MathCastle

Calculus

Complete course booklet • 14 topics

Contents

1 Core Ideas in Plain Terms

Derivatives and integrals, in everyday language

1.1 The Derivative = Instantaneous Rate

Slope at a single instant

The **derivative** $f'(x)$ is the slope of the graph right at a point — how fast the output is changing at that exact instant. If f is your position over time, f' is your speed. The power rule $(x^n)' = nx^{n-1}$ is the fast way to compute it for powers.

(diagram in the online version)

Derivative of x^2

$(x^2)' = 2x$. At $x = 3$ the slope is $2 \cdot 3 = 6$ — the curve is rising six units up per unit across at that instant.

1.2 The Integral = Accumulated Total

Adding up tiny pieces

An **integral** $\int_a^b f(x)\,dx$ adds up infinitely many thin slices to get a total — most often the *area* under the curve between a and b. If f is your speed over time, the integral is the total distance you travelled.

Area under y = 2x from 0 to 2

$\int_0^2 2x\,dx = [x^2]_0^2 = 4$. That is exactly the area of the triangle under the line — base 2, height 4, area $\frac{1}{2} \cdot 2 \cdot 4 = 4$.

(diagram in the online version)

1.3 The Fundamental Theorem, Plainly

Derivatives and integrals are opposites

The **Fundamental Theorem of Calculus** says that integrating and differentiating undo each other. To find $\int_a^b f$, find a function F whose derivative is f, then just compute $F(b) - F(a)$ — no adding up tiny slices by hand.

Using an antiderivative

To integrate $2x$: an antiderivative is x^2 (its derivative is $2x$). So $\int_1^3 2x\,dx = 3^2 - 1^2 = 8$.

1.4 Going Deeper: The Chain Rule and Why Integrals Are Areas

The chain rule: rates multiply through

If y changes 3 times as fast as u, and u changes 2 times as fast as x, then y changes 6 times as fast as x. That is the whole chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$ — differentiate the outside, keep the inside, then multiply by the inside's rate.

Chain rule in action

$\frac{d}{dx}\left(x^2+1\right)^3 = 3\left(x^2+1\right)^2 \cdot 2x = 6x\left(x^2+1\right)^2.$

Why the integral measures area

Slice the region under a curve into thin vertical strips. Each strip is nearly a rectangle: height $f(x)$, tiny width. The definite integral $\int_a^b f(x)\,dx$ is what those rectangle areas add up to as the strips get thinner — which is also why integrating a *rate* (like speed) recovers a *total* (like distance). That derivative–integral link is the **Fundamental Theorem of Calculus**: $\int_a^b F'(x)\,dx = F(b) - F(a)$.

Tangent line to a curve (multi-step)

Find the tangent line to $y = x^2$ at $x = 3$. Step 1 — **power rule**: $y' = 2x$. Step 2 — the slope there is $2 \cdot 3 = 6$. Step 3 — point-slope form through $(3, 9)$: $y - 9 = 6(x - 3)$, so $y = 6x - 9$.

1.5 Problem-Solving Playbook

Rate, total, or slope? Name it first

Before touching symbols, say what the question IS: a **rate or slope** → differentiate; an **accumulated total or area** → integrate. And always check an integral by differentiating your antiderivative — it should give back the integrand.

Worked: integrate, then check

$\int_0^2 (3x^2 + 2)\,dx$. Antiderivative: $x^3 + 2x$. Evaluate: $(8 + 4) - 0 = 12$. Check by the **power**

rule: $\dfrac{d}{dx}(x^3 + 2x) = 3x^2 + 2$ — the integrand, so the answer stands.

1.6 How to Find Limits

Five moves, in the order to try them

1. Plug in. If nothing breaks, that's the limit. **2. Factor and cancel** when plug-in gives $\frac{0}{0}$ — both parts share the root, so divide it out. **3. Multiply by the conjugate** when a square root blocks the factoring. **4. L'Hôpital's rule** is a way: on a genuine $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, take the derivative of the top and the bottom separately and try again. **5. Squeeze** an oscillating expression between two bounds that agree.

The ladder on one problem

$\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$: plug-in gives $\frac{0}{0}$; the conjugate turns it into $\dfrac{1}{\sqrt{x}+3} \to \dfrac{1}{6}$. (L'Hôpital also works: $\dfrac{1/(2\sqrt{x})}{1} \to \dfrac{1}{6}$ — two rungs of the ladder, same answer.)

When $L'Hpital$ is NOT allowed

The rule needs an actual $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Using it on $\lim_{x\to 0} \frac{x+1}{x+2} = \frac{1}{2}$ (not an indeterminate form) gives the WRONG answer 1. Check the form first, every time.

2 Infinite Sequences and Series

Everything you need for AP Calculus BC, Topic 10

2.1 Sequences: Convergence, Divergence & Limits

What a sequence limit means

A **sequence** $\{a_n\}$ is an ordered list $a_1, a_2, a_3, \ldots$ We say $\{a_n\}$ **converges** to L if $\lim\limits_{n\to\infty} a_n = L$ (a finite number); otherwise it **diverges**. To find the limit, treat n as a continuous variable and use the same tools as limits at infinity: dominant terms, dividing by the highest power, or L'Hôpital's Rule.

Rational and root sequences

$$\lim_{n\to\infty}\frac{3n^2-n}{5n^2+7}=\frac{3}{5},\qquad \lim_{n\to\infty}\frac{\ln n}{n}\overset{\text{L'H}}{=}\lim_{n\to\infty}\frac{1/n}{1}=0,\qquad \lim_{n\to\infty}\left(1+\tfrac{2}{n}\right)^n=e^2.$$

The first divides by n^2; the middle uses L'Hôpital; the last is the definition of e^x.

Tip: Useful facts — $\lim_{n\to\infty} r^n=0$ if $|r|<1$; $n^{1/n}\to 1$; $\frac{n^k}{n!}\to 0$ and $\frac{r^n}{n!}\to 0$ for any fixed k,r (factorials beat everything). A bounded, monotonic sequence always converges.

2.2 Series, Partial Sums, Geometric Series & the nth-Term Test

Series and partial sums

A **series** is the sum $\displaystyle\sum_{n=1}^{\infty} a_n$. Its value is the limit of the **partial sums** $S_N=\sum_{n=1}^{N} a_n$: the series converges to S if $\displaystyle\lim_{N\to\infty} S_N=S$. Do not confuse the sequence of terms a_n with the sequence of partial sums S_N.

Geometric series

$\displaystyle\sum_{n=0}^{\infty} ar^n=a+ar+ar^2+\cdots$ **converges if and only if** $|r|<1$, and then

$$\sum_{n=0}^{\infty} ar^n=\frac{a}{1-r}\qquad (a \text{ is the } \textit{first} \text{ term}, r \text{ the common ratio}).$$

If $|r|\ge 1$ it diverges.

Summing a geometric series

$$\sum_{n=1}^{\infty}\frac{2}{3^n}=\frac{2/3}{1-\frac{1}{3}}=\frac{2/3}{2/3}=1,\qquad \sum_{n=0}^{\infty}5\left(-\tfrac{1}{2}\right)^n=\frac{5}{1-(-\frac{1}{2})}=\frac{5}{3/2}=\frac{10}{3}.$$

For the first, the first term (at $n=1$) is $\frac{2}{3}$ and $r=\frac{1}{3}$.

The nth-Term Test (Test for Divergence)

If $\displaystyle\lim_{n\to\infty} a_n\ne 0$ (or does not exist), then $\sum a_n$ **diverges**. *Warning:* if $\lim a_n=0$, the test is **inconclusive** — the series may converge or diverge. It can never prove convergence.

Divergence test

$\sum_{n=1}^{\infty} \frac{n}{2n+1}$ diverges because $\lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \neq 0$. But for $\sum \frac{1}{n}$, $\lim \frac{1}{n} = 0$, so the test says nothing (in fact it diverges — see the p-series).

Tip: Always run the nth-term test *first*. If the terms don't shrink to 0, you're done — it diverges. If they do go to 0, you must try another test.

2.3 The Integral Test & p-Series

Integral Test

If f is positive, continuous, and *decreasing* for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)\,dx \text{ either both converge or both diverge.}$$

The integral's *value* is **not** the sum; it only decides convergence.

p-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges if** $p > 1$ and **diverges if** $p \leq 1$. The case $p = 1$, $\sum \frac{1}{n}$, is the **harmonic series**, which diverges.

Integral test in action

Test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$. With $f(x) = \frac{1}{x^2+1}$ (positive, decreasing),

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \big[\arctan x\big]_1^b = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \text{ (finite),}$$

so the series **converges**. (The sum is not $\frac{\pi}{4}$; the integral only certifies convergence.)

Tip: $\sum \frac{1}{n^2}$, $\sum \frac{1}{n^3}$ converge; $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ and $\sum \frac{1}{n}$ diverge. Memorize the $p > 1$ threshold — it powers most comparison tests.

2.4 Comparison & Limit Comparison Tests

Direct Comparison Test

Suppose $0 \le a_n \le b_n$ for all large n.

- If $\sum b_n$ **converges**, then $\sum a_n$ converges (smaller than convergent).
- If $\sum a_n$ **diverges**, then $\sum b_n$ diverges (bigger than divergent).

Compare against a known geometric or p-series.

Limit Comparison Test

If $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$ (a finite *positive* number), then $\sum a_n$ and $\sum b_n$ **both converge or both diverge**. Choose b_n from the dominant-term behavior of a_n.

Direct comparison

$\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ converges: since $\frac{1}{n^2+n} \le \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ $(p=2)$ converges.

Limit comparison

Test $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n+4}$. For large n the terms behave like $\frac{2n}{n^3} = \frac{2}{n^2}$, so compare with $b_n = \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\frac{2n+1}{n^3-n+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2n^3+n^2}{n^3-n+4} = 2 \quad (0 < 2 < \infty).$$

Since $\sum \frac{1}{n^2}$ converges, the series **converges**.

Tip: Pick the comparison series by keeping only the highest-degree term of numerator and denominator. Limit comparison is more forgiving than direct comparison when the inequality direction is awkward.

2.5 The Ratio Test & the Root Test

Ratio Test

Let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$.

$L < 1 \Rightarrow$ converges (absolutely); $L > 1$ (or ∞) $\Rightarrow$ diverges; $L = 1 \Rightarrow$ inconclusive.

Best when a_n contains **factorials** or n**th powers**.

Root Test

Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{1/n}$. Same conclusions as the ratio test ($L < 1$ converges, $L > 1$ diverges, $L = 1$ inconclusive). Best when a_n is raised to the nth power, e.g. $\left(\frac{n}{2n+1}\right)^n$.

Ratio test with a factorial

Test $\sum_{n=1}^{\infty} \frac{2^n}{n!}$:

$$L = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1,$$

so it **converges**. (In fact it sums to $e^2 - 1$.)

Root test

Test $\sum_{n=1}^{\infty} \left(\frac{3n+1}{4n-1}\right)^n$:

$$L = \lim_{n\to\infty} \left(\frac{3n+1}{4n-1}\right) = \frac{3}{4} < 1 \;\Rightarrow\; \textbf{converges}.$$

Tip: If the ratio/root test gives $L = 1$, switch tests (often a p-series or comparison). Ratio-test $L = 1$ happens for *every* p-series, so it can never settle those.

2.6 Alternating Series, Error Bound & Absolute vs. Conditional Convergence

Alternating Series Test (AST)

A series $\sum(-1)^n b_n$ with $b_n > 0$ **converges** if both hold:

(1) b_n is *decreasing* ($b_{n+1} \le b_n$), and

(2) $\lim_{n\to\infty} b_n = 0$.

Alternating Series Error Bound

If an alternating series meets the AST conditions and converges to S, then the error after N terms is bounded by the *first omitted term*:

$$|S - S_N| \le b_{N+1}.$$

Absolute vs. conditional convergence

$\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges (this forces $\sum a_n$ to converge). It converges **conditionally** if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Conditional convergence & error

The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the AST ($\frac{1}{n} \downarrow 0$), but $\sum \frac{1}{n}$ diverges, so it converges **conditionally**. To approximate its sum within 0.01, we need $b_{N+1} = \frac{1}{N+1} \le 0.01$, i.e. $N \ge 99$ terms.

Tip: To classify: test $\sum |a_n|$ first (ratio/comparison/p-series). Converges $\Rightarrow$ absolute. Diverges but the alternating series itself passes the AST $\Rightarrow$ conditional.

2.7 Power Series: Radius & Interval of Convergence

Power series and its radius

A **power series** centered at c is $\displaystyle\sum_{n=0}^{\infty} a_n (x-c)^n$. There is a **radius of convergence** R so it converges for $|x-c| < R$ and diverges for $|x-c| > R$. Find R with the **ratio test**: solve $\displaystyle\lim_{n\to\infty} \left| \frac{a_{n+1}(x-c)^{n+1}}{a_n (x-c)^n} \right| < 1$.

Interval of convergence — check the endpoints!

The set of x where the series converges is the **interval of convergence**, $(c-R,\ c+R)$ possibly including one or both endpoints. The ratio test never decides the endpoints ($L = 1$ there): substitute each endpoint and test the resulting numeric series separately.

Full interval of convergence

Find the interval for $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$. Ratio test:

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right| = |x-2| < 1 \;\Rightarrow\; R = 1,\; 1 < x < 3.$$

Endpoints: at $x = 3$, $\sum \frac{1}{n}$ diverges (harmonic). At $x = 1$, $\sum \frac{(-1)^n}{n}$ converges (AST). Interval: $\boxed{[1, 3)}$.

Differentiating & integrating a power series

Within the open interval of convergence you may differentiate or integrate **term by term**, and R stays the same (endpoints may change):

$$f(x) = \sum a_n (x-c)^n \;\Rightarrow\; f'(x) = \sum n\, a_n (x-c)^{n-1}, \qquad \int f\, dx = C + \sum \frac{a_n}{n+1}(x-c)^{n+1}.$$

Tip: $R = \infty$ means it converges for all x (e.g. $e^x, \sin x, \cos x$); $R = 0$ means only at $x = c$. Endpoints require a fresh test every time — never assume.

2.8 Taylor & Maclaurin Series; the Lagrange Error Bound

Taylor and Maclaurin series

The **Taylor series** of f centered at c is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

A **Maclaurin series** is the special case $c = 0$.

Common Maclaurin series (memorize)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \text{(all } x\text{)}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \text{(all } x\text{)}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad \text{(all } x\text{)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad (|x| < 1)$$

Building a Taylor polynomial

Third-degree Maclaurin polynomial of $f(x) = \ln(1+x)$: $f(0) = 0$, $f' = \frac{1}{1+x}$ so $f'(0) = 1$, $f'' = -\frac{1}{(1+x)^2}$ so $f''(0) = -1$, $f''' = \frac{2}{(1+x)^3}$ so $f'''(0) = 2$. Thus

$$P_3(x) = 0 + 1 \cdot x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 = x - \frac{x^2}{2} + \frac{x^3}{3}.$$

Lagrange Error Bound

If $P_n(x)$ is the nth-degree Taylor polynomial about c, the remainder satisfies

$$|R_n(x)| = |f(x) - P_n(x)| \le \frac{M}{(n+1)!}\,|x-c|^{n+1},$$

where M is any bound on $|f^{(n+1)}|$ between c and x. This measures how good the polynomial approximation is.

Lagrange error bound applied

Approximate $\cos(0.2)$ with $P_2(x) = 1 - \frac{x^2}{2}$. Here $c = 0$, $n = 2$, and $|f^{(3)}(x)| = |\sin x| \le 1$, so $M = 1$:

$$|R_2(0.2)| \le \frac{1}{3!}|0.2|^3 = \frac{0.008}{6} \approx 0.00133.$$

So $P_2(0.2) = 1 - 0.02 = 0.98$ is accurate to about 0.0013.

2.9 Which Convergence Test Should I Use?

A decision checklist

1. **nth-term test** — does $a_n \to 0$? If not, it diverges. (Always start here.)
2. **Geometric?** form ar^n: converges iff $|r| < 1$, sum $\frac{a}{1-r}$.

3. **p-series?** form $\frac{1}{n^p}$: converges iff $p > 1$.
4. **Factorials or nth powers?** Use the **ratio** (or **root**) test.
5. **Looks like a p-series/geometric?** Use **direct** or **limit comparison**.
6. **Positive, decreasing, easily integrated?** Use the **integral test**.
7. **Alternating $(-1)^n b_n$?** Use the **AST**; then check $\sum |a_n|$ for absolute vs. conditional.

2.10 Going Deeper: Advanced Series

Building new series from known ones (substitution)

Substitute into a known Maclaurin series instead of computing derivatives. Replace x by any expression:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}, \qquad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} \ (|x| < 1).$$

Building by differentiating / integrating

Integrate $\dfrac{1}{1+x^2} = \sum (-1)^n x^{2n}$ term by term to get $\arctan x$:

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \ (|x| \le 1).$$

Similarly integrating $\frac{1}{1-x}$ gives $-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$.

Telescoping series

Use partial fractions so most terms cancel:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right).$$

$S_N = 1 - \frac{1}{N+1} \to 1$, so the sum is $\boxed{1}$. The value comes straight from the partial-sum limit, not a named test.

Using a series to evaluate a tricky limit

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) - x}{x^3} = \lim_{x \to 0} \left(-\frac{1}{6} + \frac{x^2}{120} - \cdots \right) = -\frac{1}{6}.$$

Substituting the Maclaurin series makes the $\frac{0}{0}$ form transparent.

Using a series to evaluate a non-elementary integral

$\int e^{-x^2}\,dx$ has no elementary antiderivative, but integrate the series term by term:

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \cdots$$

The result is an alternating series, so its error bound is the first omitted term.

Endpoint analysis: the four possibilities

After finding R by the ratio test, each endpoint is tested separately and independently. A single power series can produce any of $(c-R, c+R)$, $[c-R, c+R)$, $(c-R, c+R]$, or $[c-R, c+R]$. Typical tools at an endpoint: the p-series test, the nth-term test, or the alternating series test. Never assume symmetry.

Lagrange bound to guarantee accuracy

How many terms of the $\sin x$ series approximate $\sin(1)$ within 10^{-4}? All derivatives of sin are bounded by $M = 1$, so with $c = 0$, $x = 1$:

$$|R_n(1)| \le \frac{1}{(n+1)!}.$$

Since $\frac{1}{7!} = \frac{1}{5040} \approx 0.0002$ and $\frac{1}{8!} = \frac{1}{40320} \approx 0.0000248 < 10^{-4}$, taking $n = 7$ (through the x^7 term) guarantees the accuracy.

Big picture: Nearly every hard series problem reduces to a *known* one — a geometric series, a p-series, or one of the four Maclaurin series — reached by algebra, substitution, differentiation, or integration. Recognize the skeleton, then transform.

3 Parametric, Polar and Vector-Valued Functions

Everything you need for AP Calculus BC, Topic 11

3.1 Parametric Equations & Their Derivatives

Parametric curves and slope

A **parametric curve** is given by $x = f(t)$ and $y = g(t)$, where the parameter t (often time) traces out the points (x, y). To find the slope of the tangent line, differentiate each coordinate with respect to t and divide:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad \text{(provided } dx/dt \neq 0\text{)}.$$

The **second derivative** is the t-derivative of the first derivative, again divided by dx/dt:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}.$$

Graphing by eliminating the parameter

Sketch $x = 3\cos t,\ y = 2\sin t$, $0 \leq t \leq 2\pi$. Since $\cos t = \frac{x}{3}$ and $\sin t = \frac{y}{2}$, use $\cos^2 t + \sin^2 t = 1$:

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse with x-radius 3 and y-radius 2, traced counterclockwise starting at $(3, 0)$.

First and second derivatives

Let $x = 2t + 1,\ y = t^2 - t$. Then $\dfrac{dx}{dt} = 2$ and $\dfrac{dy}{dt} = 2t - 1$, so

$$\frac{dy}{dx} = \frac{2t-1}{2} = t - \tfrac{1}{2}.$$

For the second derivative, differentiate $\frac{dy}{dx}$ in t (getting 1) and divide by $\frac{dx}{dt} = 2$:

$$\frac{d^2y}{dx^2} = \frac{1}{2} > 0 \quad \Rightarrow \quad \text{the curve is concave up everywhere.}$$

Tip: Do *not* divide $\frac{dy}{dx}$ by x when finding $\frac{d^2y}{dx^2}$. Always differentiate the slope with respect to t, then divide by $\frac{dx}{dt}$.

3.2 Arc Length of a Parametric Curve

Arc length

For a smooth curve $x = f(t),\ y = g(t)$ traced once for $a \leq t \leq b$, the arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The square root is the *speed*; arc length accumulates distance along the curve.

A computable arc length

Find the length of $x = t,\ y = \frac{2}{3}t^{3/2}$ for $0 \le t \le 3$. Here $\frac{dx}{dt} = 1$ and $\frac{dy}{dt} = t^{1/2}$, so

$$L = \int_0^3 \sqrt{1+t}\,dt = \left[\tfrac{2}{3}(1+t)^{3/2}\right]_0^3 = \tfrac{2}{3}(8-1) = \frac{14}{3}.$$

Tip: If the curve is retraced or the parameter reverses, arc length still adds up (it is never negative), but you must integrate over an interval that traces the curve *once*.

3.3 Vector-Valued Functions & Motion

Position, velocity, acceleration, speed

A moving particle has **position** $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then

$$\text{velocity } \mathbf{v}(t) = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle, \qquad \text{acceleration } \mathbf{a}(t) = \mathbf{r}''(t) = \langle x''(t), y''(t)\rangle.$$

$$\text{speed} = |\mathbf{v}(t)| = \sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}, \qquad \text{total distance} = \int_a^b |\mathbf{v}(t)|\,dt.$$

Velocity, acceleration, and speed

Let $\mathbf{r}(t) = \langle t^3 - 3t,\ t^2\rangle$. Then $\mathbf{v}(t) = \langle 3t^2 - 3,\ 2t\rangle$ and $\mathbf{a}(t) = \langle 6t,\ 2\rangle$. At $t = 2$:

$$\mathbf{v}(2) = \langle 9, 4\rangle, \quad \mathbf{a}(2) = \langle 12, 2\rangle, \quad \text{speed} = |\mathbf{v}(2)| = \sqrt{9^2 + 4^2} = \sqrt{97}.$$

Total distance traveled

A particle has position $\mathbf{r}(t) = \left\langle \frac{t^3}{3} - t,\ t^2\right\rangle$. Then $x' = t^2 - 1,\ y' = 2t$, and the integrand is a perfect square:

$$(t^2-1)^2 + (2t)^2 = t^4 + 2t^2 + 1 = (t^2+1)^2 \ \Rightarrow\ |\mathbf{v}(t)| = t^2 + 1.$$

Total distance on $0 \le t \le 3$:

$$\int_0^3 (t^2+1)\,dt = \left[\tfrac{t^3}{3} + t\right]_0^3 = 9 + 3 = 12.$$

Tip: *Total distance* $= \int |\mathbf{v}|\, dt$ (always ≥ 0). *Displacement* $= \mathbf{r}(b) - \mathbf{r}(a)$ is a vector; its length can be much smaller if the particle doubles back.

3.4 Polar Coordinates & Graphs

Polar↔rectangular and a curve catalog

A polar point (r, θ) satisfies

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}.$$

Common polar graphs:

- $r = a$: circle of radius a centered at the origin.
- $r = a \pm b\cos\theta$ or $a \pm b\sin\theta$: **cardioid** if $a = b$, otherwise a **limaçon**.
- $r = a\cos(n\theta)$ or $a\sin(n\theta)$: a **rose** with n petals if n is odd, $2n$ petals if n is even.

Converting a point and an equation

The point $(r, \theta) = \left(2, \frac{\pi}{3}\right)$ becomes $x = 2\cos\frac{\pi}{3} = 1,\ y = 2\sin\frac{\pi}{3} = \sqrt{3}$, i.e. $(1, \sqrt{3})$. Convert $r = 4\cos\theta$: multiply by r to get $r^2 = 4r\cos\theta$, so $x^2 + y^2 = 4x$, i.e. $(x-2)^2 + y^2 = 4$ — a circle of radius 2 centered at $(2, 0)$.

Cardioid $r = 1 + \cos\theta$: maximum $r = 2$ at $\theta = 0$, and $r = 0$ (the pole) at $\theta = \pi$.

Tip: To graph by hand, make a table of $\theta = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots$ and plot (r, θ). Watch for $r < 0$: that point is plotted in the *opposite* direction, at angle $\theta + \pi$.

3.5 Slope & Area in Polar

Slope of a polar curve

For $r = f(\theta)$, write $x = r\cos\theta,\ y = r\sin\theta$ and use the parametric rule:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}.$$

Slope at a point

For $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$: $r = 1$ and $r' = -\sin\theta = -1$. Then

$$\frac{dy}{d\theta} = r'\sin\theta + r\cos\theta = (-1)(1) + (1)(0) = -1, \quad \frac{dx}{d\theta} = r'\cos\theta - r\sin\theta = (-1)(0) - (1)(1) = -1,$$

so $\dfrac{dy}{dx} = \dfrac{-1}{-1} = 1$.

Area in polar

The area swept by $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

The area **between** an outer curve $r = R(\theta)$ and an inner curve $r = \rho(\theta)$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(R^2 - \rho^2\right) d\theta.$$

Area enclosed by a cardioid

Area inside $r = 1 + \cos\theta$:

$$A = \frac{1}{2}\int_0^{2\pi} (1 + \cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{2\pi} \left(1 + 2\cos\theta + \cos^2\theta\right) d\theta = \frac{1}{2}\left(2\pi + 0 + \pi\right) = \frac{3\pi}{2}.$$

(We used $\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$.)

Four-petaled rose $r = 2\cos 2\theta$ ($n = 2$ is even, so $2n = 4$ petals).

Tip: Choosing the limits α, β is the hard part. For one petal of a rose, integrate between two consecutive θ-values where $r = 0$.

3.6 Arc Length of a Polar Curve

Polar arc length

For $r = f(\theta)$ on $\alpha \le \theta \le \beta$,

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Arc length of a circle

For $r = 3$ (a circle of radius 3), $\frac{dr}{d\theta} = 0$, so on $0 \le \theta \le 2\pi$:

$$L = \int_0^{2\pi} \sqrt{3^2 + 0^2}\, d\theta = \int_0^{2\pi} 3\, d\theta = 6\pi,$$

which is exactly the circumference $2\pi(3)$.

Tip: Simplify $r^2 + (r')^2$ *before* integrating. Trig identities (especially $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$) often collapse the square root, as in the cardioid below.

3.7 Going Deeper: Advanced Topics

Motion along a curve: speed and total distance

A particle moves with position $\mathbf{r}(t) = \langle t - \sin t,\ 1 - \cos t\rangle$ (a cycloid). Then $\mathbf{v}(t) = \langle 1 - \cos t,\ \sin t\rangle$ and

$$|\mathbf{v}| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = 2\left|\sin\tfrac{t}{2}\right|.$$

On one arch $0 \le t \le 2\pi$, $\sin\frac{t}{2} \ge 0$, so the total distance is

$$\int_0^{2\pi} 2\sin\tfrac{t}{2}\, dt = \Big[-4\cos\tfrac{t}{2}\Big]_0^{2\pi} = -4(-1) - (-4)(1) = 8.$$

Polar area between two curves (find the intersection first)

Find the area inside $r = 3\sin\theta$ *and* outside $r = 1 + \sin\theta$. Set them equal:

$$3\sin\theta = 1 + \sin\theta \;\Rightarrow\; \sin\theta = \tfrac{1}{2} \;\Rightarrow\; \theta = \tfrac{\pi}{6},\ \tfrac{5\pi}{6}.$$

Between these angles $3\sin\theta$ is the outer curve, so

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \Big[(3\sin\theta)^2 - (1 + \sin\theta)^2\Big]\, d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \big(3 - 4\cos 2\theta - 2\sin\theta\big)\, d\theta.$$

An antiderivative is $3\theta - 2\sin 2\theta + 2\cos\theta$; evaluating gives $\frac{1}{2}\big(\frac{5\pi}{2} - \frac{\pi}{2}\big) = \pi$.

Tangent lines at the pole

When a polar curve reaches the pole ($r = 0$) at $\theta = \theta_0$, the tangent line there is simply the line $\theta = \theta_0$. For the rose $r = \cos 2\theta$, $r = 0$ when $2\theta = \frac{\pi}{2} + k\pi$, i.e. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. These four lines are the tangents to the petals at the origin.

Converting a harder polar equation

Convert $r = 6\cos\theta + 4\sin\theta$. Multiply by r:

$$r^2 = 6r\cos\theta + 4r\sin\theta \;\Rightarrow\; x^2 + y^2 = 6x + 4y.$$

Complete the square: $(x-3)^2 + (y-2)^2 = 13$, a circle centered at $(3, 2)$ with radius $\sqrt{13}$.

Parametric second derivative & concavity

Let $x = t^2 + 1,\; y = t^3 - 3t$. Then $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2 - 3$, and

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \tfrac{1}{t}\right).$$

Differentiate in t: $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2}\left(1 + \frac{1}{t^2}\right)$. Divide by $\frac{dx}{dt} = 2t$:

$$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}\left(1 + \frac{1}{t^2}\right)}{2t} = \frac{3(t^2+1)}{4t^3}.$$

Since $3(t^2 + 1) > 0$, the sign matches t^3: **concave up for $t > 0$, concave down for $t < 0$.**

Arc length of a cardioid

For $r = 1 + \cos\theta,\; r' = -\sin\theta$, and

$$r^2 + (r')^2 = (1+\cos\theta)^2 + \sin^2\theta = 2 + 2\cos\theta = 4\cos^2\tfrac{\theta}{2}.$$

So $\sqrt{r^2 + (r')^2} = 2\left|\cos\frac{\theta}{2}\right|$, and the full perimeter is

$$L = \int_0^{2\pi} 2\left|\cos\tfrac{\theta}{2}\right|\,d\theta = 8.$$

Big picture: Parametric, vector, and polar problems all reduce to the same toolkit — differentiate each coordinate in the parameter, then combine. Slopes divide by dx/dt (or $dx/d\theta$); lengths and distances integrate a square-root speed; polar areas integrate $\frac{1}{2}r^2$. Set up carefully, keep answers exact, and always ask whether the interval traces the curve once.

4 Limits and Continuity

Everything you need for AP Calculus BC, Topic 1

4.1 Limit Notation & the Intuitive Idea

What a limit means

The statement $\lim_{x \to a} f(x) = L$ means: as x gets arbitrarily close to a (from both sides) but $x \neq a$, the outputs $f(x)$ get arbitrarily close to L. A limit describes where a function is *heading*, not necessarily its value at a. In fact $f(a)$ may be undefined, or different from L.

You can estimate a limit from a **table** (plug in values of x close to a from both sides) or from a **graph** (trace the curve toward $x = a$ from the left and the right).

Estimating from a table

Estimate $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	4.001	4.01	4.1

Both sides approach 4, so the limit is 4 (even though $f(2)$ is undefined).

Tip: The value $f(a)$ is irrelevant to $\lim_{x \to a} f(x)$. Only the behavior *near* a matters.

4.2 Limit Laws & Evaluating Algebraically

Limit laws

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then limits distribute over sums, differences, products, quotients (denominator $\neq 0$), constant multiples, and powers/roots. Consequently, for any polynomial or rational function that is defined at a, you may use **direct substitution**.

Strategy when direct substitution gives $\frac{0}{0}$ (indeterminate):

- **Factor** and cancel the common factor.
- **Rationalize** (multiply by the conjugate) when a square root appears.
- **Combine a complex fraction** into a single fraction, then simplify.

Factoring

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6.$$

Rationalizing (conjugate)

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}=\lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}=\frac{1}{4}.$$

Complex fraction

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2}=\lim_{x\to 2}\frac{-(x-2)}{2x(x-2)}=\lim_{x\to 2}\frac{-1}{2x}=-\frac{1}{4}.$$

Tip: $\frac{0}{0}$ is *indeterminate*, not an answer — it signals you must simplify. But $\frac{\text{(nonzero)}}{0}$ signals an *infinite* limit or a vertical asymptote.

4.3 One-Sided Limits & When a Limit Fails to Exist

One-sided limits

$\lim_{x\to a^-} f(x)$ uses only $x<a$ (approach from the left); $\lim_{x\to a^+} f(x)$ uses only $x>a$ (approach from the right). The two-sided limit exists **if and only if** both one-sided limits exist and are *equal*:

$$\lim_{x\to a} f(x)=L \iff \lim_{x\to a^-} f(x)=\lim_{x\to a^+} f(x)=L.$$

A limit **fails to exist** (DNE) when: the left and right limits disagree (a *jump*); the function grows without bound ($\pm\infty$); or the function oscillates without settling (e.g. $\sin(1/x)$ near 0).

Jump discontinuity at $x=2$: $\lim_{x\to 2^-} f=1$, $\lim_{x\to 2^+} f=3$, so $\lim_{x\to 2} f$ DNE.

4.4 Infinite Limits & Limits at Infinity

Vertical vs. horizontal asymptotes

An **infinite limit**, e.g. $\lim_{x\to a} f(x)=+\infty$, means f grows without bound near $x=a$; the line $x=a$ is a **vertical asymptote**. A **limit at infinity**, e.g. $\lim_{x\to\infty} f(x)=L$, describes end behavior; the line $y=L$ is a **horizontal asymptote**.

Rational end behavior $\dfrac{p(x)}{q(x)}$ as $x\to\pm\infty$: compare degrees.

- deg top $<$ deg bottom $\Rightarrow$ limit 0.
- deg top $=$ deg bottom $\Rightarrow$ limit $=$ ratio of leading coefficients.
- deg top $>$ deg bottom $\Rightarrow$ limit $=\pm\infty$ (no horizontal asymptote).

Rational limit at infinity

$$\lim_{x\to\infty}\frac{3x^2-5x}{6x^2+1}=\lim_{x\to\infty}\frac{3-\frac{5}{x}}{6+\frac{1}{x^2}}=\frac{3}{6}=\frac{1}{2}.$$

Radical end behavior (mind the sign)

For $x\to+\infty$, $\sqrt{x^2}=x$, but for $x\to-\infty$, $\sqrt{x^2}=-x=|x|$.

$$\lim_{x\to\infty}\frac{\sqrt{9x^2+1}}{x}=\lim_{x\to\infty}\frac{|x|\sqrt{9+\frac{1}{x^2}}}{x}=3,\qquad \lim_{x\to-\infty}\frac{\sqrt{9x^2+1}}{x}=-3.$$

Horizontal asymptote: $\lim\limits_{x\to\pm\infty}\dfrac{3x^2}{x^2+1}=3.$

4.5 Continuity at a Point & Types of Discontinuity

The three conditions for continuity at $x=a$

f is continuous at a if and only if all three hold:

(1) $f(a)$ is defined;

(2) $\lim\limits_{x\to a} f(x)$ exists;

(3) $\lim\limits_{x\to a} f(x)=f(a)$.

Types of discontinuity:

- **Removable** (hole): $\lim_{x\to a} f$ exists but $\neq f(a)$ (or $f(a)$ undefined). Can be "patched."
- **Jump:** left and right limits exist but differ.
- **Infinite:** the function blows up (vertical asymptote).

Removable discontinuity: $\lim_{x\to 2} f$ exists but $f(2)$ sits off the curve.

4.6 IVT, Squeeze Theorem & the Special Trig Limit

Intermediate Value Theorem (IVT)

If f is continuous on $[a,b]$ and N is any value between $f(a)$ and $f(b)$, then there exists at least one c in (a,b) with $f(c)=N$. Common use: show a root exists by finding a sign change ($f(a)<0<f(b)$).

IVT root existence

Let $f(x) = x^3 + x - 1$ (continuous). $f(0) = -1 < 0$ and $f(1) = 1 > 0$, so by IVT there is a $c \in (0, 1)$ with $f(c) = 0$.

Squeeze (Sandwich) Theorem

If $g(x) \le f(x) \le h(x)$ near a and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$. Classic use: $\lim_{x\to 0} x^2 \sin(1/x) = 0$ since $-x^2 \le x^2 \sin(1/x) \le x^2$.

The special limit

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \qquad \text{and consequently} \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

Tip: $\dfrac{\sin x}{x} \to 1$ only as $x \to 0$ with the *same* argument on top and bottom. Match them: $\dfrac{\sin(5x)}{x} = 5 \cdot \dfrac{\sin(5x)}{5x} \to 5$.

4.7 Going Deeper: Advanced Limit Ideas

The formal ε–δ definition

$\lim\limits_{x\to a} f(x) = L$ means: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - L| < \varepsilon.$$

In words: no matter how tight a tolerance ε you demand around L, you can find an input window δ around a that keeps $f(x)$ inside that tolerance.

An ε–δ argument

Prove $\lim_{x\to 3}(2x - 1) = 5$. Given $\varepsilon > 0$, we need $|(2x-1) - 5| < \varepsilon$, i.e. $2|x - 3| < \varepsilon$. Choose $\delta = \varepsilon/2$. Then $0 < |x - 3| < \delta$ forces $|(2x - 1) - 5| = 2|x - 3| < 2\delta = \varepsilon$. ■

Piecewise continuity: solve for a parameter

Find k making f continuous everywhere, where

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1. \end{cases}$$

For $x \neq 1$, $f(x) = x + 1$, so $\lim_{x\to 1} f(x) = 2$. Continuity requires $f(1) = 2$, hence $k = 2$.

Matching the argument: $\frac{\sin(\textbf{stuff})}{\textbf{stuff}}$

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)} = \lim_{x\to 0} \frac{\frac{\sin(3x)}{3x}\cdot 3x}{\frac{\sin(5x)}{5x}\cdot 5x} = \frac{1\cdot 3}{1\cdot 5} = \frac{3}{5}.$$

The $\frac{1-\cos x}{x}$ family

Multiply by the conjugate $1 + \cos x$:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x} = 1 \cdot \frac{0}{2} = 0.$$

A related result: $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}.$

$L'Hpital$ preview

If $\lim \frac{f}{g}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and f, g are differentiable, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Example: $\lim_{x\to 0} \frac{\sin x}{x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$. (You will prove this rigorously later; it is a fast check now.)

Limits with the floor function $\lfloor x \rfloor$

$\lfloor x \rfloor$ is the greatest integer $\leq x$, so it jumps at every integer.

$$\lim_{x\to 2^-} \lfloor x \rfloor = 1, \qquad \lim_{x\to 2^+} \lfloor x \rfloor = 2, \qquad \lim_{x\to 2} \lfloor x \rfloor \text{ DNE.}$$

But at a non-integer, e.g. $\lim_{x\to 2.5} \lfloor x \rfloor = 2$ (it is locally constant).

Big picture: Every limit technique answers one question — *does the function settle to a single finite value as x approaches a?* Substitution first; if $\frac{0}{0}$, simplify (factor / conjugate / common denominator / trig identity); if still stuck, L'Hôpital or Squeeze.

5 The Derivative

Definition, differentiability, and the basic rules of differentiation

5.1 Rates of Change and the Tangent Line

Average vs. instantaneous rate of change

The **average rate of change** of f on $[a, b]$ is the slope of the *secant* line through the endpoints:

$$\frac{f(b) - f(a)}{b - a}.$$

The **instantaneous rate of change** at $x = a$ is the slope of the *tangent* line at that point. It is the limit of the average rates as the interval shrinks to a single point, and we call it the **derivative** $f'(a)$.

The tangent line at $x = a$ touches the curve at $\big(a, f(a)\big)$ and has slope $f'(a)$. Its equation is

$$y = f(a) + f'(a)\,(x - a).$$

Worked example: average rate of change

Find the average rate of change of $f(x) = x^2$ on $[1, 3]$.

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = \frac{8}{2} = 4.$$

So a secant line over this interval has slope 4.

Tip: "Average" $\Rightarrow$ secant $\Rightarrow$ two points. "Instantaneous" $\Rightarrow$ tangent $\Rightarrow$ one point (a limit).

5.2 The Limit Definition of the Derivative

Definition

The derivative of f **as a function** is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists. The derivative **at a point** $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The **alternate form** of the derivative at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notations: $f'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}[f(x)]$, and y' all mean the derivative.

Worked example: derivative from the definition

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

Tip: The whole point of algebra in these problems is to cancel the h in the denominator *before* letting $h \to 0$; otherwise you get $\frac{0}{0}$.

5.3 Differentiability and Continuity

When does the derivative fail to exist?

If f is **differentiable** at a, then f is **continuous** at a. The converse is false: a continuous function need not be differentiable. A function fails to be differentiable at a point where its graph has a

- **corner** (slopes from the two sides disagree), e.g. $y = |x|$ at $x = 0$;
- **cusp** (slopes head toward $+\infty$ and $-\infty$), e.g. $y = x^{2/3}$ at $x = 0$;
- **vertical tangent** (slope is infinite), e.g. $y = x^{1/3}$ at $x = 0$;
- **discontinuity** (jump, hole, or asymptote).

Worked example: a corner

$f(x) = |x|$ is continuous at $x = 0$, but the slope from the left is -1 and the slope from the right is $+1$. Since these one-sided derivatives disagree, $f'(0)$ does not exist.

Tip: "Differentiable $\Rightarrow$ continuous" is a one-way street. To *disprove* differentiability, either show a break (not continuous) or show the left- and right-hand slopes differ.

5.4 Power, Constant, Constant-Multiple, and Sum/Difference Rules

Worked example: power and sum rules

$$\frac{d}{dx}\left[3x^4 - 2x^2 + 7\right] = 3(4x^3) - 2(2x) + 0 = 12x^3 - 4x.$$

Rewrite roots and fractions as powers first:

$$\frac{d}{dx}\left[\sqrt{x}\right] = \frac{d}{dx}\left[x^{1/2}\right] = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Tip: Before differentiating, rewrite $\dfrac{1}{x^k}$ as x^{-k} and $\sqrt[m]{x^k}$ as $x^{k/m}$ so the power rule applies directly.

5.5 Derivatives of Exponential, Logarithmic, and Trig Functions

Worked example: combine with the basic rules

$$\frac{d}{dx}\left[3\sin x - 2\cos x + e^x\right] = 3\cos x - 2(-\sin x) + e^x = 3\cos x + 2\sin x + e^x.$$

Tip: The two "co-" functions (cos, cot, csc) each pick up a *minus sign* when differentiated. Watch signs carefully.

5.6 Product Rule, Quotient Rule, and Higher-Order Derivatives

Products and quotients

$$\underbrace{(fg)' = f'g + fg'}_{\text{Product Rule}} \qquad \underbrace{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}_{\text{Quotient Rule}}$$

Higher-order derivatives come from differentiating again: $f'' = (f')'$, $f''' = (f'')'$, and so on. Notation: $f''(x)$, $\dfrac{d^2y}{dx^2}$.

Worked example: product and quotient rules

Product:

$$\frac{d}{dx}\left[x^2 \sin x\right] = (2x)\sin x + x^2 \cos x = 2x \sin x + x^2 \cos x.$$

Quotient:

$$\frac{d}{dx}\left[\frac{\sin x}{x}\right] = \frac{(\cos x)(x) - (\sin x)(1)}{x^2} = \frac{x \cos x - \sin x}{x^2}.$$

5.7 Interpreting f' and f''

The sign of f' tells you whether f is increasing ($f' > 0$) or decreasing ($f' < 0$); where $f' = 0$ and changes sign, f has a local max or min. The sign of f'' tells you concavity ($f'' > 0$ concave up, $f'' < 0$ concave down). Below, $f(x) = x^2 - 1$ (blue) has a minimum exactly where its derivative $f'(x) = 2x$ (red) crosses zero.

Tip: For a quotient, some students remember "low d-high minus high d-low, over low squared." Never divide the two derivatives—always use the full formula.

5.8 Going Deeper: Advanced Derivative Ideas

1. Derivative of $\sqrt{x}$ from the definition

Multiply by the conjugate to clear the root:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

This agrees with the power rule on $x^{1/2}$.

2. Derivative of $\frac{1}{x}$ from the definition

Combine the fractions in the numerator first:

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-h}{h\,x(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}.$$

Again this matches the power rule on x^{-1}.

3. A harder definition problem: $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \lim_{h\to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\big(\sqrt{x} + \sqrt{x+h}\big)} = -\frac{1}{2x^{3/2}}.$$

Consistent with the power rule on $x^{-1/2}$: $-\frac{1}{2}x^{-3/2}$.

4. One-sided derivatives and a piecewise function

The **one-sided derivatives** at a are

$$f'_-(a) = \lim_{h\to 0^-} \frac{f(a+h) - f(a)}{h}, \qquad f'_+(a) = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h}.$$

f is differentiable at a exactly when both exist *and* are equal. This is how you solve for parameters.

4b. Solving for parameters

Find a, b so that

$$f(x) = \begin{cases} x^2, & x \le 1, \\ ax + b, & x > 1, \end{cases} \qquad \text{is differentiable at } x = 1.$$

Continuity at $x = 1$: the pieces must match, $1^2 = a(1) + b$, so $a + b = 1$.
Matching slopes at $x = 1$: derivative of x^2 is $2x$, giving 2 at $x = 1$; derivative of $ax + b$ is a.
So $a = 2$, and then $b = 1 - a = -1$.
Thus $a = 2,\ b = -1$.

5. Proving the Product Rule

Add and subtract $f(x+h)g(x)$ in the numerator:

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)\big[g(x+h) - g(x)\big] + g(x)\big[f(x+h) - f(x)\big]}{h} \\
&= \lim_{h\to 0} f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= f(x)g'(x) + g(x)f'(x).
\end{aligned}$$

(We used that f is continuous, so $f(x+h) \to f(x)$.)

6. Proving the Quotient Rule

Let $Q = \dfrac{f}{g}$, so $f = Qg$. Differentiate both sides with the product rule: $f' = Q'g + Qg'$. Solve for Q' and substitute $Q = \dfrac{f}{g}$:

$$Q' = \frac{f' - Qg'}{g} = \frac{f' - \frac{f}{g}g'}{g} = \frac{f'g - fg'}{g^2}.$$

7. The symmetric difference quotient

Calculators often estimate $f'(x)$ with the **symmetric difference quotient**

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}.$$

It uses points on both sides and is usually more accurate. For $f(x) = x^2$ it is even *exact* for every h:

$$\frac{(x+h)^2 - (x-h)^2}{2h} = \frac{4xh}{2h} = 2x.$$

Warning: this quotient can return a finite value even where f' does not exist. For $f(x) = |x|$ at $x = 0$ it gives $\frac{|h|-|-h|}{2h} = 0$, yet $f'(0)$ truly does not exist—so a numerical value alone never proves differentiability.

6 Chain, Implicit and Inverse Differentiation

The chain rule, implicit differentiation, inverse and inverse-trig derivatives, and logarithmic differentiation

6.1 The Chain Rule

The idea

If $y = f(g(x))$ is a composition (an "outside" function of an "inside" function), then

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x).$$

Differentiate the outside function, leaving the inside alone, then multiply by the derivative of the inside. In Leibniz notation with $y = f(u)$ and $u = g(x)$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Basic composition: $y = (3x^2+1)^5$

Outside is u^5, inside is $u = 3x^2+1$.

$$\begin{aligned}\frac{dy}{dx} &= 5(3x^2+1)^4 \cdot \frac{d}{dx}(3x^2+1) \\ &= 5(3x^2+1)^4 \cdot 6x = 30x\,(3x^2+1)^4.\end{aligned}$$

Chain with a trig outside: $y = \sin(\sqrt{x^2+1}\,)$

Two layers: $\sin(\,\cdot\,)$ around $\sqrt{\cdot}$ around x^2+1.

$$\begin{aligned}\frac{dy}{dx} &= \cos\bigl(\sqrt{x^2+1}\bigr) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x \\ &= \frac{x\cos\bigl(\sqrt{x^2+1}\bigr)}{\sqrt{x^2+1}}.\end{aligned}$$

Tip: Work from the *outermost* layer inward and multiply a new factor for each layer. You are never "done" until you have taken the derivative of the innermost expression.

6.2 Combining the chain rule with product and quotient rules

Mixing rules

Real problems layer rules together. Identify the *top-level* structure first (is it a product? a quotient? a composition?), apply that rule, and use the chain rule wherever a composition appears inside.

Product × chain: $y = x^2\sin(3x)$

Top level is a product $x^2 \cdot \sin(3x)$.

$$\begin{aligned}\frac{dy}{dx} &= (2x)\sin(3x) + x^2 \cdot \underbrace{3\cos(3x)}_{\text{chain}} \\ &= 2x\sin(3x) + 3x^2\cos(3x).\end{aligned}$$

Quotient × chain: $y = \dfrac{e^{2x}}{x}$

$$\frac{dy}{dx} = \frac{(2e^{2x}) \cdot x - e^{2x} \cdot 1}{x^2} = \frac{e^{2x}(2x-1)}{x^2}.$$

6.3 Implicit Differentiation

When y is not solved for

When a curve is defined by an equation like $x^2 + y^2 = 25$ (you cannot easily write $y = f(x)$), treat y as a function of x and differentiate **both sides** with respect to x. Every time you differentiate a y, the chain rule forces an extra factor of $\frac{dy}{dx}$. Then solve algebraically for $\frac{dy}{dx}$.

Circle: $x^2 + y^2 = 25$

Differentiate both sides with respect to x:

$$\begin{aligned} 2x + 2y\,\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y}. \end{aligned}$$

At the point $(3, 4)$ the slope is $-\frac{3}{4}$, so the tangent line is

$$y - 4 = -\tfrac{3}{4}\,(x - 3).$$

Folium: $x^3 + y^3 = 6xy$

Differentiate; the right side needs the product rule.

$$\begin{aligned} 3x^2 + 3y^2\,\frac{dy}{dx} &= 6y + 6x\,\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= 6y - 3x^2 \\ \frac{dy}{dx} &= \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}. \end{aligned}$$

Tip: A tangent line is *horizontal* where $\frac{dy}{dx} = 0$ (numerator $= 0$) and *vertical* where $\frac{dy}{dx}$ is undefined (denominator $= 0$).

6.4 Derivatives of Inverse Functions

The reciprocal-slope rule

If f is one-to-one and differentiable, its inverse satisfies

$$(f^{-1})'(a) = \frac{1}{f'\big(f^{-1}(a)\big)}.$$

The slope of f^{-1} at $x = a$ is the reciprocal of the slope of f at the corresponding point $f^{-1}(a)$. To use it: find the input b with $f(b) = a$ (so $f^{-1}(a) = b$), then compute $1/f'(b)$.

$f(x) = x^3 + 2x + 1$, find $(f^{-1})'(1)$

First find b with $f(b) = 1$: testing $b = 0$ gives $f(0) = 1$, so $f^{-1}(1) = 0$. Then

$$f'(x) = 3x^2 + 2, \qquad f'(0) = 2,$$

so $(f^{-1})'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

Tip: You never need a formula for f^{-1} itself. You only need one matched pair (b, a) with $f(b) = a$ and the value $f'(b)$.

6.5 Derivatives of the Inverse Trig Functions

The six formulas

$$\begin{aligned}
\frac{d}{dx}\arcsin x &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx}\arccos x &= -\frac{1}{\sqrt{1-x^2}},\\
\frac{d}{dx}\arctan x &= \frac{1}{1+x^2}, & \frac{d}{dx}\operatorname{arccot} x &= -\frac{1}{1+x^2},\\
\frac{d}{dx}\operatorname{arcsec} x &= \frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx}\operatorname{arccsc} x &= -\frac{1}{|x|\sqrt{x^2-1}}.
\end{aligned}$$

The "co-" partners just carry a minus sign. Combine each with the chain rule for a composite argument.

$y = \arctan(x^2)$

Argument $u = x^2$:

$$\frac{dy}{dx} = \frac{1}{1+(x^2)^2}\cdot 2x = \frac{2x}{1+x^4}.$$

$y = \arcsin(3x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 = \frac{3}{\sqrt{1-9x^2}}.$$

Tip: arcsin and arccos share the radical $\sqrt{1-x^2}$; arctan and arccot share $1+x^2$; the "sec/csc" pair share $|x|\sqrt{x^2-1}$.

6.6 Logarithmic Differentiation

Take a log first

For messy products, quotients, and powers—especially a **variable base raised to a variable power** like x^x—take the natural log of both sides, use log laws to break it apart, then differentiate implicitly:

$$\ln y = (\text{simpler expression}) \implies \frac{y'}{y} = (\cdots) \implies y' = y \cdot (\cdots).$$

Remember $\ln(ab) = \ln a + \ln b$, $\ln \frac{a}{b} = \ln a - \ln b$, $\ln(a^p) = p \ln a$.

Product/quotient: $y = \dfrac{x^2\sqrt{x+1}}{(2x-3)^4}$

$$\begin{aligned} \ln y &= 2\ln x + \tfrac{1}{2}\ln(x+1) - 4\ln(2x-3) \\ \frac{y'}{y} &= \frac{2}{x} + \frac{1}{2(x+1)} - \frac{8}{2x-3} \\ y' &= \frac{x^2\sqrt{x+1}}{(2x-3)^4}\left(\frac{2}{x} + \frac{1}{2(x+1)} - \frac{8}{2x-3}\right). \end{aligned}$$

Variable base and exponent: $y = x^x$

The ordinary power rule and exponential rule both fail here—the base *and* exponent vary.

$$\begin{aligned} \ln y &= x \ln x \\ \frac{y'}{y} &= 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 \\ y' &= x^x(\ln x + 1). \end{aligned}$$

Tip: Any expression of the form (function)$^{\text{(function)}}$ must be handled with logarithms (or by rewriting as $e^{g \ln f}$). You cannot use the power rule when the exponent contains x.

6.7 Second Derivatives via Implicit Differentiation

Differentiate twice

To get $\dfrac{d^2y}{dx^2}$ implicitly: first find $\dfrac{dy}{dx}$, then differentiate that expression again with respect to x (using the quotient/product/chain rules), and **substitute your formula for** $\dfrac{dy}{dx}$ wherever it appears. Finally simplify using the original equation.

$x^2 + y^2 = 1$

From $2x + 2yy' = 0$ we get $y' = -\dfrac{x}{y}$. Differentiate again with the quotient rule:

$$\begin{aligned} y'' &= -\frac{(1)\,y - x\,y'}{y^2} = -\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2} \\ &= -\frac{y + \dfrac{x^2}{y}}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}, \end{aligned}$$

using $x^2 + y^2 = 1$ in the last step.

Tip: Your y'' formula may legally contain y'; always substitute the earlier y' result so the final answer is in terms of x and y only.

6.8 Going Deeper: Advanced Techniques

1. Three-layer nested chain rule

For a triple composition, multiply one derivative factor per layer, outside to inside. For $y = \big(\tan(3x^2)\big)^4$:

$$\frac{dy}{dx} = 4\tan^3(3x^2)\cdot\sec^2(3x^2)\cdot 6x = 24x\tan^3(3x^2)\sec^2(3x^2).$$

And for $y = \sin\big(\cos(\tan x)\big)$:

$$\frac{dy}{dx} = \cos\big(\cos(\tan x)\big)\cdot\big(-\sin(\tan x)\big)\cdot\sec^2 x.$$

2. Deriving an inverse-trig derivative

Let $y = \arctan x$, so $\tan y = x$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Differentiate implicitly:

$$\begin{aligned} \sec^2 y \cdot \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}. \end{aligned}$$

The same trick gives arcsin: from $\sin y = x$, $\cos y\, y' = 1$, so $y' = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$.

3. Implicit second derivative (harder)

For $x^2 - y^2 = 1$: differentiate to get $2x - 2yy' = 0$, so $y' = \dfrac{x}{y}$. Then

$$y'' = \frac{(1)y - x\,y'}{y^2} = \frac{y - x \cdot \frac{x}{y}}{y^2} = \frac{y^2 - x^2}{y^3} = -\frac{1}{y^3},$$

since $x^2 - y^2 = 1 \Rightarrow y^2 - x^2 = -1$.

4. Logarithmic differentiation of $x^{\sin x}$

Variable base *and* variable exponent:

$$\begin{aligned} \ln y &= \sin x\, \ln x \\ \frac{y'}{y} &= \cos x\, \ln x + \sin x \cdot \frac{1}{x} \\ y' &= x^{\sin x}\left(\cos x\, \ln x + \frac{\sin x}{x}\right). \end{aligned}$$

5. A related-rates preview

Implicit differentiation with respect to *time* t is exactly the engine behind related rates. A sphere with $V = \frac{4}{3}\pi r^3$ gives, differentiating both sides in t,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Every quantity that changes gets its own "$\dfrac{d(\cdot)}{dt}$" factor by the chain rule—the same bookkeeping as putting $\dfrac{dy}{dx}$ on each y.

6. Inverse function derivative from a graph/pair

If $f(2) = 5$ and $f'(2) = \frac{1}{3}$, then the point $(2, 5)$ on f becomes $(5, 2)$ on f^{-1}, and

$$(f^{-1})'(5) = \frac{1}{f'(2)} = \frac{1}{1/3} = 3.$$

Geometrically, reflecting the graph across $y = x$ reflects each tangent line, inverting its slope.

7. Generalized power rule and $e^{g \ln f}$

Any power with a constant exponent still uses the chain rule: $\dfrac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} g'(x)$. Any power with a *variable* exponent can be rewritten to avoid logarithmic differentiation:

$$f(x)^{g(x)} = e^{g(x) \ln f(x)}, \qquad \frac{d}{dx} f^g = f^g \left(g' \ln f + g \cdot \frac{f'}{f} \right).$$

This matches the log-differentiation result and is a useful check.

8. Tangent-line practice on an implicit curve

For $x^2 + xy + y^2 = 7$ find the tangent at $(1, 2)$. Differentiating,

$$2x + y + xy' + 2yy' = 0 \;\Rightarrow\; y' = -\frac{2x + y}{x + 2y}.$$

At $(1, 2)$: $y' = -\dfrac{2+2}{1+4} = -\dfrac{4}{5}$, so the tangent line is $y - 2 = -\dfrac{4}{5}(x - 1)$.

7 Contextual Applications of Differentiation

Rates in context, motion, related rates, linear approximation & L'H ^opital

7.1 Interpreting the Derivative in Context

The meaning of $f'(a)$

If $y = f(x)$ describes a real quantity, then $f'(x) = \dfrac{dy}{dx}$ is the **instantaneous rate of change** of y with respect to x.

- **Value at a point:** $f'(a)$ is the rate of change of y *at the instant* $x = a$.
- **Units:** the units of f' are always (units of y) *per* (units of x). If V is in liters and t in seconds, then $\dfrac{dV}{dt}$ is in liters per second.

- **Sign:** $f'(a) > 0$ means y is increasing at $x = a$; $f'(a) < 0$ means decreasing.
- **Second derivative:** $f''(a)$ tells whether the *rate itself* is increasing or decreasing (concavity).

Reading a rate with units

Let $H(t)$ be the temperature (in °F) of a cup of coffee t minutes after it is poured. Suppose $H(10) = 140$ and $H'(10) = -3$.

Interpretation: At $t = 10$ minutes, the coffee is 140°F and its temperature is *decreasing* at a rate of 3 °F per minute. The units of H' are °F/min, and the negative sign means the coffee is cooling.

Estimate: Using the rate, $H(11) \approx H(10) + H'(10)(1) = 140 - 3 = 137$°F.

Tip. To interpret a derivative on the AP exam, always name three things: (1) the *value/rate*, (2) the correct *units*, and (3) *when* (the input value). "At $x = a$, y is increasing/decreasing at a rate of ____ units per ____."

7.2 Straight-Line (Rectilinear) Motion

Position, velocity, speed, acceleration

For a particle moving along a line with position $s(t)$:

- **Velocity** $v(t) = s'(t)$ (has a sign: direction of motion).
- **Speed** $= |v(t)|$ (always ≥ 0; "how fast").
- **Acceleration** $a(t) = v'(t) = s''(t)$.
- **At rest** when $v(t) = 0$; the particle may change direction there.
- **Speeding up** when v and a have the *same* sign; **slowing down** when they have *opposite* signs.
- **Displacement** on $[a, b]$ is $s(b) - s(a)$ (net change). **Total distance** is found by splitting at every time $v = 0$ and adding the absolute changes.

Full motion snapshot

A particle has position $s(t) = t^3 - 6t^2 + 9t$ meters, $t \geq 0$ seconds.

$v(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, so $v = 0$ at $t = 1, 3$. $a(t) = 6t - 12 = 6(t-2)$.

Speeding up / slowing down (compare signs of v and a):

Total distance on $[0, 4]$**:** $s(0) = 0,\ s(1) = 4,\ s(3) = 0,\ s(4) = 4.$

$$\text{distance} = |4 - 0| + |0 - 4| + |4 - 0| = 12 \text{ m}, \qquad \text{displacement} = s(4) - s(0) = 4 \text{ m}.$$

Tip. "Speeding up" is *not* the same as "moving in the positive direction." A particle moving in the negative direction speeds up when its acceleration is *also* negative. Use signs of v and a, never signs of s.

7.3 Related Rates

The three-step setup

Related-rates problems link the rates of change of two or more quantities that vary with time.

1. **Relate:** write an equation connecting the variables (geometry, similar triangles, Pythagorean theorem, area/volume formulas). Eliminate variables you have no rate for.
2. **Differentiate** both sides implicitly with respect to t (chain rule on every variable).
3. **Substitute** the known instantaneous values *last*, then solve for the unknown rate.

Never plug in the moving numbers before differentiating.

Sliding ladder

A 10-ft ladder leans against a wall. The base is pulled away at $\dfrac{dx}{dt} = 2$ ft/s. How fast is the top sliding down when the base is 6 ft from the wall?

Relate: $x^2 + y^2 = 100$. At $x = 6$: $y = \sqrt{100 - 36} = 8$.

Differentiate: $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$.

Substitute: $2(6)(2) + 2(8)\dfrac{dy}{dt} = 0 \Rightarrow 24 + 16\dfrac{dy}{dt} = 0 \Rightarrow \dfrac{dy}{dt} = -\dfrac{3}{2}$ ft/s.

The top is sliding *down* at 1.5 ft/s.

Tip. A negative answer for a rate is meaningful: it says the quantity is *decreasing*. Report the sign and the units, and answer the question that was asked ("sliding down at 1.5 ft/s").

7.4 Linear Approximation & Differentials

Tangent-line approximation

Near $x = a$ a differentiable function is close to its tangent line. The **linearization** of f at a is

$$L(x) = f(a) + f'(a)(x - a), \qquad f(x) \approx L(x) \text{ for } x \text{ near } a.$$

Differentials: for a small change dx, the estimated change in y is $dy = f'(x)\,dx$, so $\Delta y \approx dy$.

Over/under-estimate (use concavity):

- If $f'' > 0$ on the interval (concave up), the tangent line lies *below* the curve $\Rightarrow L(x)$ is an **under**estimate.
- If $f'' < 0$ (concave down), the tangent line lies *above* the curve $\Rightarrow L(x)$ is an **over**estimate.

Approximating $\sqrt{9.1}$ (and is it over or under?)

Let $f(x) = \sqrt{x}$, $a = 9$. Then $f(9) = 3$ and $f'(x) = \dfrac{1}{2\sqrt{x}}$, so $f'(9) = \dfrac{1}{6}$.

$$L(x) = 3 + \tfrac{1}{6}(x - 9) \Rightarrow \sqrt{9.1} \approx L(9.1) = 3 + \tfrac{1}{6}(0.1) = 3.0166\overline{6}.$$

Since $f''(x) = -\dfrac{1}{4}x^{-3/2} < 0$, f is concave down, so the tangent line sits above the curve and this is an **overestimate**. (Actual: $\sqrt{9.1} = 3.01662\ldots$)

Tip. To decide over- vs. under-estimate, you only need the *sign of f''* between a and the point you plug in — concave down $\Rightarrow$ over, concave up $\Rightarrow$ under.

7.5 *L'Hpital's* Rule

When and how

If $\displaystyle\lim_{x\to c}\frac{f(x)}{g(x)}$ gives the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the right-hand limit exists (differentiate top and bottom *separately* — not the quotient rule). Repeat if still indeterminate.

Other forms must be rewritten first:

- $0\cdot\infty$: rewrite as $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ by moving a factor to the denominator.

- $\infty - \infty$: combine into a single fraction.
- 1^∞, 0^0, ∞^0: set $y =$ (expression), take ln, find $\lim \ln y$, then exponentiate.

Warning: only apply the rule to a genuine $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Check the form each time.

A $\frac{0}{0}$ limit needing the rule twice

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} \ \left(\tfrac{0}{0}\right) = \lim_{x\to 0} \frac{e^x - 1}{2x} \ \left(\tfrac{0}{0}\right) = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}.$$

Each step was checked to still be $\frac{0}{0}$ before applying the rule again.

A $0 \cdot \infty$ product

$$\lim_{x\to 0^+} x \ln x \ (0 \cdot (-\infty)) = \lim_{x\to 0^+} \frac{\ln x}{1/x} \ \left(\tfrac{-\infty}{\infty}\right) = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0.$$

Tip. L'Hôpital is for *indeterminate* forms only. A form like $\frac{5}{0}$, $\frac{0}{5}$, or $\frac{\infty}{0}$ is *not* indeterminate — evaluate it directly (the answer may be 0, $\pm\infty$, or DNE).

7.6 Going Deeper: Advanced Applications

1. Newton's Method (root-finding by tangent lines)

To approximate a root of $f(x) = 0$, start with a guess x_0 and iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Each step follows the tangent line at x_n down to the x-axis.

Example ($\sqrt{2}$ as a root of $f(x) = x^2 - 2$): $f'(x) = 2x$. With $x_0 = 1.5$,

$$x_1 = 1.5 - \frac{(1.5)^2 - 2}{2(1.5)} = 1.5 - \frac{0.25}{3} = 1.41\overline{6}, \quad x_2 = 1.41\overline{6} - \frac{(1.41\overline{6})^2 - 2}{2(1.41\overline{6})} \approx 1.41422.$$

This converges rapidly ($\sqrt{2} = 1.41421\ldots$). It can fail if $f'(x_n) = 0$ or the guess is poor.

2. Accuracy of the linear approximation

The error $E(x) = f(x) - L(x)$ of a tangent-line estimate is controlled by concavity: its sign is opposite the estimate's bias (concave up $\Rightarrow$ L under $\Rightarrow$ $E > 0$). The farther x is from a and the larger $|f''|$, the bigger the error — E grows roughly like $\frac{1}{2}f''(a)(x - a)^2$. So a good estimate needs x close to a *and* small curvature. Always state "over" or "under" by checking the sign of f'' between a and x.

3. Indeterminate powers via logarithms

For 1^∞, 0^0, ∞^0, let y equal the expression and take ln.

Example (1^∞): $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x$. Let $y=\left(1+\frac{3}{x}\right)^x$, so

$$\ln y = x\ln\left(1+\tfrac{3}{x}\right) = \frac{\ln(1+3/x)}{1/x}\ \left(\tfrac{0}{0}\right) \xrightarrow{\text{L'H}} \frac{\frac{-3/x^2}{1+3/x}}{-1/x^2} = \frac{3}{1+3/x} \to 3.$$

Therefore $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^x = e^3$.

4. Multi-step related rates: draining cone

Water drains from an inverted cone (height 12 ft, top radius 6 ft) at $\dfrac{dV}{dt} = -4$ ft^3/min. Find $\dfrac{dh}{dt}$ when $h=6$ ft.

Relate & reduce: similar triangles give $\dfrac{r}{h}=\dfrac{6}{12}=\dfrac{1}{2}$, so $r=\dfrac{h}{2}$ and

$$V=\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}.$$

Differentiate: $\dfrac{dV}{dt}=\dfrac{\pi h^2}{4}\dfrac{dh}{dt}$. **Substitute** $h=6$: $-4=\dfrac{\pi(36)}{4}\dfrac{dh}{dt}=9\pi\dfrac{dh}{dt}$, so $\dfrac{dh}{dt}=-\dfrac{4}{9\pi}\approx -0.141$ ft/min (falling).

5. Related rates with an angle

A balloon rises straight up at 3 m/s. An observer stands 50 m away. How fast is the angle of elevation θ increasing when the balloon is 50 m high?

Relate: $\tan\theta=\dfrac{h}{50}$. **Differentiate:** $\sec^2\theta\,\dfrac{d\theta}{dt}=\dfrac{1}{50}\dfrac{dh}{dt}$. At $h=50$, $\theta=45°$ so $\sec^2\theta=2$: $2\dfrac{d\theta}{dt}=\dfrac{1}{50}(3)$, giving $\dfrac{d\theta}{dt}=\dfrac{3}{100}=0.03$ rad/s.

6. Rectilinear motion: the sign-chart method

For a full motion analysis on $[a,b]$:

1. Find $v=s'$ and $a=s''$; solve $v=0$ and $a=0$.

2. Build a number line; mark the sign of v (direction) and of a on each subinterval.
3. Same signs $\Rightarrow$ speeding up; opposite signs $\Rightarrow$ slowing down.
4. For **total distance**, split $[a, b]$ at every time $v = 0$ and add $|s(t_{i+1}) - s(t_i)|$; for **displacement**, just compute $s(b) - s(a)$.

8 Analytical Applications of Differentiation

Using f' and f'' to analyze and sketch functions

8.1 The Mean Value Theorem & Rolle's Theorem

Mean Value Theorem (MVT)

If f is **continuous** on the closed interval $[a, b]$ and **differentiable** on the open interval (a, b), then there exists at least one c in (a, b) with

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

That is, the instantaneous rate at some interior point equals the average rate of change over $[a, b]$: some tangent line is parallel to the secant line.

Rolle's Theorem (special case)

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there is a c in (a, b) with $f'(c) = 0$.

Applying the MVT

Let $f(x) = x^2$ on $[1, 3]$. It is a polynomial, so continuous and differentiable everywhere. The average rate is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4.$$

Set $f'(c) = 2c = 4$, so $c = 2$, which lies in $(1, 3)$. ✓

Applying Rolle's Theorem

Let $f(x) = x^2 - 2x$ on $[0, 2]$. Then $f(0) = 0 = f(2)$, and f is a polynomial, so Rolle applies. Solving $f'(c) = 2c - 2 = 0$ gives $c = 1$ in $(0, 2)$. ✓

Tip: Always *verify the hypotheses first.* For $f(x) = |x|$ on $[-1, 1]$ we have $f(-1) = f(1)$, but f is not differentiable at 0, so Rolle's Theorem does *not* apply.

8.2 Increasing/Decreasing & the First Derivative Test

Monotonicity and local extrema

On an interval where $f'(x) > 0$, f is **increasing**; where $f'(x) < 0$, f is **decreasing**. **First Derivative Test:** at a critical number c (where $f'(c) = 0$ or f' is undefined):

- f' changes $+ \to -$ at $c \Rightarrow$ local **maximum**;
- f' changes $- \to +$ at $c \Rightarrow$ local **minimum**;
- no sign change $\Rightarrow$ neither.

Sign chart for $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$, so critical numbers are $x = -1, 1$.

Local max at $x = -1$ with $f(-1) = 2$; local min at $x = 1$ with $f(1) = -2$.

Tip: Critical numbers only *name candidates.* A sign change of f' is what confirms an extremum. Test one point in each interval.

8.3 Concavity, the Second Derivative Test & Inflection Points

Concavity and inflection

Where $f''(x) > 0$, f is **concave up** ($\smile$); where $f''(x) < 0$, f is **concave down** ($\frown$). An **inflection point** is where concavity changes—so f'' must change sign there. **Second Derivative Test:** if $f'(c) = 0$, then $f''(c) > 0 \Rightarrow$ local min, $f''(c) < 0 \Rightarrow$ local max, and $f''(c) = 0$ is *inconclusive.*

Concavity of $f(x) = x^3 - 3x$

$f''(x) = 6x$. Then $f'' < 0$ for $x < 0$ (concave down) and $f'' > 0$ for $x > 0$ (concave up); the concavity changes at $x = 0$, so $(0, 0)$ is an inflection point. Using the Second Derivative Test at the critical numbers: $f''(-1) = -6 < 0$ (local max) and $f''(1) = 6 > 0$ (local min), matching the First Derivative Test.

$f'' = 0$ does not guarantee inflection

For $f(x) = x^4$, $f''(x) = 12x^2$, which is 0 at $x = 0$ but never changes sign ($f'' \geq 0$). So $x = 0$ is *not* an inflection point—f is concave up throughout.

Tip: An inflection point requires an actual *sign change* of f'', not merely $f'' = 0$.

8.4 The Extreme Value Theorem & Absolute Extrema (Candidates' Test)

Extreme Value Theorem (EVT)

If f is **continuous** on a **closed** interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$. **Candidates' Test:** the absolute extrema occur at (i) critical numbers in (a, b) or (ii) the endpoints. Evaluate f at every candidate and compare.

Candidates' Test for $f(x) = x^2 - 4x + 3$ on $[0, 3]$

$f'(x) = 2x - 4 = 0 \Rightarrow x = 2$ (in the interval). Evaluate all candidates:

$$f(0) = 3, \qquad f(2) = -1, \qquad f(3) = 0.$$

Absolute maximum 3 at $x = 0$; absolute minimum -1 at $x = 2$.

Tip: On a closed interval you compare *numbers*, not signs. Never forget the endpoints—they are candidates too.

8.5 Optimization (Applied Max/Min)

A reliable setup

1. Draw a picture; name variables.
2. Write the quantity to optimize (the *objective*).
3. Use a *constraint* to reduce to one variable; state the domain.
4. Differentiate, find critical numbers, and justify the extremum (1st/2nd derivative test or endpoints).

Largest open-top box from a 12×12 sheet

Cut squares of side x from each corner and fold. The volume is

$$V(x) = x(12-2x)^2, \qquad 0 < x < 6.$$

Then $V'(x) = (12-2x)^2 + x \cdot 2(12-2x)(-2) = (12-2x)(12-6x)$. Setting $V' = 0$: $x = 6$ (gives $V = 0$) or $x = 2$. Since V' changes $+ \to -$ at $x = 2$, it is a maximum:

$$V(2) = 2(12-4)^2 = 2 \cdot 64 = 128 \text{ cubic units.}$$

Tip: Always state the domain from the physical situation, then *justify* that your critical number gives the max or min—do not just assert it.

8.6 Curve Sketching & Connecting f, f', f''

Sketch of $f(x) = x^3 - 3x$

Combining the work above: increasing on $(-\infty, -1)$ and $(1, \infty)$, decreasing on $(-1, 1)$; local max $(-1, 2)$, local min $(1, -2)$; concave down for $x < 0$, up for $x > 0$; inflection $(0, 0)$.

The graphs of f, f', and f'' together

For $f(x) = \frac{1}{3}x^3 - x$ we get $f'(x) = x^2 - 1$ and $f''(x) = 2x$. Notice: f has a local max where f' crosses zero going $+ \to -$ $(x = -1)$; f has an inflection where f'' crosses zero and f' bottoms out $(x = 0)$.

Tip: When reading a graph of f': the *height* of f' tells the slope of f; the *zeros* of f' mark candidate extrema of f; where f' is *increasing*, f is concave up.

8.7 Going Deeper: Advanced Analysis

1. A proof-flavored MVT application

Claim: If $f'(x) = 0$ for every x in an interval I, then f is constant on I. **Why:** Pick any two points $a < b$ in I. By the MVT there is a c with

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0,$$

so $f(b) = f(a)$. Since a, b were arbitrary, f takes one value throughout I. This is the theorem behind "antiderivatives differ by a constant."

2. MVT as an inequality machine

Show $|\sin a - \sin b| \le |a - b|$ for all a, b. Apply the MVT to $g(x) = \sin x$ on $[a, b]$: there is a c with

$$\frac{\sin b - \sin a}{b - a} = \cos c.$$

Taking absolute values, $\dfrac{|\sin b - \sin a|}{|b - a|} = |\cos c| \le 1$, so $|\sin a - \sin b| \le |a - b|$.

3. Reading f from a graph of f'

Suppose the graph shown is f' (not f), with $f'(x) = x^2 - 1$.

Then f is increasing where $f' > 0$: on $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 1)$. So f has a local max at $x = -1$ and a local min at $x = 1$. Since f' is decreasing for $x < 0$ and increasing for $x > 0$, f is concave down then concave up, with an inflection point at $x = 0$.

4. Inflection from a sign change of f''

For $f(x) = x^4 - 4x^3$, $f''(x) = 12x^2 - 24x = 12x(x - 2)$. Sign of f'': positive for $x < 0$, negative on $(0, 2)$, positive for $x > 2$. Two sign changes give *two* inflection points, at $x = 0$ and $x = 2$. Contrast with $f(x) = x^4$, where $f'' = 12x^2 \ge 0$ never changes sign—no inflection despite $f''(0) = 0$.

5. Why the Candidates' Test works

The EVT guarantees a continuous f attains its max and min on $[a, b]$. Wherever such an extremum occurs at an *interior* point where f is differentiable, Fermat's Theorem forces $f' = 0$ there. So an absolute extremum can only hide at (i) an interior point with $f' = 0$ or f' undefined, or (ii) an endpoint. Checking that finite list of candidates is therefore guaranteed to find the answer.

6. Harder optimization with a constraint

Minimize the surface area of a closed cylinder of fixed volume V. Constraint: $\pi r^2 h = V \Rightarrow h = \dfrac{V}{\pi r^2}$. Objective:

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2V}{r}.$$

Then $S'(r) = 4\pi r - \dfrac{2V}{r^2} = 0 \Rightarrow r^3 = \dfrac{V}{2\pi} \Rightarrow r = \sqrt[3]{\dfrac{V}{2\pi}}$. Since $S'' = 4\pi + \dfrac{4V}{r^3} > 0$, this is a minimum. Substituting back gives $h = 2r$: the most economical can is as tall as it is wide.

7. Optimization by distance-squared

Closest point on $y = \frac{1}{x}$ $(x > 0)$ to the origin. Minimize $D^2 = x^2 + \dfrac{1}{x^2}$ (minimizing D^2 minimizes D). Then $\dfrac{d(D^2)}{dx} = 2x - \dfrac{2}{x^3} = 0 \Rightarrow x^4 = 1 \Rightarrow x = 1$. The point is $(1, 1)$ with distance $\sqrt{2}$.

Big picture: f' controls *slope* (rising/falling and where it turns); f'' controls *bending* (concavity and inflection). Every result in this topic—MVT, the derivative tests, EVT, optimization, curve sketching—is a way of turning information about f' and f'' into conclusions about f.

9 Integration and the Fundamental Theorem

Antiderivatives, Riemann sums, the definite integral, and the FTC

9.1 Antiderivatives & Indefinite Integrals

Definition

F is an **antiderivative** of f if $F'(x) = f(x)$. Because any two antiderivatives differ by a constant, the **indefinite integral** collects them all:

$$\int f(x)\,dx = F(x) + C, \qquad \text{where } F'(x) = f(x).$$

The $+C$ is the *constant of integration*; never omit it on an indefinite integral.

Basic rules (each verified by differentiating the right side):

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int k\,dx = kx + C \qquad \int \sec x \tan x\,dx = \sec x + C$$

Integration is linear: $\displaystyle\int \big[af(x) + bg(x)\big]\,dx = a\int f\,dx + b\int g\,dx$.

Worked: polynomial antiderivative

Find $\displaystyle\int \left(3x^2 - 4x + 5\right) dx$.

$$= 3 \cdot \frac{x^3}{3} - 4 \cdot \frac{x^2}{2} + 5x + C = x^3 - 2x^2 + 5x + C.$$

Check: $\frac{d}{dx}\left(x^3 - 2x^2 + 5x\right) = 3x^2 - 4x + 5$. ✓

Worked: exponential and reciprocal

Find $\displaystyle\int \left(2e^x + \frac{1}{x} - \sqrt{x}\right) dx$. Write $\sqrt{x} = x^{1/2}$:

$$= 2e^x + \ln|x| - \frac{x^{3/2}}{3/2} + C = 2e^x + \ln|x| - \frac{2}{3}x^{3/2} + C.$$

9.2 Riemann Sums & the Trapezoidal Rule

Approximating area with rectangles

Partition $[a, b]$ into n subintervals of width $\Delta x = \dfrac{b-a}{n}$. A **Riemann sum** adds up rectangle areas $f(x_i^*)\,\Delta x$, where the sample point x_i^* is the:

- **left** endpoint (LRAM), **right** endpoint (RRAM), or **midpoint** (MRAM).

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

Trapezoidal Rule. Replace each rectangle top with a slanted line (a trapezoid):

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big].$$

The interior values are doubled; the two endpoints count once.

Over- vs. under-estimate (for a positive function):

- Increasing: LRAM *underestimates*, RRAM *overestimates*.
- Decreasing: LRAM *overestimates*, RRAM *underestimates*.
- Concave up: Trapezoid *overestimates*, Midpoint *underestimates*.

- Concave down: Trapezoid *underestimates*, Midpoint *overestimates*.

Worked: right Riemann sum

Estimate $\displaystyle\int_0^2 x^2\,dx$ with $n = 4$ (RRAM). Here $\Delta x = \frac{2-0}{4} = 0.5$; right endpoints $0.5, 1, 1.5, 2$:

$$0.5\big[(0.5)^2 + (1)^2 + (1.5)^2 + (2)^2\big] = 0.5\big[0.25 + 1 + 2.25 + 4\big] = 0.5(7.5) = 3.75.$$

Since x^2 is increasing, RRAM *overestimates* the exact value $\frac{8}{3} \approx 2.667$.

9.3 The Definite Integral & Its Properties

Definite integral = limit of Riemann sums = signed area

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

It equals the **signed area**: area above the x-axis counts positive, area below counts negative.

Properties of definite integrals:

$$\int_a^a f\,dx = 0 \qquad \int_b^a f\,dx = -\int_a^b f\,dx$$

$$\int_a^b \big[f \pm g\big]\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx \qquad \int_a^b kf\,dx = k\int_a^b f\,dx$$

$$\int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx$$

Worked: definite integral by geometry

$\displaystyle\int_0^3 (2x)\,dx$ is the area of a triangle with base 3 and height $2(3) = 6$: $\frac{1}{2}(3)(6) = 9$. Confirm later with the FTC.

9.4 The Fundamental Theorem of Calculus

FTC Part 1 — accumulation functions

If f is continuous and $g(x) = \int_a^x f(t)\,dt$, then

$$g'(x) = f(x).$$

Differentiation undoes integration. With a variable upper limit $u(x)$, use the **chain rule**:

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\cdot u'(x).$$

FTC Part 2 — evaluation

If F is *any* antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \Big[F(x)\Big]_a^b.$$

Worked: FTC Part 2

$\int_1^3 2x\,dx = \Big[x^2\Big]_1^3 = 3^2 - 1^2 = 9 - 1 = 8.$

Worked: FTC Part 1 with the chain rule

Let $g(x) = \int_0^{x^2} \sin t\,dt$. Then with $u = x^2$, $u' = 2x$:

$$g'(x) = \sin(x^2)\cdot 2x = 2x\sin(x^2).$$

9.5 u-Substitution

Reversing the chain rule

If $u = g(x)$ then $du = g'(x)\,dx$. Substituting turns a hard integral into a basic one:

$$\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du.$$

Indefinite: substitute back to x at the end. **Definite:** either substitute back, *or* change the limits to u-values and skip the back-substitution.

Worked: indefinite u-sub

$\int 2x\,(x^2+1)^3\,dx$. Let $u = x^2+1 \Rightarrow du = 2x\,dx$:

$$\int u^3\,du = \frac{u^4}{4} + C = \frac{(x^2+1)^4}{4} + C.$$

Worked: definite u-sub, changing the limits

$\int_0^2 x\,e^{x^2}\,dx$. Let $u = x^2 \Rightarrow du = 2x\,dx$, so $x\,dx = \frac{1}{2}\,du$. Limits: $x = 0 \to u = 0$, $x = 2 \to u = 4$:

$$\frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}\Big[e^u\Big]_0^4 = \frac{1}{2}\big(e^4 - 1\big).$$

9.6 Average Value & the Net Change Theorem

Average value of a function

The average value of f on $[a,b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

By the Mean Value Theorem for Integrals there is a c in $[a,b]$ with $f(c) = f_{\text{avg}}$.

Net Change Theorem

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

E.g. $\int_a^b v(t)\,dt$ is *displacement*, while $\int_a^b |v(t)|\,dt$ is *total distance.*

Worked: average value

Average value of $f(x) = x^2$ on $[0,3]$:

$$\frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{1}{3}\cdot\frac{27}{3} = 3.$$

Tip. If given a velocity $v(t)$ and a starting position $s(a)$, the position is $s(b) = s(a) + \int_a^b v(t)\,dt$. Displacement can be negative; distance never is.

9.7 Going Deeper: Advanced Integration

FTC Part 1 with variable bounds on BOTH ends

Find $g'(x)$ for $g(x) = \int_x^{x^3} \cos(t^2)\,dt$. Split at any constant and apply the chain rule to each end:

$$g'(x) = \cos\left((x^3)^2\right) \cdot 3x^2 - \cos(x^2) \cdot 1 = 3x^2 \cos(x^6) - \cos(x^2).$$

The lower variable limit contributes with a *minus* sign.

Riemann sum from a table

f is increasing; selected values below. Estimate $\int_0^6 f(x)\,dx$ using a left sum with the given subintervals.

Widths $2, 3, 1$ using left values $3, 7, 12$:

$$2(3) + 3(7) + 1(12) = 6 + 21 + 12 = 39.$$

Because f is increasing, this left sum *underestimates* the true integral.

Definite integral by geometry (signed area)

Evaluate $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$. The graph $y = \sqrt{4 - x^2}$ is the upper half of a circle of radius 2:

$$= \tfrac{1}{2}\pi(2)^2 = 2\pi.$$

Average value application

A particle has velocity $v(t) = 6t - t^2$ m/s on $[0, 6]$. Its average velocity is

$$\frac{1}{6}\int_0^6 (6t - t^2)\,dt = \frac{1}{6}\left[3t^2 - \frac{t^3}{3}\right]_0^6 = \frac{1}{6}\Big(108 - 72\Big) = 6 \text{ m/s}.$$

A limit expressed as a definite integral

Recognize the Riemann sum limit and write it as an integral:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \int_0^1 x^2\,dx = \frac{1}{3}.$$

Here $\Delta x = \frac{1}{n}$ and the sample point $x_i = \frac{i}{n}$ ranges over $[0, 1]$.

Accumulation function analysis

Let $g(x) = \displaystyle\int_0^x f(t)\,dt$ where f is the piecewise-linear graph shown (triangle up, then down). Then:

- g *increases* where $f > 0$ and *decreases* where $f < 0$.
- g has a local *max* where f changes $+ \to -$; the max of g is the total positive area.
- g is concave up where f is increasing ($g'' = f' > 0$).

Net change from a rate

Water flows into a tank at $r(t) = 4 + 2t$ L/min. The amount added from $t = 0$ to $t = 5$ is

$$\int_0^5 (4 + 2t)\,dt = \Big[4t + t^2\Big]_0^5 = 20 + 25 = 45 \text{ liters.}$$

10 Techniques of Integration

Everything you need for AP Calculus BC, Topic 7

10.1 u-Substitution Review

The idea: reverse the chain rule

If you can write an integrand in the form $f(g(x))\,g'(x)$, let $u = g(x)$ so that $du = g'(x)\,dx$. The integral becomes $\displaystyle\int f(u)\,du$. Look for an "inside" function whose derivative also appears (up to a constant factor). For a definite integral, either change the limits to u-values or convert back to x before substituting the original limits.

Trickier substitutions to watch for: when only *part* of x gets absorbed, you may need to solve for the leftover. For $\displaystyle\int x\sqrt{x-1}\,dx$, let $u = x - 1$, so $x = u + 1$ and $dx = du$.

Solving for the leftover factor

$$\int x\sqrt{x-1}\,dx = \int (u+1)\sqrt{u}\,du = \int \left(u^{3/2}+u^{1/2}\right)du$$
$$= \tfrac{2}{5}u^{5/2}+\tfrac{2}{3}u^{3/2}+C = \tfrac{2}{5}(x-1)^{5/2}+\tfrac{2}{3}(x-1)^{3/2}+C.$$

A substitution that clears a denominator

$$\int \frac{\ln x}{x}\,dx, \quad u=\ln x,\ du=\tfrac{1}{x}\,dx \implies \int u\,du = \tfrac{1}{2}u^2+C=\tfrac{1}{2}(\ln x)^2+C.$$

Tip: After substituting, *every* x and dx must disappear. If an x is left over, solve the u-equation for x and substitute it in — don't guess.

10.2 Integration by Parts

The formula and LIATE

Integration by parts reverses the product rule:

$$\int u\,dv = uv - \int v\,du.$$

Choose u to be the factor that gets *simpler* when differentiated. The **LIATE** order tells you which factor to call u (earlier = better choice for u):

Logarithmic $\to$ **I**nverse trig $\to$ **A**lgebraic $\to$ **T**rig $\to$ **E**xponential.

Whatever is left (together with dx) becomes dv.

A full parts calculation

Evaluate $\int xe^x\,dx$. By LIATE, Algebraic beats Exponential, so $u=x$, $dv=e^x\,dx$.

$$\begin{aligned} u &= x &&\Rightarrow && du = dx,\\ dv &= e^x\,dx &&\Rightarrow && v=e^x. \end{aligned}$$
$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Check: $\frac{d}{dx}\left(xe^x-e^x\right) = e^x + xe^x - e^x = xe^x$. ✓

Repeated parts & the tabular method. When u is a polynomial, you apply parts once per degree. The *tabular* method organizes this: differentiate u down to 0 in one column, antidifferentiate dv in the other, and multiply along the diagonals with alternating signs $+, -, +, \ldots$

Tabular method for $\int x^2 \cos x \, dx$

sign	$D\ (u)$	$I\ (dv)$
$+$	x^2	$\cos x$
$-$	$2x$	$\sin x$
$+$	2	$-\cos x$
$-$	0	$-\sin x$

Multiply along diagonals with the signs:

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Cyclic parts. For $\int e^x \sin x \, dx$ neither factor simplifies, but applying parts twice reproduces the original integral, which you solve for algebraically.

Cyclic parts: $\int e^x \sin x \, dx$

Let $I = \int e^x \sin x \, dx$. Take $u = \sin x,\ dv = e^x dx$: $\quad I = e^x \sin x - \int e^x \cos x \, dx$. Apply parts again to $\int e^x \cos x \, dx$ with $u = \cos x$:

$$\begin{aligned} I &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ I &= e^x \sin x - e^x \cos x - I \\ 2I &= e^x (\sin x - \cos x) \ \Rightarrow\ I = \tfrac{1}{2} e^x (\sin x - \cos x) + C. \end{aligned}$$

Tip: For $\int \ln x \, dx$ and $\int \arctan x \, dx$ there is no obvious dv — take $dv = dx$ (so $v = x$) and let u be the whole function. Result: $\int \ln x \, dx = x \ln x - x + C$.

10.3 Rational Functions by Partial Fractions

Decomposing a proper rational function

To integrate $\dfrac{P(x)}{Q(x)}$ with $\deg P < \deg Q$: factor $Q(x)$, then write the fraction as a sum of simpler pieces.

- Each **distinct linear factor** $(ax + b)$ contributes $\dfrac{A}{ax + b}$.

- A **repeated linear factor** $(ax+b)^k$ contributes $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_k}{(ax+b)^k}$.

If $\deg P \geq \deg Q$, do **polynomial long division** first. Clear denominators, then solve for the constants (plug in convenient x-values or match coefficients).

Full partial-fraction integral (distinct linear factors)

Evaluate $\displaystyle\int \frac{5x-4}{x^2-x-2}\,dx$. Factor: $x^2-x-2=(x-2)(x+1)$.

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1} \;\Rightarrow\; 5x-4 = A(x+1)+B(x-2).$$

Let $x=2$: $6=3A \Rightarrow A=2$. Let $x=-1$: $-9=-3B \Rightarrow B=3$. So

$$\int \Big(\frac{2}{x-2} + \frac{3}{x+1}\Big)dx = 2\ln|x-2| + 3\ln|x+1| + C.$$

A repeated linear factor

$$\frac{3x+1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}, \quad 3x+1 = A(x+1)+B.$$

$x=-1:$ $B=-2$. Matching x: $A=3$. Then

$$\int \frac{3x+1}{(x+1)^2}\,dx = 3\ln|x+1| + \frac{2}{x+1} + C.$$

Tip: Always check that the fraction is *proper* ($\deg P < \deg Q$) before decomposing. Otherwise divide first — the quotient integrates as a polynomial and the remainder gets the partial fractions.

10.4 Trigonometric Integrals

Powers of sine and cosine

For $\displaystyle\int \sin^m x \cos^n x\,dx$:

- If a power is **odd**, peel off one factor and convert the rest with $\sin^2 x + \cos^2 x = 1$, then substitute.

- If both powers are **even**, use the power-reduction identities

$$\sin^2 x = \tfrac{1-\cos 2x}{2}, \qquad \cos^2 x = \tfrac{1+\cos 2x}{2}.$$

Odd power — substitute

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx \overset{u=\cos x}{=} -\int (1-u^2)\,du = -\cos x + \tfrac{1}{3}\cos^3 x + C.$$

Even power — power reduction

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Tip: The double-angle identity $\sin 2x = 2\sin x\cos x$ lets you rewrite $\int \sin x\cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x + C$ instantly.

10.5 Trigonometric Substitution (a first look)

Which substitution to use

When a root of a quadratic appears, trade x for a trig function so the Pythagorean identity collapses the root:

$$\sqrt{a^2-x^2}: \ x = a\sin\theta, \qquad \sqrt{a^2+x^2}: \ x = a\tan\theta, \qquad \sqrt{x^2-a^2}: \ x = a\sec\theta.$$

After substituting, integrate in θ, then use a right triangle to convert back to x.

$\int \frac{dx}{\sqrt{9-x^2}}$ with $x = 3\sin\theta$

Then $dx = 3\cos\theta\,d\theta$ and $\sqrt{9-x^2} = 3\cos\theta$, so

$$\int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \arcsin\!\left(\frac{x}{3}\right) + C.$$

Tip: Many $\sqrt{a^2-x^2}$ and $\sqrt{a^2+x^2}$ integrals are faster with the standard arcsin/arctan formulas. Reach for a full trig sub when the root is multiplied by other powers of x.

10.6 Improper Integrals

Rewrite as a limit

An integral is **improper** if a limit of integration is infinite, or if the integrand blows up somewhere on the interval. Handle it by replacing the bad endpoint with a limit:

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, \qquad \int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx \quad \text{(vertical asymptote at } b\text{)}.$$

If the limit is a finite number, the integral **converges**; if it is $\pm\infty$ or does not exist, it **diverges**. Split an integral with two trouble spots into two pieces — *both* must converge.

Full improper integral (infinite limit) — converges

$$\begin{aligned}\int_1^{\infty}\frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t x^{-2}\,dx = \lim_{t\to\infty}\Big[-\frac{1}{x}\Big]_1^t \\ &= \lim_{t\to\infty}\Big(-\frac{1}{t}+1\Big) = 0+1 = 1. \quad (\textbf{converges to } 1)\end{aligned}$$

Unbounded integrand — diverges

$\frac{1}{x}\to\infty$ as $x\to 0^+$, so

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\big[\ln x\big]_t^1 = \lim_{t\to 0^+}(0-\ln t) = +\infty. \quad (\textbf{diverges})$$

***p*-integral test:** $\displaystyle\int_1^{\infty}\frac{dx}{x^p}$ converges $\iff p>1$; $\displaystyle\int_0^1\frac{dx}{x^p}$ converges $\iff p<1$.

10.7 Going Deeper: Advanced Techniques

1. Reduction formulas

Repeated integration by parts produces a formula that lowers a power by a fixed step. For example,

$$\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx.$$

Apply it repeatedly until you reach $\int \sin x\,dx$ or $\int dx$. The same pattern gives reduction formulas for $\cos^n x$, $\sec^n x$, $x^n e^x$, and $(\ln x)^n$.

2. Deriving cyclic parts in general

For $\int e^{ax}\sin(bx)\,dx$, two applications of parts give

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}\big(a\sin bx - b\cos bx\big)}{a^2+b^2} + C.$$

The denominator a^2+b^2 is the fingerprint of a cyclic "solve-for-I" problem.

3. Partial fractions with an irreducible quadratic

An irreducible factor (x^2+px+q) contributes a *linear* numerator $\dfrac{Bx+C}{x^2+px+q}$. Evaluate:

$$\frac{2x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

$x=0:\ A=1$. Matching: $A+B=0 \Rightarrow B=-1$, and $C=2$. So

$$\int \frac{2x+1}{x(x^2+1)}\,dx = \ln|x| - \tfrac{1}{2}\ln(x^2+1) + 2\arctan x + C,$$

using $\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)$ and $\int \frac{1}{x^2+1}dx = \arctan x$.

4. The comparison test for improper integrals

If $0 \le f(x) \le g(x)$ for all $x \ge a$, then:

$$\int_a^\infty g\,dx \text{ converges} \;\Rightarrow\; \int_a^\infty f\,dx \text{ converges}; \qquad \int_a^\infty f\,dx \text{ diverges} \;\Rightarrow\; \int_a^\infty g\,dx \text{ diverges}.$$

Use it when an antiderivative is hard to find but the integrand is squeezed by a p-integral.

5. Comparison test in action

Does $\displaystyle\int_1^\infty \frac{1}{x^2+x}\,dx$ converge? For $x \ge 1$, $x^2+x > x^2 > 0$, so $0 < \frac{1}{x^2+x} < \frac{1}{x^2}$. Since $\int_1^\infty \frac{dx}{x^2}$ converges (a p-integral with $p=2>1$), the smaller integral **converges** by comparison.

6. A full trig substitution with a leftover power

Evaluate $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ with $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$, $\sqrt{4-x^2} = 2\cos\theta$:

$$\begin{aligned}\int \frac{4\sin^2\theta}{2\cos\theta}\,2\cos\theta\,d\theta = \int 4\sin^2\theta\,d\theta = 2\int (1-\cos 2\theta)\,d\theta \\ = 2\theta - \sin 2\theta + C = 2\theta - 2\sin\theta\cos\theta + C \\ = 2\arcsin\left(\frac{x}{2}\right) - \frac{x\sqrt{4-x^2}}{2} + C,\end{aligned}$$

reading $\sin\theta = \frac{x}{2}$, $\cos\theta = \frac{\sqrt{4-x^2}}{2}$ off the reference triangle.

7. Completing the square before substituting

When a quadratic has no rational roots, complete the square to expose a standard form. For $\int \frac{dx}{x^2+2x+5}$, write $x^2+2x+5 = (x+1)^2+4$, then let $u = x+1$:

$$\int \frac{du}{u^2+4} = \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C = \frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C.$$

Big picture: Before integrating, run a quick checklist — (1) does a simple u-sub work? (2) is it a product suited to parts? (3) is it a rational function (partial fractions)? (4) powers of trig? (5) a root of a quadratic (trig sub / complete the square)? Matching the *form* to the *method* is the whole game.

11 Applications of Integration

Everything you need for AP Calculus BC, Topic 8

11.1 Area Between Two Curves

Area formulas

If $f(x) \geq g(x)$ on $[a,b]$, the area of the region between the curves is

$$A = \int_a^b \big[\,\underbrace{f(x)}_{\text{top}} - \underbrace{g(x)}_{\text{bottom}}\,\big]\,dx.$$

If instead the region is bounded left/right by $x = v(y)$ (right) and $x = u(y)$ (left) for $c \leq y \leq d$, integrate in y:

$$A = \int_c^d \Big[\underbrace{v(y)}_{\text{right}} - \underbrace{u(y)}_{\text{left}} \Big]\, dy.$$

Choosing dx vs. dy: integrate in x when the region has a clear top curve and bottom curve; integrate in y when it has a clear right curve and left curve (this often avoids splitting into two integrals). Always find the intersection points first — they are your limits.

Integrating in x

Find the area between $y = 2x$ and $y = x^2$. Intersections: $2x = x^2 \Rightarrow x = 0, 2$. Top is $2x$:

$$A = \int_0^2 (2x - x^2)\, dx = \left[x^2 - \tfrac{x^3}{3}\right]_0^2 = 4 - \tfrac{8}{3} = \tfrac{4}{3}.$$

Integrating in y (avoids splitting)

Find the area bounded by $x = y^2$ and $x = y + 2$. Intersections: $y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1, 2$. Right curve is $x = y + 2$:

$$A = \int_{-1}^2 \left[(y+2) - y^2\right] dy = \left[\tfrac{y^2}{2} + 2y - \tfrac{y^3}{3}\right]_{-1}^2 = \tfrac{10}{3} - \left(-\tfrac{7}{6}\right) = \tfrac{9}{2}.$$

Doing this in x would require two integrals because the bottom boundary changes.

Tip: If curves cross inside $[a, b]$, the region flips top/bottom — split at each crossing and integrate $|\,\text{top} - \text{bottom}\,|$ piece by piece.

11.2 Average Value, Accumulation & Net Change

Average value of a function

The average value of f on $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

By the Mean Value Theorem for Integrals, if f is continuous there is at least one $c \in (a, b)$ with $f(c) = f_{\text{avg}}$.

Average value

Average value of $f(x) = x^2$ on $[0, 3]$:

$$f_{\text{avg}} = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\left[\tfrac{x^3}{3}\right]_0^3 = \frac{1}{3}\cdot 9 = 3.$$

Accumulation & net change

If $F'(t) = r(t)$ is a rate of change, the net change of F over $[a, b]$ is

$$F(b) - F(a) = \int_a^b r(t)\,dt.$$

For a particle with velocity $v(t)$:

$$\textbf{displacement} = \int_a^b v(t)\,dt, \qquad \textbf{total distance} = \int_a^b |v(t)|\,dt.$$

Displacement vs. total distance

A particle has velocity $v(t) = t^2 - 4$ (m/s) on $[0, 3]$. Since $v(t) = 0$ at $t = 2$, with $v < 0$ on $[0, 2]$ and $v > 0$ on $[2, 3]$:

$$\text{displacement} = \int_0^3 (t^2 - 4)\,dt = \left[\tfrac{t^3}{3} - 4t\right]_0^3 = 9 - 12 = -3 \text{ m}.$$

$$\text{total distance} = \int_0^2 -(t^2 - 4)\,dt + \int_2^3 (t^2 - 4)\,dt = \tfrac{16}{3} + \tfrac{7}{3} = \tfrac{23}{3} \text{ m}.$$

Tip: Position at time b is $s(b) = s(a) + \int_a^b v(t)\,dt$. Speed is $|v(t)|$; total distance integrates *speed*, so you must break the integral at every sign change of v.

11.3 Volumes of Revolution: Disks & Washers

Disk and washer method (slices ⊥ to the axis)

Revolving a region about a horizontal axis and slicing perpendicular to it gives circular cross-sections.

$$\textbf{Disk: } V = \pi\int_a^b \big[R(x)\big]^2\,dx, \qquad \textbf{Washer: } V = \pi\int_a^b \Big(\big[R(x)\big]^2 - \big[r(x)\big]^2\Big)\,dx,$$

where R is the outer radius (far edge from axis) and r the inner radius (near edge). For revolution about a vertical axis, integrate in y with radii measured horizontally.

A representative disk of radius $R = \sqrt{x}$ generated by revolving the shaded region about the x-axis.

Disk

Region under $y = \sqrt{x}$ from $x = 0$ to 4, revolved about the x-axis. Radius $R = \sqrt{x}$:

$$V = \pi \int_0^4 (\sqrt{x})^2 \, dx = \pi \int_0^4 x \, dx = \pi \left[\tfrac{x^2}{2} \right]_0^4 = 8\pi.$$

Washer

Region between $y = 2x$ (top) and $y = x^2$ (bottom) on $[0, 2]$, revolved about the x-axis. Outer $R = 2x$, inner $r = x^2$:

$$V = \pi \int_0^2 \left[(2x)^2 - (x^2)^2 \right] dx = \pi \int_0^2 (4x^2 - x^4) \, dx = \pi \left[\tfrac{4x^3}{3} - \tfrac{x^5}{5} \right]_0^2 = \tfrac{64\pi}{15}.$$

Tip: When revolving about a line $y = k$ (not the x-axis), each radius is a *distance*: $R = |\,\text{curve} - k\,|$. Square the distance, not the raw function value.

11.4 Volumes of Revolution: Cylindrical Shells

Shell method (slices $\parallel$ to the axis)

Slicing parallel to the axis of revolution produces thin cylindrical shells:

$$V = 2\pi \int_a^b (\text{radius})(\text{height}) \, dx.$$

For revolution about the y-axis with a region described by x: radius $= x$, height $=$ top $-$ bottom. For revolution about the x-axis, integrate in y: radius $= y$, height $=$ right $-$ left.

Shells about the y-axis

Region under $y = \sqrt{x}$ on $[0, 4]$, revolved about the y-axis. Shell radius $= x$, height $= \sqrt{x}$:

$$V = 2\pi \int_0^4 x\sqrt{x} \, dx = 2\pi \int_0^4 x^{3/2} \, dx = 2\pi \left[\tfrac{2}{5} x^{5/2} \right]_0^4 = 2\pi \cdot \tfrac{64}{5} = \tfrac{128\pi}{5}.$$

Tip: *Disks/washers* slice perpendicular to the axis; *shells* slice parallel to it. Use shells when perpendicular slices would force you to solve $y = f(x)$ for x or would create a washer with an awkward inner radius.

11.5 Volumes by Known Cross-Sections

Cross-section method

If every slice perpendicular to the x-axis has area $A(x)$, then

$$V = \int_a^b A(x)\,dx.$$

Common cross-sections built on a segment of length s (usually $s = \text{top} - \text{bottom}$ of the base region):

$$\text{Square: } A = s^2, \quad \text{Semicircle (diameter } s\text{): } A = \tfrac{\pi}{8}s^2, \quad \text{Equilateral } \triangle: \; A = \tfrac{\sqrt{3}}{4}s^2.$$

Base region under $y = \sqrt{x}$; a square cross-section (drawn in perspective) stands on the slice of length $s = \sqrt{x}$.

Square cross-sections

The base is the region under $y = \sqrt{x}$ on $[0, 4]$. Cross-sections $\perp$ to the x-axis are squares with side $s = \sqrt{x}$:

$$V = \int_0^4 (\sqrt{x})^2\,dx = \int_0^4 x\,dx = \left[\tfrac{x^2}{2}\right]_0^4 = 8.$$

Semicircular cross-sections

The base is bounded by $y = x$, $x = 2$, and the x-axis. Cross-sections $\perp$ to the x-axis are semicircles with diameter $s = x$, so $A = \frac{\pi}{8}x^2$:

$$V = \int_0^2 \tfrac{\pi}{8}x^2\,dx = \tfrac{\pi}{8}\left[\tfrac{x^3}{3}\right]_0^2 = \tfrac{\pi}{8} \cdot \tfrac{8}{3} = \tfrac{\pi}{3}.$$

Tip: No π appears for square or triangular cross-sections — only circular/semicircular ones carry a π. Read whether the base segment is the *diameter* or the *radius* of a circular slice; they differ by a factor of 4 in area.

11.6 Arc Length of $y = f(x)$

Arc length formula

The length of the smooth curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx.$$

It comes from summing hypotenuses $\sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dy/dx)^2}\, dx$ of tiny right triangles along the curve.

Arc length that simplifies to a perfect square

Find the length of $y = \frac{1}{3}(x^2+2)^{3/2}$ on $[0, 3]$. Here $y' = x\sqrt{x^2+2}$, so

$$1 + (y')^2 = 1 + x^2(x^2+2) = x^4 + 2x^2 + 1 = (x^2+1)^2.$$

$$L = \int_0^3 \sqrt{(x^2+1)^2}\, dx = \int_0^3 (x^2+1)\, dx = \left[\tfrac{x^3}{3} + x\right]_0^3 = 9 + 3 = 12.$$

Tip: Textbook arc-length problems are engineered so $1 + (y')^2$ becomes a perfect square. If it does not, you may only be asked to *set up* the integral — or to evaluate it numerically on a calculator.

11.7 Going Deeper: Advanced Applications

Surface area of a solid of revolution

Revolving $y = f(x) \geq 0$ on $[a, b]$ about the x-axis produces a surface of area

$$S = \int_a^b 2\pi\, y\, \sqrt{1 + (y')^2}\, dx = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx.$$

About the y-axis (region in the first quadrant), $S = \displaystyle\int 2\pi\, x\, \sqrt{1 + (y')^2}\, dx$. Think "circumference $2\pi r$ times arc-length element."

A surface-area setup

Revolve $y = \sqrt{x}$, $1 \leq x \leq 4$, about the x-axis. With $y' = \frac{1}{2\sqrt{x}}$, $1 + (y')^2 = 1 + \frac{1}{4x} = \frac{4x+1}{4x}$, so

$$S = 2\pi \int_1^4 \sqrt{x} \cdot \sqrt{\tfrac{4x+1}{4x}}\, dx = 2\pi \int_1^4 \tfrac{1}{2}\sqrt{4x+1}\, dx = \pi\left[\tfrac{1}{6}(4x+1)^{3/2}\right]_1^4 = \tfrac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).$$

Work: variable force and Hooke's law

Work done by a variable force $F(x)$ moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx.$$

For a spring, Hooke's law gives $F(x) = kx$, so $W = \int_0^d kx\,dx = \frac{1}{2}kd^2$.

Work: pumping a fluid

To pump fluid out of a tank, slice the fluid into horizontal layers at height y of area $A(y)$ and thickness dy. A layer has weight $\rho g\,A(y)\,dy$ and must be lifted a distance $D(y)$:

$$W = \int_c^d \rho g\,A(y)\,D(y)\,dy,$$

where ρg is the fluid's weight density (for water, $\rho g \approx 9800\ \mathrm{N/m^3}$, or $62.4\ \mathrm{lb/ft^3}$).

Mass from a density function

A thin rod along $[0, 4]$ has linear density $\rho(x) = 3 + 2x$ (kg/m). Its mass is the accumulation of density:

$$m = \int_0^4 (3 + 2x)\,dx = \Big[3x + x^2\Big]_0^4 = 12 + 16 = 28 \text{ kg}.$$

The same idea gives total population from a population-density function, or total rainfall from a rate.

Choosing disks/washers vs. shells wisely

Match the slice to the axis:

- Slice **perpendicular** to the axis $\Rightarrow$ disks/washers; the variable of integration matches the axis direction.
- Slice **parallel** to the axis $\Rightarrow$ shells.

Prefer whichever keeps your radii/heights as *single* functions and avoids solving the boundary curve for the other variable. Revolving a region under $y = f(x)$ about the y-axis is usually one clean shell integral $2\pi \int x f(x)\,dx$, whereas washers would require inverting f.

Cross-sections on a non-trivial base

The base is the region between $y = \sqrt{x}$ and $y = \frac{x}{2}$ (they meet at $x = 0$ and $x = 4$). Cross-sections $\perp$ to the x-axis are squares whose side is the vertical gap $s = \sqrt{x} - \frac{x}{2}$:

$$V = \int_0^4 \left(\sqrt{x} - \tfrac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \tfrac{x^2}{4}\right) dx = \left[\tfrac{x^2}{2} - \tfrac{2}{5}x^{5/2} + \tfrac{x^3}{12}\right]_0^4 = 8 - \tfrac{64}{5} + \tfrac{16}{3} = \tfrac{8}{15}.$$

The side length is the *difference* of the two boundary curves, exactly as in an area problem.

Arc length of a parametric curve (BC preview)

For a curve given by $x = x(t)$, $y = y(t)$ on $[\alpha, \beta]$, the arc length is

$$L = \int_\alpha^\beta \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt.$$

Example: for $x = \cos t$, $y = \sin t$, $\sqrt{(-\sin t)^2 + (\cos t)^2} = 1$, so one full loop $t \in [0, 2\pi]$ has length $\int_0^{2\pi} 1\, dt = 2\pi$ — the circumference of the unit circle, as expected. This unifies with $y = f(x)$: taking $t = x$ recovers $\sqrt{1 + (dy/dx)^2}$.

Big picture: Every application here is the same move — slice the quantity into thin pieces, write the piece as (something) $\cdot\, dx$, and integrate. Area, volume, distance, work, mass, and length are all definite integrals of a rate or a cross-sectional measure.

12 Differential Equations

Everything you need for AP Calculus BC, Topic 9

12.1 Verifying Solutions & Modeling

What a differential equation is

A **differential equation** (DE) is an equation involving an unknown function and its derivatives, such as $\dfrac{dy}{dx} = 2y$ or $\dfrac{dP}{dt} = kP$. A **solution** is a function $y = f(x)$ that makes the equation true when substituted in. The **general solution** contains an arbitrary constant C (a whole family of curves); a **particular solution** is pinned down by an **initial condition** like $y(0) = 3$.

To **verify** a proposed solution, differentiate it and substitute both y and its derivative into the DE; check that the two sides are identical.

Verifying a solution

Show $y = 3e^{2x}$ solves $\dfrac{dy}{dx} = 2y$ with $y(0) = 3$.

$$\frac{dy}{dx} = 3 \cdot 2e^{2x} = 6e^{2x}, \qquad 2y = 2 \cdot 3e^{2x} = 6e^{2x}.$$

The two sides agree, and $y(0) = 3e^0 = 3$. ✓

Modeling. Translate "rate of change" language directly into a derivative. "The rate of change of P is proportional to P" becomes $\dfrac{dP}{dt} = kP$. "The temperature changes at a rate proportional to the difference from the room temperature 70°" becomes $\dfrac{dT}{dt} = -k(T - 70)$.

Tip: The order of a DE is the highest derivative it contains. The number of arbitrary constants in the general solution equals the order (first order $\Rightarrow$ one constant C).

12.2 Slope Fields

Reading a slope field

A first-order DE $\dfrac{dy}{dx} = F(x, y)$ gives the *slope* of the solution curve at every point. A **slope field** draws a short segment with that slope at a grid of points. To sketch a solution through a given point, start there and "flow" along the segments, always keeping the curve tangent to the nearby marks.

Slope field for $\dfrac{dy}{dx} = x$ *with the solution through* $(0, -1)$*, namely* $y = \frac{1}{2}x^2 - 1$*.* Segments are horizontal along the y-axis (where $x = 0$) and steepen as $|x|$ grows.

Matching a slope field to a DE. Check special features:

- Where are the segments *horizontal*? Set $F(x, y) = 0$.
- Does the slope depend on x only, y only, or both? (If segments in a row look identical, the slope depends on x only.)
- Is the slope positive or negative in each region?

Slope field for $\dfrac{dy}{dx} = y$ *with the solution* $y = e^x$ *through* $(0, 1)$*.* Now segments are horizontal along the x-axis (where $y = 0$) and depend only on y.

Tip: Equilibrium (constant) solutions appear as *horizontal rows* of flat segments. For $\dfrac{dy}{dx} = y$ the line $y = 0$ is an equilibrium.

12.3 Separable Differential Equations & IVPs

The separation method

A DE is **separable** if it can be written $\dfrac{dy}{dx} = g(x)\,h(y)$. Separate the variables so each side involves only one letter, then integrate:

$$\frac{1}{h(y)}\,dy = g(x)\,dx \quad \Longrightarrow \quad \int \frac{1}{h(y)}\,dy = \int g(x)\,dx.$$

Add a single constant $+C$, then apply the initial condition to solve for C (an **initial-value problem**, IVP).

A full separable IVP

Solve $\dfrac{dy}{dx} = xy$ with $y(0) = 3$.

$$\frac{1}{y}\,dy = x\,dx \implies \int \frac{1}{y}\,dy = \int x\,dx \implies \ln|y| = \frac{x^2}{2} + C.$$

Exponentiate: $y = Ae^{x^2/2}$ where $A = \pm e^C$. Apply $y(0) = 3$: $3 = Ae^0 = A$. Therefore

$$\boxed{y = 3e^{x^2/2}}.$$

Check: $y' = 3e^{x^2/2} \cdot x = x \cdot y$. ✓

Tip: Apply the initial condition as *early* as reasonable, and keep exact forms (ln, e, fractions). Fold $\pm e^C$ into a single constant A once integration is done.

12.4 Exponential Growth & Decay

The growth/decay law

The single most important DE in the course is

$$\frac{dy}{dt} = ky \quad \Longrightarrow \quad y = y_0\,e^{kt},$$

where $y_0 = y(0)$ is the initial amount. $k > 0$ gives **growth**; $k < 0$ gives **decay**.

Half-life ⇒ find k

A radioactive sample has a half-life of 5 years. Find k, then the fraction left after 12 years.

$$\tfrac{1}{2}y_0 = y_0 e^{5k} \implies e^{5k} = \tfrac{1}{2} \implies k = \frac{\ln\frac{1}{2}}{5} = -\frac{\ln 2}{5} \approx -0.1386.$$

After 12 yr: $\dfrac{y(12)}{y_0} = e^{12k} = e^{-12\ln 2/5} = 2^{-12/5} \approx 0.189$, about 18.9%.

Newton's Law of Cooling

$\dfrac{dT}{dt} = -k(T - T_s)$ (with surroundings T_s) is separable. Let $u = T - T_s$, so $\dfrac{du}{dt} = -ku$ and $u = u_0 e^{-kt}$. Therefore

$$T(t) = T_s + (T_0 - T_s)\,e^{-kt}.$$

A 200° cup in a 70° room: $T(t) = 70 + 130\,e^{-kt}$, cooling toward 70° as $t \to \infty$.

Tip: "Doubling time" and "half-life" both give k via $2 = e^{kT_d}$ or $\frac{1}{2} = e^{kT_h}$, so $k = \dfrac{\ln 2}{T_d}$ or $k = -\dfrac{\ln 2}{T_h}$.

12.5 Logistic Growth (BC)

The logistic model

When growth is limited by a **carrying capacity** M, use

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

When P is small, $1 - \frac{P}{M} \approx 1$, so growth is nearly exponential; as $P \to M$ the factor $\to 0$ and growth stalls.

The logistic facts to memorize

For $\dfrac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$:

- **Equilibria:** $P = 0$ and $P = M$ (where $\frac{dP}{dt} = 0$).
- **Long-run:** $\lim_{t\to\infty} P(t) = M$ for any P_0 with $0 < P_0 < M$.
- **Fastest growth** (inflection point of P) occurs at $P = \dfrac{M}{2}$, where the rate equals $\dfrac{kM}{4}$.

- **Solution:** $P(t) = \dfrac{M}{1 + Ae^{-kt}}$, with $A = \dfrac{M - P_0}{P_0}$.

Logistic slope field ($M = 4$) with the S-shaped solution through $(0, 1)$. Segments are flat along $P = 0$ and $P = M$; the curve is steepest as it crosses $P = M/2$.

Tip: The graph of $P(t)$ is concave up below $M/2$ and concave down above it, so the maximum *rate* of change is exactly at the halfway population $P = M/2$.

12.6 Euler's Method (BC)

Following the tangent line in steps

When a DE cannot be solved in closed form, approximate the solution numerically. Starting from (x_0, y_0) with step size h, repeatedly follow the tangent line:

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h \cdot F(x_n, y_n).$$

Each step uses the slope F at the *current* point to predict the next y.

A full Euler's method table

Approximate $y(0.6)$ for $\dfrac{dy}{dx} = x + y$, $y(0) = 1$, using $h = 0.2$.

n	x_n	y_n	$F = x_n + y_n$	$y_{n+1} = y_n + h\,F$
0	0.0	1.000	1.000	$1 + 0.2(1.000) = 1.200$
1	0.2	1.200	1.400	$1.2 + 0.2(1.400) = 1.480$
2	0.4	1.480	1.880	$1.48 + 0.2(1.880) = 1.856$
3	0.6	1.856		

So $y(0.6) \approx \boxed{1.856}$. (The exact solution gives $y(0.6) \approx 1.9436$; Euler underestimates here because the curve is concave up.)

Tip: If the solution is **concave up**, Euler's tangent lines lie below the curve, so the estimate is *too small*; if **concave down**, the estimate is *too large*. Smaller h means smaller error.

12.7 Going Deeper: Advanced Differential Equations

Mixing (tank) problems

Let $y(t)$ be the amount of a substance in a tank. Then

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}),$$

where each rate is (concentration)×(flow). If the tank is well-mixed, the outflow concentration is $\dfrac{y}{V}$ (current amount over current volume).

A mixing problem worked out

A tank holds 100 L with 5 kg of dissolved salt. Brine at 0.2 kg/L enters at 3 L/min; the mixture leaves at 3 L/min (volume stays 100 L).

$$\frac{dy}{dt} = \underbrace{(0.2)(3)}_{\text{in}} - \underbrace{\frac{y}{100}(3)}_{\text{out}} = 0.6 - 0.03y = -0.03\,(y - 20).$$

This is separable with solution $y = 20 + (5 - 20)e^{-0.03t} = 20 - 15e^{-0.03t}$. As $t \to \infty$, $y \to 20$ kg (steady state, where inflow salt balances outflow).

Why the logistic curve is S-shaped (qualitative)

Differentiate $\dfrac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ with respect to t to study $\dfrac{d^2P}{dt^2}$. It is positive for $0 < P < \frac{M}{2}$ (curve concave up, growth accelerating) and negative for $\frac{M}{2} < P < M$ (concave down, growth decelerating). The sign change at $P = \frac{M}{2}$ is the inflection point — the moment of fastest growth — which is why the graph rises, bends, and levels off toward M.

Euler's method error

Euler's method is **first order**: the accumulated (global) error is roughly proportional to the step size h. Halving h roughly halves the error but doubles the number of steps. The local error each step comes from ignoring curvature (y''), so more curvature means faster error growth. This is why the concavity of the true solution tells you the *direction* of the error.

Direction-field analysis of equilibria

An **equilibrium** solution satisfies $F(x, y) = 0$ (a constant solution). Classify it from the sign of $\dfrac{dy}{dx}$ nearby:

- **Stable** (a sink): solutions on both sides move *toward* it. For logistic growth, $P = M$ is stable.
- **Unstable** (a source): nearby solutions move *away*. For logistic growth, $P = 0$ is unstable.

A **phase line** records these arrows and predicts long-term behavior without solving the DE.

Reading a phase line

For $\dfrac{dy}{dt} = (y-1)(y-4)$, equilibria are $y = 1$ and $y = 4$. Testing signs: $\frac{dy}{dt} > 0$ for $y < 1$, < 0 for $1 < y < 4$, > 0 for $y > 4$. So $y = 1$ is **stable** (arrows point in) and $y = 4$ is **unstable** (arrows point out). A solution starting at $y(0) = 2$ decreases toward 1.

Second-order preview

Higher-order DEs involve y''. For example, $y'' = -\omega^2 y$ models simple harmonic motion, with general solution

$$y = A\cos(\omega t) + B\sin(\omega t).$$

Two arbitrary constants A, B appear because the equation is second order; two conditions (e.g. $y(0)$ and $y'(0)$) pin down a particular solution. You will meet these systematically in a later course.

Big picture: Every method answers "how does y change?" Verify by substituting; visualize with a slope field; solve exactly by separating; model growth with $y_0 e^{kt}$, bounded growth with the logistic law; and when no formula exists, approximate with Euler's method — always checking concavity to know which way the estimate leans.

13 Non-Elementary Integration

Feynman's trick, King's and Queen's rules, Glasser's master theorem, and friends

13.1 Integrals With No Antiderivative

Why a new toolkit is needed

Plenty of definite integrals have NO elementary antiderivative — $\int e^{-x^2}\,dx$, $\int \frac{\sin x}{x}\,dx$, $\int \frac{x^a-1}{\ln x}\,dx$ — yet their values over specific intervals are clean: $\sqrt{\pi}$, $\frac{\pi}{2}$, $\ln(a+1)$. The tools in this topic never find an antiderivative. They exploit SYMMETRY of the interval, PARAMETERS you can differentiate in, and SUBSTITUTIONS that map an interval onto itself. This is the calculus that competitions (Putnam especially) and physics actually use.

Feynman's trick: differentiate under the integral

Embed the integral in a family $I(a) = \int f(x,a)\,dx$, compute $I'(a) = \int \partial_a f\,dx$ (usually elementary), integrate back in a, and fix the constant from a value of a where I is known. Example:

$I(a) = \displaystyle\int_0^1 \frac{x^a - 1}{\ln x} dx$ has $I'(a) = \int_0^1 x^a dx = \frac{1}{a+1}$, so $I(a) = \ln(a+1)$ (using $I(0) = 0$). The $\ln x$ that made the integral impossible was erased by one ∂_a.

Worked: Feynman's trick on the Dirichlet integral

Compute $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$. Step 1 — add a damping parameter: $I(t) = \int_0^\infty e^{-tx} \frac{\sin x}{x} dx$, with $I(\infty) = 0$ and the target $I(0)$. Step 2 — $I'(t) = -\int_0^\infty e^{-tx} \sin x \, dx = -\frac{1}{1+t^2}$ (a Laplace transform). Step 3 — integrate: $I(t) = \frac{\pi}{2} - \arctan t$ (constant from $I(\infty) = 0$). Step 4 — $I(0) = \frac{\pi}{2}$. Every step elementary; the answer famously not.

13.2 King's Rule, Queen's Rule, and Reflection

King's rule (the reflection substitution)

$\displaystyle\int_a^b f(x) \, dx = \int_a^b f(a+b-x) \, dx$ — substituting $x \to a+b-x$ flips the interval onto itself. Its power: ADD the two forms. For $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$, the reflection swaps sin and cos, and adding gives $2I = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$, so $I = \frac{\pi}{4}$. A stray factor of x on $[0, \pi]$ becomes $\pi - x$; adding again gives $\int_0^\pi x \, g(\sin x) dx = \frac{\pi}{2} \int_0^\pi g(\sin x) dx$ — how $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$ falls out.

Queen's rule (folding a symmetric interval)

$\displaystyle\int_0^{2a} f(x) \, dx = 2 \int_0^a f(x) \, dx$ if $f(2a-x) = f(x)$, and $= 0$ if $f(2a-x) = -f(x)$. Companion (sometimes "Jack's rule"): $\int_{-a}^a f = 2 \int_0^a f$ for even f, 0 for odd f. Example: $\int_0^\pi \sin^3 x \, dx = 2 \int_0^{\pi/2} \sin^3 x \, dx = 2 \cdot \frac{2}{3} = \frac{4}{3}$; and $\int_0^{2\pi} \cos^3 x \, dx = 0$ by the odd-symmetry about π... check: $\cos^3(2\pi - x) = \cos^3 x$ so use the period instead — $\int_0^{2\pi} \cos^3 = 0$ because $\cos^3$ has zero mean over a full period.

Worked: the log-sine integral solves an equation for itself

Find $I = \int_0^{\pi/2} \ln(\sin x) dx$. Step 1 — King: $I = \int_0^{\pi/2} \ln(\cos x) dx$. Step 2 — add: $2I = \int_0^{\pi/2} \ln(\sin x \cos x) dx = \int_0^{\pi/2} \ln \frac{\sin 2x}{2} dx = \int_0^{\pi/2} \ln(\sin 2x) dx - \frac{\pi}{2} \ln 2$. Step 3 — substitute $u = 2x$ and fold with Queen: $\frac{1}{2} \int_0^\pi \ln(\sin u) du = I$. Step 4 — $2I = I - \frac{\pi}{2} \ln 2$, so $I = -\frac{\pi}{2} \ln 2$.

13.3 Glasser's Master Theorem and the 1/x Reflection

Glasser's master theorem

For any integrable f and constants $a_i > 0$, b_i: $\displaystyle\int_{-\infty}^{\infty} f\left(x - \sum_i \frac{a_i}{x - b_i}\right) dx = \int_{-\infty}^{\infty} f(x)\,dx.$

The simplest case (Cauchy–Schlmilch) is $\int f(x - \frac{1}{x})dx = \int f(x)dx$: the map $u = x - \frac{1}{x}$ covers every real value once on each half-line, and the two pieces of du add up to dx. So $\int_{-\infty}^{\infty} e^{-(x-1/x)^2} dx = \int e^{-x^2} dx = \sqrt{\pi}$ — a hideous integrand, evaluated by recognizing the pattern.

The $x \to 1/x$ reflection on $(0, \infty)$

On $(0, \infty)$ the substitution $x \to \frac{1}{x}$ maps the interval to itself. Two uses: it can show an integral equals its own negative (so it is 0: $\int_0^\infty \frac{\ln x}{1+x^2} dx = 0$), or produce a second form to add: $\int_0^\infty \frac{dx}{1+x^4}$ becomes $\int_0^\infty \frac{x^2 dx}{1+x^4}$, and adding, dividing by x^2, and substituting $u = x - \frac{1}{x}$ yields $\frac{\pi\sqrt{2}}{4}$. When the interval is $(0, \infty)$, try this before anything else.

13.4 More Named Integrals, With Proofs

Dirichlet variants and the tangent bridge

$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{1-\cos x}{x^2} dx = \frac{\pi}{2}$ (one integration by parts each lands on $\int_0^\infty \frac{\sin x}{x} dx$). Substituting $x = \tan\theta$ turns $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx$ into $-2\int_0^{\pi/2} \ln\cos\theta\, d\theta = \pi \ln 2$; and Feynman with the parameter $\arctan(ax)$ gives $\int_0^\infty \frac{\arctan x}{x(1+x^2)} dx = \frac{\pi}{2} \ln 2$.

Fresnel, the reflection formula, Wallis

$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ (the Mellin integral $\int_0^\infty t^{s-1} \sin t\,dt = \Gamma(s) \sin\frac{\pi s}{2}$ at $s = \frac{1}{2}$). $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$, which evaluates $\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}$. Wallis: $\prod_{n\geq 1} \frac{4n^2}{4n^2-1} = \frac{\pi}{2}$, from the reduction formula for $\int_0^{\pi/2} \sin^n x\,dx$ squeezed between consecutive terms. And $\int_0^1 \frac{\ln x}{1-x} dx = -\frac{\pi^2}{6}$ by expanding $\frac{1}{1-x}$ and integrating termwise.

Proof: $\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$

Let $u = \sin^2 x$ and $dv = x^{-2}dx$, so $v = -\frac{1}{x}$. The boundary term $\left[-\frac{\sin^2 x}{x}\right]_0^\infty$ vanishes at both ends (near 0, $\sin^2 x \approx x^2$; at infinity it is bounded over x). What remains is $\int_0^\infty \frac{2 \sin x \cos x}{x} dx = \int_0^\infty \frac{\sin 2x}{x} dx$, and the substitution $t = 2x$ leaves $\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$. ■

Proof: the Wallis product

Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$. Integration by parts gives $I_n = \frac{n-1}{n} I_{n-2}$, so $I_{2n} = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$ and

$I_{2n+1} = \frac{(2n)!!}{(2n+1)!!}$. Since $\sin^{2n+1} \le \sin^{2n} \le \sin^{2n-1}$ on $[0, \frac{\pi}{2}]$, $I_{2n+1} \le I_{2n} \le I_{2n-1}$, and $\frac{I_{2n-1}}{I_{2n+1}} = \frac{2n+1}{2n} \to 1$ squeezes $\frac{I_{2n}}{I_{2n+1}} \to 1$. Writing that ratio out: $\frac{\pi}{2} \cdot \frac{(2n-1)!!\,(2n+1)!!}{((2n)!!)^2} \to 1$, which rearranges to $\prod_{k=1}^{n} \frac{4k^2}{4k^2-1} \to \frac{\pi}{2}$. ■

13.5 The Named Integrals: Frullani, Beta, Gamma, Weierstrass

Frullani

$\displaystyle\int_0^\infty \frac{f(ax)-f(bx)}{x}dx = \big(f(0)-f(\infty)\big)\ln\frac{b}{a}$. With $f = e^{-t}$: $\int_0^\infty \frac{e^{-x}-e^{-3x}}{x}dx = \ln 3$. Proof idea: write $\frac{f(ax)-f(bx)}{x} = -\int_a^b f'(tx)\,dt$ and swap the order of integration — Feynman's idea once more.

Beta and Gamma

$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx = n!$ (integrate by parts n times), and $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$. So $\int_0^1 x^3(1-x)^2dx = \frac{3!\,2!}{6!} = \frac{1}{60}$ instantly, and $\int_0^\infty x^4e^{-x}dx = 24$. Also $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which is the Gaussian integral wearing a different hat.

Weierstrass substitution

$t = \tan\frac{x}{2}$ turns $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2\,dt}{1+t^2}$: ANY rational function of sin and cos becomes a rational function of t, solvable by partial fractions. Example: $\int_0^\pi \frac{dx}{2+\cos x} = \int_0^\infty \frac{2\,dt}{3+t^2} = \frac{\pi}{\sqrt{3}}$.

Worked: series expansion when nothing else fits

Find $\int_0^1 \ln x \ln(1-x)dx$. Step 1 — expand $\ln(1-x) = -\sum \frac{x^n}{n}$. Step 2 — swap sum and integral: $-\sum \frac{1}{n}\int_0^1 x^n \ln x\,dx = \sum \frac{1}{n(n+1)^2}$ (the inner integral is $-\frac{1}{(n+1)^2}$ by parts). Step 3 — partial fractions $\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}$: telescoping gives 1, the last sums to $\zeta(2)-1$. Step 4 — total $2 - \frac{\pi^2}{6}$. Series turn integrals into sums you already know.

14 Course Review

Every key definition, theorem, rule, and formula — Topics 1 through 11

14.1 Limits & Continuity

Use this as a comprehensive review reference: skim a topic's concept boxes, work the example, and note the red tip. Everything you need to review all of AP Calculus BC is here.

What a limit is

$\lim_{x\to a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x close enough to a (but $x \neq a$). The value $f(a)$ is irrelevant to the limit — only the behavior *near* a matters.

Limit laws

If $\lim_{x\to a} f = L$ and $\lim_{x\to a} g = M$ (both finite), then

- $\lim(f \pm g) = L \pm M$
- $\lim(cf) = cL$
- $\lim(fg) = LM$
- $\lim \dfrac{f}{g} = \dfrac{L}{M}$ (if $M \neq 0$)
- $\lim f^n = L^n$
- $\lim \sqrt[n]{f} = \sqrt[n]{L}$ ($L \geq 0$ if n even)

For continuous functions, $\lim_{x\to a} f(x) = f(a)$ (direct substitution).

One-sided, infinite, and at-infinity limits

One-sided: $\lim_{x\to a^-} f$ (from the left) and $\lim_{x\to a^+} f$ (from the right). The two-sided limit exists $\iff$ both one-sided limits exist and are equal.

Infinite limits ($\to \pm\infty$) signal a *vertical asymptote* at $x = a$.

Limits at infinity describe end behavior and *horizontal asymptotes*: for a rational function, compare degrees — if deg(num) $<$ deg(den) the limit is 0; if equal, it is the ratio of leading coefficients; if greater, it is $\pm\infty$ (no horizontal asymptote).

Indeterminate forms

Direct substitution sometimes yields an **indeterminate form** — no immediate answer, more work required:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 0^0, \quad \infty^0, \quad 1^\infty.$$

Resolve with algebra (factor/cancel, rationalize, common denominators), the Squeeze Theorem, or L'Hôpital's Rule. Forms like $\frac{k}{0}$ ($k \neq 0$) are *not* indeterminate — they give $\pm\infty$.

Continuity & the Intermediate Value Theorem

f is **continuous at** a if all three hold: (1) $f(a)$ is defined, (2) $\lim_{x\to a} f(x)$ exists, and (3) $\lim_{x\to a} f(x) = f(a)$.

IVT: If f is continuous on $[a,b]$ and N is any value between $f(a)$ and $f(b)$, then there exists $c \in (a,b)$ with $f(c) = N$. (Guarantees a root when $f(a), f(b)$ have opposite signs.)

Squeeze Theorem and a key special limit

Squeeze: If $g(x) \le f(x) \le h(x)$ near a and $\lim_{x\to a} g = \lim_{x\to a} h = L$, then $\lim_{x\to a} f = L$.

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1} \qquad \text{and} \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0.$$

Resolving a $0/0$ form by factoring

$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x+3) = 6.$$

Substitution gives $\frac{0}{0}$; cancel the common factor first, then substitute.

Tip: Always try direct substitution first. If you get a number, that's the answer. If you get $\frac{0}{0}$, do algebra. If you get $\frac{k}{0}$ with $k \neq 0$, it's an infinite limit (check signs from each side).

14.2 The Derivative

Limit definition of the derivative

The derivative is the instantaneous rate of change / slope of the tangent line:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{or} \qquad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

Notations: $f'(x)$, $\dfrac{dy}{dx}$, y', $D_x f$, $\dfrac{d}{dx} f$.

Differentiability vs. continuity

Differentiable $\Rightarrow$ continuous, but **not** conversely. A function fails to be differentiable at *corners* (e.g. $|x|$ at 0), *cusps*, *vertical tangents* (infinite slope), and any *discontinuity*.

Basic differentiation rules

- **Constant:** $\dfrac{d}{dx}[c] = 0$
- **Power:** $\dfrac{d}{dx}[x^n] = nx^{n-1}$
- **Constant multiple:** $(cf)' = cf'$
- **Sum:** $(f \pm g)' = f' \pm g'$
- **Product:** $(fg)' = f'g + fg'$
- **Quotient:** $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$

Derivatives of the elementary functions

- $\dfrac{d}{dx}e^x = e^x$
- $\dfrac{d}{dx}a^x = a^x \ln a$
- $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$
- $\dfrac{d}{dx}\log_a x = \dfrac{1}{x \ln a}$
- $\dfrac{d}{dx}\sin x = \cos x$
- $\dfrac{d}{dx}\cos x = -\sin x$
- $\dfrac{d}{dx}\tan x = \sec^2 x$
- $\dfrac{d}{dx}\cot x = -\csc^2 x$
- $\dfrac{d}{dx}\sec x = \sec x \tan x$
- $\dfrac{d}{dx}\csc x = -\csc x \cot x$

Product and quotient rules

$$\frac{d}{dx}\left[x^2 \sin x\right] = 2x \sin x + x^2 \cos x, \qquad \frac{d}{dx}\left[\frac{x}{\cos x}\right] = \frac{\cos x + x \sin x}{\cos^2 x}.$$

Tip: For the quotient rule remember "low d-high minus high d-low, over low squared." Sign order matters — $f'g - fg'$, not the reverse.

14.3 Chain, Implicit & Inverse Differentiation

Chain Rule

The derivative of a composition $f(g(x))$ is

$$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big) \cdot g'(x) \qquad \text{(outside derivative} \times \text{inside derivative).}$$

In Leibniz form: $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$. General power rule: $\dfrac{d}{dx}[u^n] = nu^{n-1}u'$.

Implicit differentiation

When y is defined implicitly by an equation, differentiate *both sides* with respect to x, applying the chain rule to every y term (each contributes a factor $\frac{dy}{dx}$), then solve algebraically for $\frac{dy}{dx}$.

Derivative of an inverse function

If f is one-to-one and differentiable with $f(a) = b$ (so $f^{-1}(b) = a$), then

$$\big(f^{-1}\big)'(b) = \frac{1}{f'\big(f^{-1}(b)\big)} = \frac{1}{f'(a)}.$$

The slope of the inverse is the *reciprocal* of the slope of f at the corresponding point.

Inverse trigonometric derivatives

- $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arccot} x = -\dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}\operatorname{arcsec} x = \dfrac{1}{|x|\sqrt{x^2-1}}$
- $\dfrac{d}{dx}\operatorname{arccsc} x = -\dfrac{1}{|x|\sqrt{x^2-1}}$

Logarithmic differentiation

For products of many factors, or a *variable base with a variable exponent* $y = [f(x)]^{g(x)}$: take ln of both sides, use log laws to expand, differentiate implicitly, then multiply back by y.

Chain rule and logarithmic differentiation

$$\frac{d}{dx}\sin(3x^2) = \cos(3x^2)\cdot 6x = 6x\cos(3x^2).$$

For $y = x^x$: $\ln y = x\ln x \Rightarrow \frac{y'}{y} = \ln x + 1 \Rightarrow y' = x^x(\ln x + 1)$.

Tip: You cannot use the power rule on x^x (exponent isn't constant) or on 2^x (base isn't the variable). When both base and exponent vary, logarithmic differentiation is the tool.

14.4 Contextual Applications of Differentiation

Rates of change & units

A derivative $\frac{dy}{dx}$ carries units of "y-units per x-unit." In context, $f'(a)$ is the instantaneous rate of change of f at a; interpret answers with correct units (e.g. "the volume is increasing at 12 cm^3/sec").

Straight-line motion

For position $s(t)$: velocity $v(t) = s'(t)$; acceleration $a(t) = v'(t) = s''(t)$; **speed** $= |v(t)|$.

- Particle moves *right/up* when $v > 0$, *left/down* when $v < 0$, at rest when $v = 0$.
- Speeding up when v and a have the *same* sign; slowing down when *opposite* signs.
- *Displacement* $= \int_a^b v\,dt$; *total distance* $= \int_a^b |v|\,dt$.

Related rates

When several quantities change with time and are linked by an equation: differentiate the relating equation *implicitly with respect to* t, then substitute known values and solve for the unknown rate. Substitute numbers *only after* differentiating.

Linearization & differentials

The tangent-line (linear) approximation near $x = a$:

$$L(x) = f(a) + f'(a)(x - a) \approx f(x).$$

Differentials: $dy = f'(x)\,dx$ estimates the change Δy for a small change $dx = \Delta x$. Overestimates where f is concave down, underestimates where concave up.

$L'Hpital's$ Rule

If $\lim \dfrac{f}{g}$ is of form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (and $g' \neq 0$ nearby), then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

provided the right side exists. For $0 \cdot \infty$ or $\infty - \infty$, rewrite as a quotient first; for $1^\infty, 0^0, \infty^0$, take ln first.

Related rates: expanding circle

Area $A = \pi r^2$, and $\frac{dr}{dt} = 2$ cm/s. Then $\frac{dA}{dt} = 2\pi r\,\frac{dr}{dt}$. When $r = 5$: $\frac{dA}{dt} = 2\pi(5)(2) = 20\pi$ cm^2/s.

Tip: L'Hôpital applies *only* to $\frac{0}{0}$ and $\frac{\infty}{\infty}$. Check the form every time before differentiating — applying it to a determinate form gives a wrong answer.

14.5 Analytical Applications of Differentiation

Mean Value Theorem & Rolle's Theorem

MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ with

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{(instantaneous rate } = \text{ average rate).}$$

Rolle's Theorem is the special case $f(a) = f(b)$: then $f'(c) = 0$ for some c.

Increasing/decreasing & the First Derivative Test

f increases where $f' > 0$, decreases where $f' < 0$. At a critical point ($f' = 0$ or undefined), f has a

- *local max* if f' changes $+ \to -$,
- *local min* if f' changes $- \to +$,
- *neither* if f' does not change sign.

Concavity, inflection & the Second Derivative Test

f is **concave up** where $f'' > 0$, **concave down** where $f'' < 0$. An **inflection point** is where concavity changes (requires $f'' = 0$ or undefined *and* a sign change).
Second Derivative Test: at a critical point c with $f'(c) = 0$: if $f''(c) > 0$ it's a local min; if $f''(c) < 0$ a local max; if $f''(c) = 0$ the test is inconclusive.

Extreme Value Theorem & absolute extrema

EVT: a function continuous on a closed interval $[a, b]$ attains an absolute max and min there. Find them with the **Candidates Test**: evaluate f at all critical points *and* at the endpoints a, b; the largest/smallest outputs are the absolute extrema.

Optimization & connecting f, f', f''

Optimization: write the quantity to optimize as a function of one variable (use a constraint to eliminate others), find critical points, and justify the extremum.
Connections: $f' > 0 \Rightarrow f \uparrow$; $f'' > 0 \Rightarrow f$ concave up. A max of f' occurs where $f'' = 0$ changes $+ \to -$ (an inflection point of f). Reading any one of f, f', f'' constrains the others.

Optimization: largest rectangle

Maximize area $A = xy$ with perimeter $2x + 2y = 40$, so $y = 20 - x$ and $A(x) = x(20 - x) = 20x - x^2$. Then $A'(x) = 20 - 2x = 0 \Rightarrow x = 10$, and $A'' = -2 < 0$ confirms a max. The rectangle is 10×10, area 100.

Tip: On the AP exam, *justify* every extremum: cite the sign change of f' (First Derivative Test) or the sign of f'', and for absolute extrema on $[a, b]$ always include the endpoints as candidates.

14.6 Integration & the Fundamental Theorem

Antiderivative (basic) rules

- $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
- $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C$
- $\displaystyle\int \sin x\,dx = -\cos x + C$
- $\displaystyle\int \cos x\,dx = \sin x + C$
- $\displaystyle\int \sec^2 x\,dx = \tan x + C$
- $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
- $\displaystyle\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$
- $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$

Riemann sums & the Trapezoidal Rule

A definite integral is a limit of Riemann sums: $\displaystyle\int_a^b f\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$, $\Delta x = \frac{b-a}{n}$. Approximations (left/right/midpoint) sample x_i^* at those points. **Trapezoidal Rule** (equal width Δx):

$$\int_a^b f\,dx \approx \frac{\Delta x}{2}\big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big].$$

Trapezoid *overestimates* when concave up, *underestimates* when concave down.

Definite integral properties

- $\int_a^a f\,dx = 0$
- $\int_b^a f\,dx = -\int_a^b f\,dx$
- $\int_a^b cf\,dx = c\int_a^b f\,dx$
- $\int_a^b (f \pm g)\,dx = \int_a^b f\,dx \pm \int_a^b g\,dx$
- $\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$

Fundamental Theorem of Calculus

Part 1 (accumulation / derivative of an integral): if $g(x) = \int_a^x f(t)\,dt$ with f continuous, then $g'(x) = f(x)$. With a variable upper limit $u(x)$ (chain rule): $\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = f\big(u(x)\big)\,u'(x)$.

Part 2 (evaluation): if $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

u-substitution, average value, net change

u-sub reverses the chain rule: let $u = g(x)$, $du = g'(x)\,dx$, and *change the limits* for a definite integral.

Average value of f on $[a, b]$: $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Net Change Theorem: $\int_a^b F'(x)\,dx = F(b) - F(a)$ — the integral of a rate gives the total (net) accumulated change.

FTC Part 2 with u-substitution

$$\int_0^1 2x\,e^{x^2}\,dx : \quad u = x^2,\ du = 2x\,dx;\quad x{:}0 \to 1 \Rightarrow u{:}0 \to 1.$$

$$= \int_0^1 e^u\,du = \big[e^u\big]_0^1 = e - 1.$$

Tip: Never forget "$+C$" on an indefinite integral, and always change the bounds when you substitute in a *definite* integral (or convert back to x before evaluating).

14.7 Techniques of Integration

Integration by parts (LIATE)

$$\int u\,dv = uv - \int v\,du.$$

Choose u by **LIATE** priority — **L**ogarithmic, **I**nverse trig, **A**lgebraic, **T**rigonometric, **E**xponential — whichever comes first is u; the rest is dv. Repeat for higher powers; use tabular integration to organize.

Partial fractions

For a *proper* rational function (numerator degree < denominator), factor the denominator and split into simpler fractions:

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b},$$

solve for constants, then integrate each piece (giving logs / arctangents). If improper, do polynomial long division first.

Trigonometric integrals

- $\int \sin^m x \cos^n x\,dx$: if a power is *odd*, peel one factor and convert with $\sin^2 + \cos^2 = 1$.
- If both powers *even*, use $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$.
- $\int \tan^m x \sec^n x\,dx$: use $\sec^2 = 1 + \tan^2$ with $\frac{d}{dx}\tan x = \sec^2 x$.

Trigonometric substitution

- $\sqrt{a^2 - x^2}$: let $x = a\sin\theta$
- $\sqrt{a^2 + x^2}$: let $x = a\tan\theta$
- $\sqrt{x^2 - a^2}$: let $x = a\sec\theta$

Then simplify the radical with a Pythagorean identity and convert back with a reference triangle.

Improper integrals & p-integrals

Handle infinite limits or infinite discontinuities as *limits*: $\displaystyle\int_a^\infty f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$. Converges if the limit is finite, else diverges. Key result:

$$\int_1^\infty \frac{dx}{x^p} \text{ converges} \iff p > 1; \qquad \int_0^1 \frac{dx}{x^p} \text{ converges} \iff p < 1.$$

Integration by parts

$$\int x\,e^x\,dx: \;\; u = x,\;\; dv = e^x dx \Rightarrow du = dx,\;\; v = e^x.$$

$$= xe^x - \int e^x\,dx = xe^x - e^x + C = e^x(x-1) + C.$$

Tip: Before reaching for a heavy technique, check for a simple u-substitution. LIATE tells you which factor to call u; the one *later* in the list becomes dv (it must be easy to integrate).

14.8 Applications of Integration

Area between curves

Between $y = f(x)$ (top) and $y = g(x)$ (bottom) on $[a, b]$:

$$A = \int_a^b \big[\, f(x) - g(x) \,\big]\,dx = \int_a^b (\text{top} - \text{bottom})\,dx.$$

If curves are given as $x =$ functions of y, integrate (right $-$ left) dy. Find intersection points to set the limits.

Volumes of revolution

- **Disks:** $V = \pi \displaystyle\int_a^b [R(x)]^2\,dx$ (solid region against the axis).
- **Washers:** $V = \pi \displaystyle\int_a^b \big([R]^2 - [r]^2\big)\,dx$ (outer R, inner r; gap from the axis).
- **Shells:** $V = 2\pi \displaystyle\int_a^b (\text{radius})(\text{height})\,dx$ (convenient when the axis is parallel to the slices).

Volumes by known cross-sections

If cross-sections perpendicular to the x-axis have area $A(x)$:

$$V = \int_a^b A(x)\,dx.$$

Squares: $A = s^2$; equilateral triangles: $A = \frac{\sqrt{3}}{4}s^2$; semicircles: $A = \frac{\pi}{8}s^2$ (with $s =$ base length from the region).

Arc length, average value, motion

Arc length of $y = f(x)$ on $[a, b]$: $L = \displaystyle\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$.

Average value: $f_{\text{avg}} = \dfrac{1}{b-a} \displaystyle\int_a^b f\, dx$.

Motion: displacement $= \int v\, dt$; total distance $= \int |v|\, dt$; $s(t) = s(t_0) + \int_{t_0}^{t} v\, d\tau$.

Volume by washers

Region between $y = x$ and $y = x^2$ (they meet at $x = 0, 1$) revolved about the x-axis. Outer radius x, inner x^2:

$$V = \pi \int_0^1 (x^2 - x^4)\, dx = \pi \left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right]_0^1 = \pi \left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \frac{2\pi}{15}.$$

Tip: Sketch the region and a representative slice first. Slices *perpendicular* to the axis of revolution give disks/washers (dx about x-axis); slices *parallel* give shells.

14.9 Differential Equations

Slope fields

A **slope field** plots short segments with slope $\frac{dy}{dx}$ at grid points, showing the family of solution curves. A particular solution follows the segments through its initial point. Slopes depend only on where $\frac{dy}{dx}$ is evaluated.

Separable equations

If $\dfrac{dy}{dx} = g(x)h(y)$, separate and integrate:

$$\int \frac{dy}{h(y)} = \int g(x)\, dx.$$

Add $+C$ once, then apply the initial condition to solve for C and get the particular solution.

Exponential growth & decay

$\dfrac{dP}{dt} = kP$ (rate proportional to amount) solves to

$$P(t) = P_0\, e^{kt},$$

with $k > 0$ growth, $k < 0$ decay, $P_0 = P(0)$. Half-life / doubling problems reduce to solving for t.

Logistic growth

Growth limited by a carrying capacity M:

$$\frac{dP}{dt} = kP\Big(1 - \frac{P}{M}\Big).$$

Key facts: $\lim\limits_{t\to\infty} P = M$; growth is *fastest* at $P = \frac{M}{2}$ (the inflection point of $P(t)$); solutions are S-shaped. Solution: $P(t) = \dfrac{M}{1 + Ae^{-kt}}$ with $A = \dfrac{M - P_0}{P_0}$.

Euler's Method

Numerically approximate a solution from a starting point using step size h:

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h \cdot f(x_n, y_n),$$

where $f(x, y) = \frac{dy}{dx}$. Follow the tangent line one step at a time.

Separable equation with initial condition

Solve $\dfrac{dy}{dx} = xy$, $y(0) = 2$:

$$\int \frac{dy}{y} = \int x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + C \Rightarrow y = Ae^{x^2/2}.$$

$y(0) = 2 \Rightarrow A = 2$, so $y = 2e^{x^2/2}$.

Tip: For logistic growth, the maximum rate of change occurs at half the carrying capacity, $P = \frac{M}{2}$. Euler's method with $\frac{dy}{dx}$ concave up *underestimates*; concave down *overestimates*.

14.10 Infinite Sequences & Series

Geometric & p-series (the benchmarks)

Geometric: $\sum_{n=0}^{\infty} ar^n$ converges $\iff$ $|r|<1$, with sum $\dfrac{a}{1-r}$ ($a =$ first term).

p-series: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges $\iff$ $p>1$. The harmonic series ($p=1$) diverges.

The convergence tests

- **nth-Term Test:** if $\lim a_n \neq 0$, diverges. (Can only show divergence.)
- **Integral Test:** f positive, decreasing, continuous $\Rightarrow \sum a_n$ and $\int_1^\infty f\,dx$ share fate.
- **Direct Comparison:** $0 \le a_n \le b_n$; bigger converges $\Rightarrow$ smaller converges (and dually).
- **Limit Comparison:** $\lim \frac{a_n}{b_n} = L$, $0 < L < \infty \Rightarrow$ same behavior as $\sum b_n$.
- **Ratio Test:** $L = \lim \left|\frac{a_{n+1}}{a_n}\right|$; $L<1$ converges, $L>1$ diverges, $L=1$ inconclusive.
- **Root Test:** $L = \lim \sqrt[n]{|a_n|}$; same conclusions as ratio test.
- **Alternating Series Test:** $\sum(-1)^n b_n$ with $b_n \downarrow 0$ converges.

Which test should I use? (decision order)

1. **nth-term:** does $a_n \to 0$? If not $\Rightarrow$ diverges. (Always first.)
2. **Geometric** (ar^n)? converges iff $|r|<1$, sum $\frac{a}{1-r}$.
3. **p-series** ($\frac{1}{n^p}$)? converges iff $p>1$.
4. **Factorials or nth powers**? Ratio (or Root) Test.
5. **Rational / resembles a p-series**? Direct or Limit Comparison.
6. **Positive, decreasing, easy to integrate**? Integral Test.
7. **Alternating** $(-1)^n b_n$? AST; then check $\sum |a_n|$ for absolute vs. conditional.

Error bounds; absolute vs. conditional

Alternating Series Error: if AST holds, $|S - S_N| \le b_{N+1}$ (first omitted term).
Absolute convergence: $\sum |a_n|$ converges (implies $\sum a_n$ converges). **Conditional**: $\sum a_n$ converges but $\sum |a_n|$ diverges.

Power series: radius & interval

$\sum a_n (x-c)^n$ converges on $|x-c|<R$. Find R by the **ratio test**; then *test each endpoint*

separately (ratio test gives $L = 1$ there). Term-by-term differentiation/integration keeps the same R (endpoints may change).

Taylor & Maclaurin series; Lagrange error

Taylor about c: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$. **Maclaurin** is $c = 0$.

Lagrange Error Bound: $|R_n(x)| \le \frac{M}{(n+1)!}\,|x-c|^{n+1}$, where M bounds $|f^{(n+1)}|$ between c and x.

Common Maclaurin series (memorize)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (\text{all } x)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad (\text{all } x)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad (\text{all } x)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (|x| < 1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad (-1 < x \le 1)$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \qquad (|x| \le 1)$$

Ratio test with a factorial

$\sum_{n=1}^{\infty} \frac{2^n}{n!}$: $L = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$, so it **converges**.

Tip: Build unfamiliar series from the six memorized ones by *substitution*, *differentiation*, or *integration* rather than computing derivatives. E.g. $e^{-x^2} = \sum \frac{(-1)^n x^{2n}}{n!}$. Endpoints of an interval of convergence always need a fresh, separate test.

14.11 Parametric, Polar & Vector-Valued Functions

Parametric derivatives

For $x = x(t)$, $y = y(t)$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \quad \left(\tfrac{dx}{dt} \neq 0\right), \qquad \frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}.$$

The second derivative divides by $\frac{dx}{dt}$ again — *not* by $\frac{d^2x}{dt^2}$.

Parametric arc length

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Vector-valued motion

For position $\mathbf{r}(t) = \langle x(t), y(t) \rangle$:

- **Velocity** $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$; **acceleration** $\mathbf{a}(t) = \langle x''(t), y''(t) \rangle$.
- **Speed** $= |\mathbf{v}(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$.
- **Total distance** $= \displaystyle\int_{t_1}^{t_2} |\mathbf{v}(t)|\, dt$ (same integrand as arc length).
- **Position from velocity:** $x(t) = x(t_0) + \int_{t_0}^{t} x'(\tau)\, d\tau$ (and likewise y).

Polar curves: slope, area, arc length

With $x = r\cos\theta$, $y = r\sin\theta$ and $r = f(\theta)$:

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Area swept from $\theta = \alpha$ to β: $A = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2\, d\theta$; between two curves $\frac{1}{2}\int (r_{\text{out}}^2 - r_{\text{in}}^2)\, d\theta$.

Arc length: $L = \displaystyle\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Parametric slope and second derivative

Let $x = t^2$, $y = t^3$. Then $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$. For the second derivative, $\frac{d}{dt}\left(\frac{3t}{2}\right) = \frac{3}{2}$, so

$$\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}.$$

Polar area

Area inside one petal of $r = \cos(2\theta)$ (petal for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$):

$$A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2(2\theta)\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{1+\cos 4\theta}{2}\, d\theta = \frac{\pi}{8}.$$

Tip: "Speed," "total distance," and "arc length" all use the *same* integrand $\sqrt{(x')^2 + (y')^2}$. For polar area, always use $\frac{1}{2}\int r^2\, d\theta$ — and choose θ-limits that trace the region exactly once.