MathCastle

AMC 8 / MathCounts

Complete course booklet • 8 topics

Contents

1 Number Theory Toolkit

Divisibility, primes, and remainders

1.1 Divisibility Rules

The rules worth memorizing

Divisible by **3** or **9** when the digit sum is; by **4** when the last two digits form a multiple of 4; by **8** when the last three do; by **11** when the alternating digit sum (units − tens + hundreds ...) is a multiple of 11.

In plain terms. Instead of doing long division, just add or peek at a few digits. It is a shortcut to answer "does this divide evenly?" in one glance.

Example. Is 8,514 divisible by 9? Digit sum $= 8 + 5 + 1 + 4 = 18$, and 18 is a multiple of 9, so yes.

Why the digit-sum rule works

Because $10 \equiv 1 \pmod 9$, every power of ten leaves remainder 1, so a number and its digit sum leave the same remainder mod 9 (and mod 3).

In plain terms. Each "10" is just "one more than a nine," so tens, hundreds, and so on don't change the remainder when you divide by 9 — only the digits themselves do.

Example. For 23: $23 = 2 \cdot 10 + 3 = 2 \cdot 9 + 2 + 3$, so its remainder mod 9 is $2 + 3 = 5$, exactly the digit sum.

1.2 Primes and Factorization

Prime factorization is the master key

Write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then the number of divisors is $(a_1 + 1) \cdots (a_k + 1)$, and the sum of divisors is $\prod \dfrac{p_i^{a_i+1} - 1}{p_i - 1}$.

In plain terms. Break a number into its prime "building blocks." Once you know the blocks, counting or adding up all its factors is just plugging into a formula.

Example. $72 = 2^3 \cdot 3^2$, so it has $(3+1)(2+1) = 12$ divisors, and their sum is $\dfrac{2^4 - 1}{1} \cdot \dfrac{3^3 - 1}{2} = 15 \cdot 13 = 195$.

Checking primality quickly

To test whether n is prime you only need trial divisors up to $\sqrt{n}$: if $n = ab$ with $a \leq b$ then $a \leq \sqrt{n}$.

In plain terms. A factor pair always has one member no bigger than the square root, so you never need to test past $\sqrt{n}$ — that cuts the work enormously.

Example. To check 97: $\sqrt{97} < 10$, so only test $2, 3, 5, 7$. None divide it, so 97 is prime.

1.3 GCD, LCM, and Units Digits

The GCD–LCM identity

For positive integers, $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$.

In plain terms. The greatest common divisor and least common multiple always multiply to give the two numbers' product — so knowing one gives the other.

Example. $\gcd(12, 18) = 6$, so $\operatorname{lcm}(12, 18) = \dfrac{12 \cdot 18}{6} = 36$.

Units digits cycle

The last digit of a^n repeats with a period dividing 4: powers of 2 cycle $2, 4, 8, 6$; of 3 cycle $3, 9, 7, 1$.

In plain terms. Last digits go in a short repeating loop, so you just find where your exponent lands in the loop instead of computing the whole power.

Example. Last digit of 7^{100}: the cycle for 7 has length 4, and $100 \equiv 0 \pmod 4$, so it matches 7^4, whose last digit is 1.

2 Counting & Probability

Systematic counting and complements

2.1 The Fundamental Principles

Multiply independent choices

If one choice has m options and a second, independent choice has n options, the pair has $m \cdot n$ options. Overlapping or exclusive choices are handled by splitting into cases and adding.

In plain terms. When decisions don't affect each other, multiply the number of options. When they can't happen together, add.

Example. A menu with 3 mains and 4 drinks gives $3 \cdot 4 = 12$ meals.

Permutations vs. combinations

Ordered picks of k from n: ${}_nP_k = \dfrac{n!}{(n-k)!}$. Unordered picks: $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

In plain terms. If rearranging counts as different, use permutations; if it does not, use combinations. The only difference is dividing out the $k!$ orderings.

Example. Seat 3 of 8 in a row (order matters): ${}_8P_3 = 8 \cdot 7 \cdot 6 = 336$. Choose a committee of 3 (order doesn't): $\binom{8}{3} = 56$.

2.2 Complementary Counting and Probability

Count the opposite

For "at least one," compute (total) − (ways it fails). The complement usually avoids messy casework.

In plain terms. Counting what you don't want and subtracting is often far easier than counting what you do want.

Example. Probability of at least one head in 4 flips: the only failure is all tails, $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, so the answer is $1 - \frac{1}{16} = \frac{15}{16}$.

Probability is a ratio of counts

For equally likely outcomes, $P(\text{event}) = \dfrac{\text{favorable}}{\text{total}}$.

In plain terms. Probability is just "good outcomes over all outcomes" — so counting carefully is the whole task.

Example. Rolling a sum of 7 with two dice: 6 favorable pairs out of 36, so $P = \frac{6}{36} = \frac{1}{6}$.

3 Algebra & Manipulation

Equations, ratios, and factoring tricks

3.1 Setting Up and Solving

Vieta for quadratics

For $ax^2 + bx + c = 0$ with roots r, s: sum $r + s = -\dfrac{b}{a}$ and product $rs = \dfrac{c}{a}$.

In plain terms. The two numbers that solve a quadratic add up to $-b/a$ and multiply to c/a — you can read those off without solving.

Example. For $x^2 - 7x + 12 = 0$ the roots sum to 7 and multiply to 12, so they are 3 and 4.

3.2 Ratios and Clever Factoring

Ratios and proportions

A ratio $a : b$ means the amounts are ak and bk. A proportion $\frac{a}{b} = \frac{c}{d}$ cross-multiplies to $ad = bc$.

In plain terms. A ratio is a shared multiplier — turn "3 to 5" into "$3k$ and $5k$," then use the given total to find k.

Example. If red to blue is $3 : 5$ and there are 40 total, then $3k + 5k = 40$, so $k = 5$: 15 red and 25 blue.

Simon's Favorite Factoring Trick

The expression $xy + ax + by$ factors as $(x + b)(y + a) - ab$.

In plain terms. When an equation mixes a product like xy with plain x and y terms, add the one missing number to make it factor into a neat product.

Example. For $xy + 2x + 3y = 8$, add 6: $(x + 3)(y + 2) = 14$, so the integer factor pairs of 14 give the solutions.

4 Geometry Essentials

Angles, triangles, similarity, and circles

4.1 Angles and Triangles

Angle facts

A triangle's angles sum to 180°; a quadrilateral's to 360°. An exterior angle equals the sum of the two remote interior angles.

In plain terms. Angles in a shape add to a fixed total, so a missing angle is just the total minus the known ones.

Example. A triangle with angles 50° and 60° has third angle $180° - 110° = 70°$.

Special right triangles and triples

45-45-90 has sides $1 : 1 : \sqrt{2}$; 30-60-90 has $1 : \sqrt{3} : 2$. Common Pythagorean triples: 3-4-5, 5-12-13, 8-15-17, 7-24-25.

In plain terms. A few triangle shapes show up constantly — memorize their side ratios so you recognize them instantly instead of using the Pythagorean theorem every time.

Example. A right triangle with legs 6 and 8 is a doubled 3-4-5, so its hypotenuse is 10.

4.2 Similarity, Area, and Circles

Similar triangles

Equal-angle triangles have proportional sides; if the ratio is k, areas scale by k^2.

In plain terms. Same shape, different size: every length scales by the same factor, and areas by that factor squared.

Example. If a triangle is scaled by $k = 3$, a side of 5 becomes 15 and the area becomes 9 times as large.

Area and circle formulas

Triangle area $= \frac{1}{2}bh = \frac{1}{2}ab\sin C$. Circle: circumference $2\pi r$, area πr^2; an arc of angle $\theta°$ has length $\frac{\theta}{360} \cdot 2\pi r$.

In plain terms. Area of a triangle is half base times height; a circle's size formulas both start from the radius.

Example. A triangle with sides $5, 5, 6$: the altitude to the base 6 makes two 3-4-5 triangles, so height $= 4$ and area $= \frac{1}{2} \cdot 6 \cdot 4 = 12$.

5 Clever Counting & Logic

Complements, casework, and parity

5.1 Counting Without Listing

The complement trick

Sometimes the easiest way to count what you want is to count what you do NOT want: $|A| = |\text{total}| - |\text{not } A|$.

In plain terms. Count everything, count the "bad" ones, subtract.

Example. How many 3-digit numbers contain at least one 7? Total: 900. With no 7s: $8 \cdot 9 \cdot 9 = 648$. So $900 - 648 = 252$.

Casework that stays organized

Split the count into cases that do not overlap and together cover everything, then add the case counts.

In plain terms. Sort the possibilities into non-overlapping piles and count each pile.

Example. Two dice sum to 5: the cases are $(1, 4), (2, 3), (3, 2), (4, 1)$ — exactly 4 ways.

Parity (odd/even) arguments

The parity of a sum depends only on how many ODD terms it has — an even number of odd terms gives an even sum.

In plain terms. You can often decide odd vs. even without computing anything.

Example. $1 + 2 + \cdots + 9 = 45$ is odd because it contains five odd terms, and 5 is odd.

6 Why the Formulas Work

Every rule you use, actually proved

6.1 The Angle Formulas, Proved

Proof: a triangle's angles sum to $180°$

Draw the line through one vertex PARALLEL to the opposite side. The two new angles at that vertex equal the two far angles of the triangle (**alternate interior angles**), and together with the vertex angle they fill a straight line: $A + B + C = 180°$. ■

In plain terms. Slide the two bottom corners up to the top corner along parallel lines — the three angles line up along a ruler.

Example. Angles $50°$ and $60°$ force the third to be $180 - 110 = 70°$.

(diagram in the online version)

Proof: an n-gon's angles sum to $(n - 2) \cdot 180°$

Pick one vertex and draw all diagonals from it. They cut the polygon into exactly $n-2$ triangles, and the polygon's interior angles are exactly the triangles' angles combined: $(n - 2) \cdot 180°$. ■

In plain terms. Fan the shape into triangles from one corner and count them: always two fewer than the sides.

Example. A hexagon fans into 4 triangles: $4 \cdot 180° = 720°$.

Proof: exterior angles always total $360°$

Walk once around the polygon. At each corner you turn by that corner's exterior angle, and one full lap is one full turn: the turns sum to $360°$ — no matter how many sides. ■

In plain terms. Walking the fence line spins you around exactly once.

Example. A regular 20-gon: each exterior angle is $360/20 = 18°$.

6.2 The Pythagorean Theorem, Proved

Proof by rearrangement: $a^2 + b^2 = c^2$

Take a big square of side $a+b$ and tuck four copies of the right triangle into its corners. What remains uncovered is a tilted square on the hypotenuse: area $(a+b)^2 - 4 \cdot \frac{1}{2}ab = c^2$, and expanding the left side gives $a^2 + b^2 = c^2$. ■

In plain terms. Four copies of the same triangle leave a c-square of leftover space — and computing the leftover two ways forces the theorem.

Example. Legs 3 and 4: $9 + 16 = 25$, so the hypotenuse is 5.

(diagram in the online version)

Why the 3-4-5 family never ends

Scaling every side of a right triangle by k scales both sides of $a^2 + b^2 = c^2$ by k^2, so $(3k, 4k, 5k)$ is a right triangle for every k — one proof, infinitely many triangles.

In plain terms. Blow up or shrink a right triangle and it stays right.

Example. $(6, 8, 10)$ and $(30, 40, 50)$ both work automatically.

6.3 The Sum Formulas, Proved

Proof (Gauss pairing): $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$

Write the sum forwards and backwards and add columns: every column totals $n+1$, and there are n columns, so TWICE the sum is $n(n+1)$. ■

In plain terms. Pair the smallest with the largest, second-smallest with second-largest — every pair is worth the same.

Example. $1 + 2 + \cdots + 100$: fifty pairs of 101 gives 5050.

Proof by picture: the first n odd numbers sum to n^2

Each new odd number $2k+1$ wraps an L-shape around a $k \times k$ square, growing it to $(k+1) \times (k+1)$. Stacking L-shapes builds a perfect square: $1 + 3 + \cdots + (2n-1) = n^2$. ■

In plain terms. Odd numbers are exactly the layers of a growing square.

Example. $1 + 3 + 5 + 7 = 16 = 4^2$.

7 Complete Syllabus & Gap Fillers

Everything the AMC 8 tests, with the pieces that were missing

7.1 What the AMC 8 Tests

The checklist

Arithmetic and fractions/decimals/percents; ratios, rates, and unit conversion; basic number theory (divisibility, primes, LCM/GCD, remainders, digit problems); counting (lists, complements, simple casework, permutations of small sets); probability of simple events; averages and simple statistics (mean, median, mode, range); geometry (perimeter, area, angle sums, triangles, circles, coordinate basics, Pythagorean triples); patterns and sequences; logic puzzles and reading carefully. Twenty-five questions, forty minutes, no calculators.

7.2 Gap Fillers

Rates and work

Distance = rate × time; average speed is TOTAL distance over TOTAL time, never the average of speeds (60 km at 30 km/h and 60 at 60 km/h averages 40). Work problems add RATES: pipes filling in 6 and 3 hours together fill $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ per hour.

Statistics in one box

Mean = total ÷ count (so "the mean of 7 numbers is 10, remove one, mean becomes 9" means the removed number is $70 - 54 = 16$). Median is the middle of the sorted list; mode the most frequent; range max minus min. Adding a constant shifts mean/median; multiplying scales them.

Clock and calendar angles

The minute hand moves 6° per minute, the hour hand 0.5° per minute (and 30° per hour). At 3:40: minute at 240°, hour at 110°, angle 130°. Days of the week cycle mod 7; leap years add a day.

8 Formulas & Methods

Every formula the AMC 8 uses, with the methods that apply them

8.1 Arithmetic, Ratios & Percents

Percents and ratios

Percent change $= \frac{\text{new}-\text{old}}{\text{old}}$; a p% increase then p% decrease leaves $(1 - \frac{p^2}{100^2})$ of the start. Ratio $a : b : c$ with total T gives parts $\frac{aT}{a+b+c}$. "a% of b" equals "b% of a".

Convert every percent to a multiplier (30% off is ×0.7) and chain multipliers for successive changes — never add percents of different bases. Ratio problems: assign k to one part and solve for k from the given total or difference.

Rates, work, and averages

Distance $= rt$. Average speed $= \frac{\text{total distance}}{\text{total time}}$ — the harmonic mean for equal distances $(\frac{2v_1v_2}{v_1+v_2})$. Work rates ADD: $\frac{1}{t_1} + \frac{1}{t_2} = \frac{1}{t}$. Mean $= \frac{\text{sum}}{\text{count}}$; changing the mean by removing one item: removed $=$ old total $-$ new total.

Set up a table with columns rate, time, amount. For "how many of each" mixture problems, use one variable and the total. For averages, always work with TOTALS, not averages.

8.2 Number Theory & Counting

Divisibility, primes, LCM/GCD

Divisible by 3/9: digit sum; by 4/8: last two/three digits; by 11: alternating digit sum. $\gcd \cdot \text{lcm} = ab$. Number of divisors of $p^a q^b$: $(a+1)(b+1)$. Remainder cycles: last digits of powers repeat with period ≤ 4.

Factor completely before anything else. Units-digit questions: find the cycle length, reduce the exponent. "Smallest number leaving remainders": list one sequence and filter by the other condition.

Counting and probability

Multiplication principle for independent choices; permutations $n!$, with repeats $\frac{n!}{a!b!}$; combinations $\binom{n}{k}$. Complement: $P(\text{at least one}) = 1 - P(\text{none})$. Probability $= \frac{\text{favorable}}{\text{total}}$ with equally likely outcomes.

Make the sample space explicit (a table for two dice, a tree for coins). Casework by a stated criterion. When "at least" appears, use the complement.

8.3 Geometry

Angles, triangles, circles

Triangle angles sum to 180°; polygon interior sum $(n-2)180°$; exterior angles sum to 360°. Pythagorean triples 3-4-5, 5-12-13, 8-15-17, 7-24-25. Area: triangle $\frac{1}{2}bh$, trapezoid $\frac{(b_1+b_2)h}{2}$, circle πr^2, circumference $2\pi r$. Similar figures: lengths scale by k, areas by k^2, volumes by k^3.

Draw to scale and label everything known. Look for right triangles to apply Pythagoras; for overlapping shapes, compute (whole) $-$ (parts). Coordinates: distance formula $\sqrt{\Delta x^2 + \Delta y^2}$, midpoint $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Clocks, calendars, patterns

Minute hand 6°/min, hour hand 0.5°/min. Days cycle mod 7. Arithmetic sequence: $a_n = a_1 + (n-1)d$, sum $\frac{n(a_1+a_n)}{2}$; $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

For patterns, compute the first four terms and identify the rule before extrapolating; for cyclic patterns, find the period and reduce.

The universal problem-solving loop

Read twice. Restate the goal in your own words. List the givens. Pick a representation. Try small cases. Look for symmetry, an invariant, or an extremal object. Compute, then CHECK against a second method or a sanity bound.

On the AMC 8 and MathCounts the check is the whole strategy: answer choices can be tested, units can be verified, and a quick estimate ("about 30, so 28 not 280") catches most slips. Draw every geometry problem to scale. Make a table for every counting problem. Write the equation before solving it.