equilibria appear and annihilate. The signature: $f = 0$ and $f' = 0$ simultaneously.

Plotting equilibria against r gives the bifurcation diagram — the parabola $r = -y^2$ — and the whole family's fate in one picture.

Worked: harvesting a logistic population to collapse

Model $y' = y(1 - y) - h$: logistic growth minus constant harvest h.

Step 1 — equilibria: $y^2 - y + h = 0$, so $y = \frac{1 \pm \sqrt{1-4h}}{2}$. Step 2 — read the cases: $h < \frac{1}{4}$: two equilibria (upper stable — sustainable); $h = \frac{1}{4}$: they merge at $y = \frac{1}{2}$; $h > \frac{1}{4}$: NONE, $y' < 0$ everywhere, extinction in finite time. Step 3 — the moral: the maximum sustainable harvest is the peak of the growth curve, $\frac{1}{4}$ — and passing it is not a gradual decline but a saddle-node CLIFF. This one phase-line computation is the backbone of real fisheries mathematics.

Significance: tipping points are bifurcations

Fishery collapses, sudden lake eutrophication, and climate tipping points are saddle-node bifurcations in the wild: a slowly moving parameter annihilates the stable state, and the system jumps — with no gentle warning and no easy way back (hysteresis). The mathematics of this topic is how ecologists quantify "how close to the edge are we?"

Try it: Try it: a bifurcation diagram in your head

For $y' = ry - y^3$: equilibria as r varies? Work: $y(r - y^2) = 0$: always $y = 0$; for $r > 0$ also $y = \pm\sqrt{r}$. At $r = 0$ the origin loses stability and hands it to the new pair — a PITCHFORK bifurcation, the symmetric sibling of the saddle-node, and the standard model of spontaneous symmetry breaking.

13.2 Proofs & Why It Matters

Proof idea: Picard iteration

Rewrite the IVP as the integral equation $y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$ and iterate: $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds$ starting from $y_0(t) \equiv y_0$.

Continuity of $\partial f/\partial y$ makes f Lipschitz in y, so successive iterates differ by a factor $\leq L|t - t_0|$ each round — a geometric squeeze forcing convergence to a solution, and forcing any two solutions together (uniqueness) by the same estimate. For $y' = y$, the iterates are exactly the Taylor partial sums of e^t. ■

Proof: the saddle-node signature

At a bifurcation the equilibrium equation $f(y; r) = 0$ must have a DOUBLE root — a simple root moves smoothly as r varies (implicit function theorem) and cannot vanish. A double root means $f = 0$ and $f_y = 0$ together.

For $f = r + y^2$: $y^2 = -r$ and $2y = 0$ give $y = 0$, $r = 0$ — the collision point — and near it the equilibria $\pm\sqrt{-r}$ exist only on one side. Two conditions, one parameter: bifurcation points are isolated, which is why they are EVENTS. ■

13.3 Going Deeper: Worked Problems

Worked: a full phase-line analysis with a parameter

For $y' = y(y - 1)(y - r)$ with $0 < r < 1$, find and classify all equilibria and describe what happens as $r \to 1$.

Step 1 — equilibria: $y = 0$, $y = r$, $y = 1$. Step 2 — sign of f on each interval (test points): for $y < 0$: $(-)(-)(-) < 0$, moving down; on $(0, r)$: $(+)(-)(-) > 0$, moving up; on $(r, 1)$: $(+)(-)(+) < 0$, down; for $y > 1$: $(+)(+)(+) > 0$, up. Step 3 — read stability: arrows converge on r (stable) and diverge from 0 and 1 (unstable). Step 4 — as $r \to 1$ the stable equilibrium collides with the unstable one at 1 and they annihilate (a saddle-node): for $r > 1$ the region just above 0 flows all the way up unboundedly. Step 5 — verify with f': $f'(r) = r(r-1) < 0$ ✓ stable; $f'(0) = r > 0$ and $f'(1) = 1 - r > 0$ ✓ unstable.

Worked: finite-time blowup, exactly

Solve $y' = y^2$, $y(0) = 2$, and find the time at which the solution ceases to exist.

Step 1 — separate: $\dfrac{dy}{y^2} = dt$, so $-\dfrac{1}{y} = t + C$. Step 2 — data: $-\frac{1}{2} = C$, hence $y = \dfrac{1}{\frac{1}{2} - t} = \dfrac{2}{1-2t}$. Step 3 — the solution blows up as $t \to \frac{1}{2}^-$: it exists only on $[0, \frac{1}{2})$, even though $f(y) = y^2$ is perfectly smooth everywhere. Step 4 — the moral: Picard guarantees existence only LOCALLY; a smooth right-hand side that grows faster than linearly can drive solutions to infinity in finite time. Compare $y' = y$ (exists forever) — the difference between linear and quadratic growth is the difference between eternity and half a second.

14 Boundary Values & Fourier Series

Eigenvalue problems, sine series, the heat and wave equations

14.1 From Boundary Conditions to Fourier

Boundary conditions quantize

An initial-value problem specifies everything at one point; a BOUNDARY-value problem splits conditions between two ($y(0) = y(L) = 0$) — and suddenly solutions exist only for special parameter values.

$y'' + \lambda y = 0$ with $y(0) = y(\pi) = 0$: the general solution $\sin(\sqrt{\lambda}\, x)$ meets the far boundary only when $\sqrt{\lambda}$ is an integer — EIGENVALUES $\lambda_n = n^2$ with eigenfunctions $\sin nx$. A guitar string of length L can vibrate only at frequencies $\frac{n\pi c}{L}$: harmonics are a boundary-value theorem.

Fourier series: every shape from pure tones

The eigenfunctions are ORTHOGONAL ($\int_0^\pi \sin mx \sin nx\, dx = 0$ for $m \neq n$), so any reasonable f on $(0, \pi)$ expands as $f = \sum b_n \sin nx$ with each coefficient computed independently: $b_n = \frac{2}{\pi} \int_0^\pi f \sin nx\, dx$ — a projection, exactly as in linear algebra with an orthogonal basis.

The flat function $f = 1$ becomes the square-wave series $\frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$.

The heat and wave equations, mode by mode

Separation of variables turns a PDE into the eigenvalue problem plus simple time behavior.

HEAT ($u_t = ku_{xx}$): each mode decays, $u = \sum b_n e^{-kn^2 t} \sin nx$ — the n^2 means fine detail smooths almost instantly while the broad shape lingers.

WAVE ($u_{tt} = c^2 u_{xx}$): each mode oscillates at frequency nc, or equivalently d'Alembert's $f(x - ct) + g(x + ct)$ — disturbances travel at speed c unchanged. Expand the initial data, evolve each mode by its own simple rule, superpose: that is the whole method.

Significance: the spectrum is everywhere

MP3 compression, noise-cancelling headphones, MRI reconstruction, and spectroscopy all rest on expanding signals in sines and cosines. The heat-equation origin story matters too: Fourier invented the series in 1807 to model heat in a metal bar, was told the idea was rigorous nonsense, and turned out to be the most useful ”nonsense” in engineering history.

Try it: Try it: decay rates order themselves

A metal bar's initial temperature has modes $\sin x$ and $\sin 5x$ with equal amplitude. After time t, their ratio? Work: mode n decays like $e^{-n^2 kt}$, so the ratio is $e^{-(25-1)kt} = e^{-24kt}$ — the $\sin 5x$ ripple dies 24 times faster in the exponent. Fine structure vanishes almost immediately; the broad shape lingers. That IS why blurring smooths.

14.2 Proofs & Why It Matters

Proof: orthogonality of the sine family

Product-to-sum: $\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]$. Integrate over $[0, \pi]$: for $m \neq n$ both cosine integrals vanish ($\sin k\pi = 0$), giving 0; for $m = n$ the first term is $\frac{1}{2}\int_0^\pi 1\,dx = \frac{\pi}{2}$.

Hence $\int_0^\pi f \sin nx\,dx = b_n \cdot \frac{\pi}{2}$ when $f = \sum b_m \sin mx$ — every other term integrates away, isolating b_n. ■

Proof: separation of variables for heat

Try $u = X(x)T(t)$ in $u_t = ku_{xx}$: $XT' = kX''T$, so $\dfrac{T'}{kT} = \dfrac{X''}{X}$. The left side depends only on t, the right only on x — a function of t equal to a function of x must be CONSTANT, say $-\lambda$.

This splits the PDE into $X'' + \lambda X = 0$ (the boundary-value problem, quantizing $\lambda = n^2$) and $T' = -k\lambda T$ (exponential decay $e^{-kn^2 t}$). Superposition assembles the general solution because the heat equation is linear. ■

14.3 Going Deeper: Worked Problems

Worked: a full heat-equation solve

A bar of length π with ends held at 0 starts at temperature $u(x,0) = 3\sin x - \sin 3x$. Find $u(x,t)$ for $u_t = 2u_{xx}$.

Step 1 — separation gives modes $\sin nx$ (the boundary conditions quantize to integer n), each decaying like e^{-2n^2t} (the 2 is the diffusivity). Step 2 — the initial data is ALREADY a finite Fourier sine series: $b_1 = 3$, $b_3 = -1$, all others 0 — no coefficient integrals needed. Step 3 — evolve each mode: $u(x,t) = 3e^{-2t}\sin x - e^{-18t}\sin 3x$. Step 4 — check: $u_t = -6e^{-2t}\sin x + 18e^{-18t}\sin 3x$ and $2u_{xx} = 2(-3e^{-2t}\sin x + 9e^{-18t}\sin 3x)$ — equal ✓; ends: $\sin 0 = \sin 3\pi = 0$ ✓. Step 5 — by $t = 0.5$ the $\sin 3x$ ripple has shrunk by $e^{-9} \approx 0.01\%$ while the fundamental keeps $e^{-1} \approx 37\%$: the bar forgets its fine structure almost instantly.

Worked: a Fourier coefficient integral by hand

Find the Fourier sine series of $f(x) = x$ on $(0,\pi)$.

Step 1 — $b_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi x\sin nx\,dx$. Step 2 — integrate by parts with $u = x$, $dv = \sin nx\,dx$: $\left[-\dfrac{x\cos nx}{n}\right]_0^\pi + \dfrac{1}{n}\int_0^\pi \cos nx\,dx = -\dfrac{\pi\cos n\pi}{n} + 0 = -\dfrac{\pi(-1)^n}{n}$. Step 3 — so $b_n = \dfrac{2}{\pi}\cdot\dfrac{\pi(-1)^{n+1}}{n} = \dfrac{2(-1)^{n+1}}{n}$: $x = 2\left(\sin x - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \cdots\right)$. Step 4 — a famous payoff: at $x = \frac{\pi}{2}$ the series gives $\dfrac{\pi}{2} = 2\left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \cdots\right)$ — the Leibniz formula for π, falling out of a heat-flow calculation.

15 Numerical Methods: Euler to Runge–Kutta

Heun, RK4, error orders, and stability ceilings

15.1 Computing Solutions Honestly

The ladder of methods

EULER: follow the tangent, $y_{n+1} = y_n + hf(t_n, y_n)$ — global error $O(h)$. IMPROVED EULER (Heun): predict with Euler, evaluate the slope at the prediction, use the AVERAGE of the two slopes — $O(h^2)$: on $y' = y$, $y(0) = 1$, $h = 1$ it gives 2.5 where Euler gives 2 and the truth is $e \approx 2.718$.

RK4 blends four slope samples per step for $O(h^4)$ — the workhorse default. Order buys compounding accuracy: halving h improves Euler 2× but RK4 16×.

Stability: the other failure mode

Accuracy is not the only concern. On decay $y' = -\lambda y$, Euler iterates $y_{n+1} = (1-\lambda h)y_n$: if $h > \frac{2}{\lambda}$ the multiplier exceeds 1 in size and the computed solution OSCILLATES AND GROWS while the true one dies quietly.

Stiff problems (mixing fast and slow decay) force explicit methods to tiny steps for stability alone — the reason implicit methods (backward Euler: $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$, stable for every h) exist.

Worked: watching the error order with your own eyes

Solve $y' = y$, $y(0) = 1$ to $t = 1$ (truth: $e \approx 2.71828$).

Step 1 — Euler with $h = 0.5$: $(1.5)^2 = 2.25$, error 0.468. Step 2 — Euler with $h = 0.25$: $(1.25)^4 \approx 2.441$, error 0.277 — roughly HALVED, as $O(h)$ promises. Step 3 — Heun with $h = 0.5$: multiplier $1+h+\frac{h^2}{2} = 1.625$ per step, $(1.625)^2 \approx 2.641$, error 0.078. Step 4 — Heun with $h = 0.25$: error ≈ 0.022, close to a QUARTER — $O(h^2)$ verified numerically. Running two step sizes and comparing is the standard self-check every real computation should include.

Significance: most equations are solved this way

Weather forecasts, spacecraft trajectories, protein folding, and car-crash simulations never see a closed-form solution — they are RK-style steppers running billions of steps. The stability theory of this topic is not academic: an unstable integrator once meant exploding simulations and, historically, real engineering failures.

Try it: Try it: one honest Euler step

For $y' = t + y$, $y(0) = 1$, take one Euler step with $h = 0.1$, then one Heun step. Work: Euler: $y_1 = 1 + 0.1(0+1) = 1.1$. Heun: predictor slope 1; corrector slope $f(0.1, 1.1) = 1.2$; $y_1 = 1 + 0.05(1+1.2) = 1.11$.

Answer. True value: $2e^{0.1} - 0.1 - 1 \approx 1.1103$ — Heun lands within 0.0003 at a tenth the step cost of high-accuracy Euler.

15.2 Proofs & Why It Matters

Proof: Euler's global error is O(h)

One step from the true solution commits local error $y(t+h) - y(t) - hy'(t) = \frac{h^2}{2}y''(\xi)$ — the first Taylor term the tangent line drops.

Reaching a fixed time T takes $\frac{T}{h}$ steps; local errors accumulate (amplified at most by a bounded factor e^{LT} from the Lipschitz constant), totalling $\frac{T}{h} \cdot O(h^2) = O(h)$. The same bookkeeping with a sharper local error $O(h^{p+1})$ yields global order p — which is exactly how Heun ($p = 2$) and RK4 ($p = 4$) earn their labels. ■

Proof: Heun's extra order

Heun's update is $y + \frac{h}{2}[f(y) + f(y + hf(y))]$. Expand the second slope: $f(y + hf(y)) = f + hff' + O(h^2)$, so the update is $y + hf + \frac{h^2}{2}ff' + O(h^3)$.

The true solution expands as $y + hy' + \frac{h^2}{2}y'' + O(h^3)$, and $y'' = \frac{d}{dt}f(y) = f'y' = ff'$ — the h^2 terms MATCH. Local error drops to $O(h^3)$, hence global $O(h^2)$: averaging the two slopes secretly reproduces the second Taylor coefficient. ■

15.3 Going Deeper: Worked Problems

Worked: RK4 by hand, one step

Apply one RK4 step with $h = 0.2$ to $y' = y - t$, $y(0) = 2$ (exact solution $y = t + 1 + e^t$).

Step 1 — the four slopes: $k_1 = f(0, 2) = 2$; $k_2 = f(0.1, 2 + 0.1 \cdot 2) = f(0.1, 2.2) = 2.1$; $k_3 = f(0.1, 2 + 0.1 \cdot 2.1) = f(0.1, 2.21) = 2.11$; $k_4 = f(0.2, 2 + 0.2 \cdot 2.11) = f(0.2, 2.422) = 2.222$. Step 2 — weighted average: $y_1 = 2 + \dfrac{0.2}{6}(2 + 2 \cdot 2.1 + 2 \cdot 2.11 + 2.222) = 2 + \dfrac{0.2}{6}(12.642) = 2.4214$. Step 3 — exact: $y(0.2) = 1.2 + e^{0.2} = 2.42140\ldots$ — RK4 agrees to about five significant figures in ONE step; Euler would have given 2.4 (error 0.02). Step 4 — the pattern: slopes at start, two midpoint estimates, and end, weighted $1 : 2 : 2 : 1$ — Simpson's rule in disguise, which is why the order is four.

Worked: catching instability before it bites

For $y' = -50y$ (a stiff decay), find the largest Euler step that stays stable, and compare with backward Euler.

Step 1 — Euler: $y_{n+1} = (1 - 50h)y_n$; stability needs $|1 - 50h| < 1$, i.e. $h < 0.04$. With $h = 0.05$ the multiplier is -1.5: the numbers alternate sign and GROW 50% per step — nonsense for a decaying solution. Step 2 — backward Euler: $y_{n+1} = y_n - 50hy_{n+1}$, so $y_{n+1} = \dfrac{y_n}{1 + 50h}$; the multiplier is below 1 for EVERY $h > 0$ — unconditionally stable. Step 3 — the trade: backward Euler costs an algebraic solve per step (trivial here, a Newton iteration in general) but lets the step size follow accuracy instead of stability. Step 4 — this is why "stiff" is a word every simulation engineer knows: the fast-decaying mode is boring but still dictates explicit step sizes.