# MathCastle

AMC 12

Complete course booklet • 8 topics

---

## Contents

# 1 Circles & Lines

*Distances, angle bisectors, tangents, and conic meetings*

## 1.1 Distance: Point to Line

**The point-to-line distance formula**

The distance from $(x_0, y_0)$ to the line $Ax + By + C = 0$ is $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$; parallel lines $Ax + By = C_1$, $Ax + By = C_2$ sit $\dfrac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$ apart.

**In plain terms.** Plug the point into the line's left side and divide by the length of $(A, B)$ — it measures how far the point sits along the line's perpendicular (normal) direction. Never solve for the foot of the perpendicular unless the problem asks for it.

**Example.** From $(6, 7)$ to $3x + 4y - 10 = 0$: $d = \dfrac{|18 + 28 - 10|}{5} = \dfrac{36}{5}$.

**Tangency is a distance statement**

A line is tangent to a circle exactly when the center's distance to the line EQUALS the radius; it misses when the distance exceeds the radius, and cuts a chord when it is smaller.

**In plain terms.** One formula answers three different-looking questions: "is it tangent?", "what k makes it tangent?", and "how long is the chord?" (half-chord $= \sqrt{r^2 - d^2}$).

**Example.** $y = x + k$ is tangent to $x^2 + y^2 = 8$ when $\dfrac{|k|}{\sqrt{2}} = 2\sqrt{2}$, i.e. $k = \pm 4$.

## 1.2 The Bisector Between Two Lines

**Angle bisectors as an equidistance locus**

The bisectors of the angles between $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ satisfy $\dfrac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \dfrac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$ — two perpendicular lines, one per sign.

**In plain terms.** A bisector is just the set of points at EQUAL distance from both lines, so set the two distance formulas equal. The $\pm$ gives both bisectors; pick the acute or obtuse one by testing a point.

**Example.** For $y = \frac{3}{4}x$ and the $x$-axis: $\dfrac{|3x - 4y|}{5} = |y|$ gives $y = \frac{1}{3}x$ (acute) and $y = -3x$ (obtuse) — note the two bisectors are perpendicular.

**Symmetry shortcut**

If swapping $x \leftrightarrow y$ (or another visible reflection) exchanges the two lines, then the mirror line of that reflection is automatically one of the bisectors.

**In plain terms.** Before computing, look for a symmetry that hands you the bisector for free.

**Example.** $3x + 4y = 12$ and $4x + 3y = 12$ swap under $x \leftrightarrow y$, so $y = x$ bisects them — no algebra needed.

## 1.3 Circles: Chords, Tangents, Power

### The three right-triangle facts

Chord at distance $d$ from the center: length $2\sqrt{r^2 - d^2}$. Tangent from an external point at distance $d$: length $\sqrt{d^2 - r^2}$. Common external tangent of circles $(r_1, r_2)$ with centers $d$ apart: $\sqrt{d^2 - (r_2 - r_1)^2}$ (internal: $\sqrt{d^2 - (r_1 + r_2)^2}$).

**In plain terms.** Every one of these is the SAME move: drop the radius to the chord or tangency point, get a right triangle, apply Pythagoras. Learn the move, not four formulas.

**Example.** Tangent from $(13, 0)$ to $x^2 + y^2 = 25$: $\sqrt{169 - 25} = 12$.

### Nearest point of a circle to a line

If the center is at distance $d > r$ from a line, the circle's nearest point to the line is at distance $d - r$ and the farthest at $d + r$ — both along the perpendicular from the center.

**In plain terms.** Circles turn min/max distance questions into "center distance, then adjust by the radius."

**Example.** From $x^2 + y^2 = 9$ to $3x + 4y = 25$: center distance 5, so the minimum is $5 - 3 = 2$.

## 1.4 Circles Meet Parabolas

### Intersecting a circle with a conic

Substitute the parabola into the circle and solve the resulting equation in $u = x^2$ (or $y$): each POSITIVE root $u$ contributes two points ($x = \pm\sqrt{u}$), $u = 0$ one point, and negative roots none.

**In plain terms.** The whole art is bookkeeping: a quadratic in $u$ can produce $0, 1, 2, 3,$ or $4$ intersection points depending on the signs of its roots — count carefully instead of guessing from a sketch.

**Example.** $y = x^2 - 4$ into $x^2 + y^2 = 4$: $u^2 - 7u + 12 = 0$ gives $u = 3, 4$, both positive — four intersection points.

### Focus and directrix

The parabola $x^2 = 4py$ has focus $(0, p)$ and directrix $y = -p$; every point is equidistant from the two. In the form $y = ax^2$, $p = \dfrac{1}{4a}$.

**In plain terms.** When a contest problem says "focus", convert to $x^2 = 4py$ first — the geometry (equal distances) is what problems actually use.

**Example.** $y = \frac{x^2}{8}$ is $x^2 = 8y$: focus $(0, 2)$, directrix $y = -2$.

# 2 Trigonometry & Complex Numbers

*Identities, de Moivre, and roots of unity*

## 2.1 Trigonometric Identities

**The identities you cannot do without**

Pythagorean: $\sin^2\theta + \cos^2\theta = 1$. Double angle: $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 1 - 2\sin^2\theta$. Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos C$.

**In plain terms.** A handful of identities convert between sines and cosines and relate a triangle's sides to its angles.

**Example.** If $\sin\theta = \frac{3}{5}$ (acute), then $\cos\theta = \frac{4}{5}$ and $\sin 2\theta = 2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$.

## 2.2 Complex Numbers

**Polar form and de Moivre**

Write $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$; then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

**In plain terms.** A complex number is a length and an angle. Raising to a power multiplies the length and just adds up the angle.

**Example.** $(1 + i)^8$: here $r = \sqrt{2}$, $\theta = 45°$, so $r^8 = 16$ and $8\theta = 360°$, giving 16.

**Roots of unity**

The solutions of $z^n = 1$ are $e^{2\pi ik/n}$, equally spaced on the unit circle, and for $n > 1$ they sum to 0.

**In plain terms.** The $n$th roots of 1 are $n$ evenly-spaced points around a circle; being symmetric, they cancel to zero.

**Example.** The cube roots of 1 are $1, \omega, \omega^2$ with $1 + \omega + \omega^2 = 0$.

# 3 Advanced Algebra

*Logs, inequalities, and functions*

## 3.1 Logarithms

**Logarithm laws**

$\log_b(xy) = \log_b x + \log_b y$, $\log_b(x^n) = n \log_b x$, and $\log_b x = \dfrac{\log x}{\log b}$.

**In plain terms.** Logs turn multiplication into addition and pull exponents out front, which untangles equations with a variable in the exponent.

**Example.** $\log_2 40 - \log_2 5 = \log_2 \frac{40}{5} = \log_2 8 = 3$.

### 3.2 Inequalities and Functions

**AM–GM inequality**

For nonnegative reals, $\dfrac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}$, equality when all equal.

**In plain terms.** The average is always at least the geometric mean, so "fixed sum" or "fixed product" extremes happen when the numbers are equal.

**Example.** For $x > 0$, $x + \frac{9}{x} \geq 2\sqrt{x \cdot \frac{9}{x}} = 6$, with the minimum 6 at $x = 3$.

## 4 Sequences, Series & Recursion

*Sums, telescoping, and recurrences*

### 4.1 Standard Sums

**Arithmetic and geometric series**

Arithmetic: $S_n = \frac{n}{2}(a_1 + a_n)$. Geometric: $S_n = a_1 \dfrac{1 - r^n}{1 - r}$, and if $|r| < 1$, $S_\infty = \dfrac{a_1}{1 - r}$.

**In plain terms.** For a steadily-growing list use the arithmetic sum; for one that multiplies each step use the geometric sum.

**Example.** $1 + \frac{1}{2} + \frac{1}{4} + \cdots = \dfrac{1}{1 - 1/2} = 2$.

**Telescoping**

If each term is $f(k) - f(k+1)$, the sum collapses to $f(1) - f(n+1)$.

**In plain terms.** Write each piece as a difference so neighboring pieces cancel, leaving only the first and last.

**Example.** $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}$.

# 5 Logs & Complex Numbers

*Change of base, De Moivre, roots of unity*

## 5.1 Two Powerful Toolkits

**Change of base — and telescoping logs**

$\log_a b = \dfrac{\ln b}{\ln a}$, so products of logarithms telescope: $\log_a b \cdot \log_b c = \log_a c$.

**In plain terms.** Rewrite every log over one base and watch middle terms cancel.

**Example.** $\log_2 3 \cdot \log_3 4 = \log_2 4 = 2$.

**De Moivre's theorem**

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ — powers of a unit complex number just multiply the angle.

**In plain terms.** Raising to a power spins the point around the unit circle $n$ times as fast.

**Example.** $(\cos 45° + i\sin 45°)^2 = \cos 90° + i\sin 90° = i$.

*(diagram in the online version)*

**Roots of unity sum to zero**

The $n$ solutions of $z^n = 1$ are evenly spaced on the unit circle, and their sum is 0 for $n \geq 2$.

**In plain terms.** Evenly spread arrows around a circle cancel out.

**Example.** For cube roots of unity: $1 + \omega + \omega^2 = 0$, which is why $\omega + \omega^2 = -1$.

# 6 Proofs and Derivations

*Trig identities, the binomial theorem, and log rules — from scratch*

## 6.1 The Trig Identities, Derived

**Proof:** $\sin^2\theta + \cos^2\theta = 1$

The unit-circle point $(\cos\theta, \sin\theta)$ lies at distance 1 from the origin, and the distance formula squares to $\cos^2\theta + \sin^2\theta = 1$ — the Pythagorean theorem wearing trig notation. Dividing by $\cos^2\theta$ yields $1 + \tan^2\theta = \sec^2\theta$. ■

**In plain terms.** It is the Pythagorean theorem on a radius-1 circle. Every "Pythagorean identity" is this one line, re-divided.

**Example.** $\sin\theta = \frac{3}{5}$ (Quadrant I) forces $\cos\theta = \frac{4}{5}$: no triangle needed.

*(diagram in the online version)*

**Derivation of the Law of Cosines**

Place the triangle with $C$ at the origin and side $a$ along the $x$-axis: $B = (a, 0)$ and $A = (b\cos C, b\sin C)$. The distance $AB = c$ squares to $c^2 = (b\cos C - a)^2 + (b\sin C)^2 = a^2 + b^2 - 2ab\cos C$ after applying $\sin^2 + \cos^2 = 1$. ■

**In plain terms.** Drop the triangle onto coordinates and use the distance formula once — the identity mops up.

**Example.** Sides $5, 7$ with a $60°$ angle between: $c^2 = 25 + 49 - 70 \cdot \frac{1}{2} = 39$.

**Derivation of the Law of Sines**

The area of a triangle is $\frac{1}{2}ab\sin C$ — computed from ANY vertex. Equating the three versions, $\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$, and dividing through by $\frac{1}{2}abc$ gives $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$. ■

**In plain terms.** One area, three formulas for it — setting them equal is the whole proof.

**Example.** A $30°$ angle opposite side 4: every other side is $\dfrac{\text{its opposite's sine}}{\sin 30°} \cdot 4 \cdot \frac{1}{2} \ldots$ — i.e. $\frac{a}{\sin A}$ is the constant 8.

**Double angle from the addition formula**

Set $\beta = \alpha$ in $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ to get $\sin 2\alpha = 2\sin\alpha\cos\alpha$; the same move on cosine gives $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. Every double- and half-angle identity is the addition formula talking to itself. ■

**In plain terms.** Learn ONE formula (addition); the rest are its special cases.

**Example.** $\sin 2\theta$ with $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$: $2 \cdot \frac{3}{5} \cdot \frac{4}{5} = \frac{24}{25}$.

## 6.2 The Binomial Theorem, Proved

**Proof by counting committees**

Expanding $(1 + x)^n$ means choosing, from each of the $n$ factors, either the 1 or the $x$. A term $x^k$ appears once for every way to pick WHICH $k$ factors contribute an $x$ — that is $\binom{n}{k}$ ways. Hence $(1 + x)^n = \sum_k \binom{n}{k} x^k$. ■

**In plain terms.** The coefficient counts choices, so it must be "n choose k" — the theorem is combinatorics, not algebra.

**Example.** The $x^2$ coefficient in $(1+x)^5$ is $\binom{5}{2} = 10$.

**Two identities that fall out immediately**

Set $x = 1$: $\sum_k \binom{n}{k} = 2^n$ (every subset picks in or out). Set $x = -1$: the alternating sum is 0 — so odd-sized and even-sized subsets are equally many. ■

**In plain terms.** Plugging numbers into a PROVED identity mass-produces new facts.

**Example.** $\binom{10}{0} + \binom{10}{2} + \cdots = \frac{2^{10}}{2} = 512$.

### 6.3 The Log Rules, Derived

**Product and power rules from exponent laws**

Let $m = \log_b x$, $n = \log_b y$, so $x = b^m$, $y = b^n$. Then $xy = b^{m+n}$ says exactly $\log_b(xy) = m + n$, and $x^p = b^{mp}$ says $\log_b(x^p) = p \log_b x$. Logs ARE exponents, so exponent laws translate one-for-one. ■

**In plain terms.** Every log rule is an exponent rule read backwards.

**Example.** $\log_2 8 + \log_2 4 = 3 + 2 = 5 = \log_2 32$.

**Derivation of change of base**

From $x = b^{\log_b x}$, take $\log_c$ of both sides and pull the exponent down (power rule): $\log_c x = \log_b x \cdot \log_c b$. Divide: $\log_b x = \dfrac{\log_c x}{\log_c b}$. ■

**In plain terms.** One log in any base is two logs in the base your calculator likes.

**Example.** $\log_2 5 = \dfrac{\ln 5}{\ln 2} \approx 2.32$.

## 7 Complete Syllabus & Gap Fillers

*Everything the AMC 12 tests, with the pieces that were missing*

### 7.1 What the AMC 12 Tests

**The checklist**

Everything on the AMC 10, plus: trigonometry (identities, sum/product formulas, law of sines/cosines, trig equations); logarithms and exponential equations; complex numbers (de Moivre, roots of unity, modulus geometry); polynomials in depth (Vieta for cubics/quartics, symmetric sums, roots of unity filter); conic sections; sequences and series including telescoping products and recursions with characteristic equations; combinatorics with generating-function intuition; probability with geometric probability and expected value; functions (composition, inverses, functional-equation style). Twenty-five questions, seventy-five minutes.

### 7.2 Gap Fillers

**Conics on the AMC 12**

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $c^2 = a^2 - b^2$ (foci), hyperbola with $c^2 = a^2 + b^2$, parabola $y^2 = 4px$ with focus $(p, 0)$. Reflective properties and the focus–directrix definition appear as "sum of distances" problems in disguise.

**Geometric probability**

Probability = favorable area ÷ total area. Two numbers in $[0, 1]$ with $|x - y| < \frac{1}{4}$: the complement is two right triangles of legs $\frac{3}{4}$, so $1 - \frac{9}{16} = \frac{7}{16}$. Draw the region; the answer is usually a polygon or a circle piece.

**Complex numbers as geometry**

Multiplication rotates and scales; $|z - a|$ is a distance; roots of $z^n = r^n$ form a regular $n$-gon of circumradius $r$ (the roots of $z^6 = 64$ enclose area $6\sqrt{3}$). Sums of roots of unity vanish, which collapses many trig sums.

## 8 Formulas & Methods

*The AMC 12 formula sheet: trig, logs, complex numbers, and beyond*

### 8.1 Trigonometry & Complex Numbers

**Identities**

$\sin^2 + \cos^2 = 1$; $\sin(a \pm b)$, $\cos(a \pm b)$, $\tan(a \pm b)$; double angle $\sin 2a = 2 \sin a \cos a$, $\cos 2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$; half angle; product-to-sum $2 \sin a \cos b = \sin(a + b) + \sin(a - b)$; sum-to-product. $R$-form: $a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi)$.

Convert everything to sines and cosines of one angle, or to a single trig function via the $R$-form. Exact values: 15°, 18°, 36°, 75° from half/difference formulas and the golden ratio.

**Triangles**

Law of sines $\frac{a}{\sin A} = 2R$; law of cosines; area $\frac{1}{2}ab \sin C$; Area $= \frac{abc}{4R} = rs$. Stewart, Apollonius (median), angle bisector length $\sqrt{bc\left(1 - \frac{a^2}{(b+c)^2}\right)}$.

Given SSS use cosines; ASA/AAS use sines; SSA check for the ambiguous case. Two area formulas equated solve for the unknown.

**Complex numbers**

$|z|^2 = z\bar{z}$; $z = re^{i\theta}$; de Moivre $(re^{i\theta})^n = r^n e^{in\theta}$; $n$th roots of unity sum to 0 and are equally spaced; $x^n - 1 = \prod(x - \omega^k)$; $|z - a|$ is a distance; multiplication rotates by arg.

Convert to polar for powers and roots. Use $\omega^3 = 1$, $1+\omega+\omega^2 = 0$ to collapse sums. Treat geometry problems on regular polygons as roots-of-unity problems.

### 8.2 Algebra & Polynomials

**Logs and exponentials**

All log rules; $\log_a b \cdot \log_b c = \log_a c$; $a^{\log_a x} = x$; $x^{\log_b y} = y^{\log_b x}$. Exponential equations: match bases, or substitute $u = a^x$.

Take logs of products; exponentiate sums of logs. Check the domain ($x > 0$ inside a log) after solving.

**Polynomials in depth**

Vieta for any degree; Newton's sums $p_k = e_1 p_{k-1} - e_2 p_{k-2} + \cdots$; remainder theorem $P(a)$; factor theorem; conjugate root theorem; rational root theorem; roots-of-unity filter for coefficient sums; Descartes' rule for sign counts. Sum of coefficients $= P(1)$; alternating sum $= P(-1)$.

To evaluate symmetric expressions of roots, never find the roots. To count real roots, use derivatives (Rolle) and sign changes. Substitute $x = 1, -1, i, \omega$ to extract coefficient information.

**Sequences, series, recursion**

Characteristic equation for linear recurrences; generating functions $\sum a_n x^n$ turn recurrences into algebra; $\sum nx^n = \frac{x}{(1-x)^2}$; telescoping products; periodic recurrences (compute a cycle). Arithmetico-geometric sums by the shift-and-subtract trick.

Compute the first six terms of any recursion before theorizing. Look for a period. For sums with $n \cdot r^n$, differentiate the geometric series or subtract $rS$ from $S$.

### 8.3 Geometry, Counting & Probability

**Conics and coordinates**

Circle $(x-h)^2+(y-k)^2 = r^2$; parabola $y^2 = 4px$ (focus $(p, 0)$); ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $c^2 = a^2 - b^2$;

hyperbola $c^2 = a^2 + b^2$, asymptotes $y = \pm\frac{b}{a}x$. Distance from a point to a line; reflection across a line; rotation by $90°$: $(x, y) \to (-y, x)$.

Complete the square to identify a conic. Use the focus–directrix definition for "distance to a point equals distance to a line" problems.

**Advanced counting and probability**

Inclusion–exclusion for $n$ sets; derangements $D_n \approx \frac{n!}{e}$; Catalan $\frac{1}{n+1}\binom{2n}{n}$; Vandermonde $\sum \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$; hockey stick $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$. Geometric probability with areas; expected value by indicators; first-step equations for expected waiting times.

Translate "no two adjacent" into gaps, "paths not crossing" into Catalan, "sum of products" into Vandermonde. For expected values, write the linearity sum before anything else.

**The universal problem-solving loop**

Read twice. Restate the goal in your own words. List the givens. Pick a representation. Try small cases. Look for symmetry, an invariant, or an extremal object. Compute, then CHECK against a second method or a sanity bound.

At AMC 10/12 and AIME level, the second method is what separates a 4-second guess from a 12-second certainty: compute a probability two ways (complement and direct), a length via two theorems, a count via a recurrence and a formula. Casework must be organized by a stated criterion so nothing is double-counted; algebra must be checked by substituting back.