MathCastle

AMC 10

Complete course booklet • 9 topics

Contents

1 Circles & Lines

Distances, angle bisectors, tangents, and conic meetings

1.1 Distance: Point to Line

The point-to-line distance formula

The distance from (x_0, y_0) to the line $Ax + By + C = 0$ is $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$; parallel lines $Ax + By = C_1$, $Ax + By = C_2$ sit $\dfrac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$ apart.

In plain terms. Plug the point into the line's left side and divide by the length of (A, B) — it measures how far the point sits along the line's perpendicular (normal) direction. Never solve for the foot of the perpendicular unless the problem asks for it.

Example. From $(6, 7)$ to $3x + 4y - 10 = 0$: $d = \dfrac{|18 + 28 - 10|}{5} = \dfrac{36}{5}$.

Tangency is a distance statement

A line is tangent to a circle exactly when the center's distance to the line EQUALS the radius; it misses when the distance exceeds the radius, and cuts a chord when it is smaller.

In plain terms. One formula answers three different-looking questions: "is it tangent?", "what k makes it tangent?", and "how long is the chord?" (half-chord = $\sqrt{r^2 - d^2}$).

Example. $y = x + k$ is tangent to $x^2 + y^2 = 8$ when $\dfrac{|k|}{\sqrt{2}} = 2\sqrt{2}$, i.e. $k = \pm 4$.

1.2 The Bisector Between Two Lines

Angle bisectors as an equidistance locus

The bisectors of the angles between $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$ satisfy $\dfrac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} = \pm \dfrac{A_2x + B_2y + C_2}{\sqrt{A_2^2 + B_2^2}}$ — two perpendicular lines, one per sign.

In plain terms. A bisector is just the set of points at EQUAL distance from both lines, so set the two distance formulas equal. The $\pm$ gives both bisectors; pick the acute or obtuse one by testing a point.

Example. For $y = \frac{3}{4}x$ and the x-axis: $\dfrac{|3x - 4y|}{5} = |y|$ gives $y = \frac{1}{3}x$ (acute) and $y = -3x$ (obtuse) — note the two bisectors are perpendicular.

Symmetry shortcut

If swapping $x \leftrightarrow y$ (or another visible reflection) exchanges the two lines, then the mirror line of that reflection is automatically one of the bisectors.

In plain terms. Before computing, look for a symmetry that hands you the bisector for free.

Example. $3x + 4y = 12$ and $4x + 3y = 12$ swap under $x \leftrightarrow y$, so $y = x$ bisects them — no algebra needed.

1.3 Circles: Chords, Tangents, Power

The three right-triangle facts

Chord at distance d from the center: length $2\sqrt{r^2 - d^2}$. Tangent from an external point at distance d: length $\sqrt{d^2 - r^2}$. Common external tangent of circles (r_1, r_2) with centers d apart: $\sqrt{d^2 - (r_2 - r_1)^2}$ (internal: $\sqrt{d^2 - (r_1 + r_2)^2}$).

In plain terms. Every one of these is the SAME move: drop the radius to the chord or tangency point, get a right triangle, apply Pythagoras. Learn the move, not four formulas.

Example. Tangent from $(13, 0)$ to $x^2 + y^2 = 25$: $\sqrt{169 - 25} = 12$.

Nearest point of a circle to a line

If the center is at distance $d > r$ from a line, the circle's nearest point to the line is at distance $d - r$ and the farthest at $d + r$ — both along the perpendicular from the center.

In plain terms. Circles turn min/max distance questions into "center distance, then adjust by the radius."

Example. From $x^2 + y^2 = 9$ to $3x + 4y = 25$: center distance 5, so the minimum is $5 - 3 = 2$.

1.4 Circles Meet Parabolas

Intersecting a circle with a conic

Substitute the parabola into the circle and solve the resulting equation in $u = x^2$ (or y): each POSITIVE root u contributes two points ($x = \pm\sqrt{u}$), $u = 0$ one point, and negative roots none.

In plain terms. The whole art is bookkeeping: a quadratic in u can produce $0, 1, 2, 3$, or 4 intersection points depending on the signs of its roots — count carefully instead of guessing from a sketch.

Example. $y = x^2 - 4$ into $x^2 + y^2 = 4$: $u^2 - 7u + 12 = 0$ gives $u = 3, 4$, both positive — four intersection points.

Focus and directrix

The parabola $x^2 = 4py$ has focus $(0, p)$ and directrix $y = -p$; every point is equidistant from the two. In the form $y = ax^2$, $p = \dfrac{1}{4a}$.

In plain terms. When a contest problem says "focus", convert to $x^2 = 4py$ first — the geometry (equal distances) is what problems actually use.

Example. $y = \frac{x^2}{8}$ is $x^2 = 8y$: focus $(0, 2)$, directrix $y = -2$.

2 Algebra & Polynomials

Vieta, the remainder theorem, and systems

2.1 Polynomial Roots

Vieta for any degree

For a monic polynomial with roots $r_1, \ldots, r_n$: $\sum r_i = -a_{n-1}$, $\sum_{i<j} r_i r_j = a_{n-2}$, and $\prod r_i = (-1)^n a_0$.

In plain terms. The coefficients ARE the symmetric combinations of the roots (with signs). So you can get sums and products of roots straight from the polynomial.

Example. For $x^3 - 6x^2 + 11x - 6$: the roots sum to 6, sum in pairs to 11, and multiply to 6 — they are $1, 2, 3$.

Remainder and factor theorems

The remainder of $P(x)$ divided by $x - a$ is $P(a)$; so $x - a$ is a factor exactly when $P(a) = 0$.

In plain terms. To find the remainder or test a factor, just plug the number in — no long division needed.

Example. $P(x) = x^2 - 5x + 6$ has $P(2) = 0$, so $x - 2$ is a factor; indeed $P(x) = (x - 2)(x - 3)$.

Manipulate without solving

Symmetric expressions reduce to $S = r + s$ and $P = rs$: $r^2 + s^2 = S^2 - 2P$ and $\frac{1}{r} + \frac{1}{s} = \frac{S}{P}$.

In plain terms. You rarely need the actual roots — rewrite the expression using only their sum and product, which Vieta hands you.

Example. If $x^2 - 3x + 1 = 0$ then $x + \frac{1}{x} = 3$, so $x^2 + \frac{1}{x^2} = 3^2 - 2 = 7$.

3 Counting: Inclusion–Exclusion & Stars and Bars

The two workhorses of AMC counting

3.1 Inclusion–Exclusion

The principle

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

In plain terms. Add each group, subtract the double-counted overlaps, then add back what you subtracted too many times.

Example. Integers 1–100 divisible by $2, 3$, or 5: $50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$.

3.2 Stars and Bars

Distributing identical objects

Nonnegative solutions of $x_1 + \cdots + x_k = n$ number $\binom{n+k-1}{k-1}$; positive solutions number $\binom{n-1}{k-1}$.

In plain terms. Line up n identical items and $k-1$ dividers; where the dividers go decides who gets what. Counting the arrangements is one binomial.

Example. Give 10 candies to 3 kids: $\binom{12}{2} = 66$ ways; if each must get one, $\binom{9}{2} = 36$.

4 Number Theory: Modular Arithmetic

Congruences, Fermat, and bases

4.1 Working Modulo n

Congruence arithmetic

$a \equiv b \pmod{n}$ means $n \mid (a - b)$. You may add, subtract, and multiply congruences.

In plain terms. Only the remainder matters, so replace big numbers by their remainders and do easy arithmetic.

Example. 3^{100} mod 5: since $3^4 = 81 \equiv 1$, $3^{100} = (3^4)^{25} \equiv 1$.

Fermat's little theorem

If p is prime and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$.

In plain terms. Raising to the power $p-1$ resets you to 1 modulo a prime, so huge exponents shrink fast.

Example. 2^{100} mod 7: $2^6 \equiv 1$, and $100 = 6 \cdot 16 + 4$, so $2^{100} \equiv 2^4 = 16 \equiv 2$.

5 Geometry: Power of a Point & Mass Points

Circle power and weighted balancing

5.1 Power of a Point

The constant product

From a point P, along any line through P meeting a circle at A, B, the product $PA \cdot PB$ is constant; for a tangent of length t, $t^2 = PA \cdot PB$.

In plain terms. No matter which line you draw through a point, the two distances to the circle multiply to the same number.

Example. Chords AB and CD cross at P with $PA = 2, PB = 6, PC = 3$: then $PD = \dfrac{2 \cdot 6}{3} = 4$.

5.2 Mass Points and Coordinates

Mass points

Hang weights at the vertices so each side balances at its division point (weight $\times$ length equal on both sides); the weights reveal all segment ratios.

In plain terms. Treat the triangle like a mobile: put weights so it balances, and the balance conditions give the ratios you want.

Example. If a cevian splits a side $2 : 3$, put weights 3 and 2 at its endpoints so $3 \cdot 2 = 2 \cdot 3$ balances at the division point.

When in doubt, use coordinates

Place the figure on axes; use the distance formula, perpendicular slopes, and the shoelace area formula.

In plain terms. Give the points coordinates and the geometry turns into algebra you can just compute.

Example. A right angle at the origin with legs on the axes makes the third vertex, lengths, and area immediate.

6 Algebraic Weapons

Factorizations and substitutions that crack hard problems

6.1 Identities That Do the Work

Difference of squares, everywhere

$a^2 - b^2 = (a-b)(a+b)$ — the most-used identity in competition algebra.

In plain terms. A difference of two squares always splits into a product, often collapsing huge arithmetic.

Example. $51^2 - 49^2 = (51-49)(51+49) = 2 \cdot 100 = 200$ — no squaring needed.

Symmetric sums via Vieta

By **Vieta's formulas**, everything symmetric in two roots can be written using $x+y$ and xy: for instance $x^2 + y^2 = (x+y)^2 - 2xy$.

In plain terms. You rarely need the roots themselves — just their sum and product.

Example. If $x + y = 6$ and $xy = 5$, then $x^2 + y^2 = 36 - 10 = 26$.

The substitution $u = x + \frac{1}{x}$

From $u = x + \frac{1}{x}$: $x^2 + \frac{1}{x^2} = u^2 - 2$ and $x^3 + \frac{1}{x^3} = u^3 - 3u$.

In plain terms. Expressions built from x and $1/x$ collapse into a polynomial in u.

Example. If $x + \frac{1}{x} = 3$, then $x^2 + \frac{1}{x^2} = 9 - 2 = 7$.

7 Where the Formulas Come From

The quadratic formula, series sums, and circle power — derived

7.1 The Quadratic Formula, Derived

Derivation by completing the square

From $ax^2+bx+c = 0$: divide by a, move the constant, and complete the square: $\left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2-4ac}{4a^2}$. Square-root both sides and solve: $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. ■

In plain terms. The formula is not magic — it is completing the square ONCE, in general, so you never have to do it again.

Example. $x^2 - 6x + 5 = 0$: $(x-3)^2 = 4$, so $x = 3 \pm 2$ — the formula gives the same 5 and 1.

(diagram in the online version)

Proof of Vieta's formulas

If r and s are the roots, the quadratic factors as $a(x-r)(x-s) = a\left(x^2-(r+s)x+rs\right)$. Matching coefficients with ax^2+bx+c: $r+s=-\dfrac{b}{a}$ and $rs=\dfrac{c}{a}$. ■

In plain terms. Multiply the factored form back out — the sum and product of the roots are sitting in the coefficients.

Example. $x^2-7x+12$: roots sum to 7 and multiply to 12 (3 and 4), read off with no solving.

Why the discriminant decides everything

In the derivation, the only square root taken is $\sqrt{b^2-4ac}$. Positive: two real roots. Zero: the $\pm$ collapses — one repeated root. Negative: no real square root — no real roots. The discriminant is simply WHAT SITS UNDER THE RADICAL. ■

In plain terms. One number tells you how many real answers exist before you solve anything.

Example. x^2+x+1: $\Delta = 1-4 = -3 < 0$, so no real roots.

7.2 The Series Formulas, Derived

Arithmetic series: proof of $S_n = \frac{n}{2}(a_1+a_n)$

Write the series forwards and backwards and add: each column gives a_1+a_n (what one end gains, the other loses), so $2S_n = n(a_1+a_n)$. ■

In plain terms. Same Gauss pairing as $1+\cdots+100$, just starting anywhere and stepping by anything.

Example. $5+8+11+\cdots+35$ (11 terms): $S=\frac{11}{2}(5+35)=220$.

Geometric series: proof of $S_n = a\dfrac{1-r^n}{1-r}$

Multiply the sum by r and subtract: $S_n - rS_n = a - ar^n$ — every middle term appears once in each copy and cancels. Divide by $1-r$. ■ For $|r|<1$, $r^n \to 0$ gives the infinite form $S=\dfrac{a}{1-r}$.

In plain terms. Shifting the whole series one step and subtracting wipes out everything but the two ends.

Example. $1+\frac{1}{2}+\frac{1}{4}+\cdots = \dfrac{1}{1-\frac{1}{2}} = 2$.

7.3 Circle Power, Proved

Proof of the inscribed angle theorem

First the case where one chord is a diameter: the triangle from the center is isosceles (two radii), so the central angle, an exterior angle of it, equals the sum of two equal base angles — twice the inscribed angle. Any other position splits (or subtracts) into two such cases. ■

In plain terms. An angle at the rim is always half the angle at the center over the same arc — proved with one isosceles triangle.

Example. An angle inscribed in a semicircle sits on a 180° arc, so it is 90° — Thales' theorem for free.

(diagram in the online version)

Proof of Power of a Point (two chords)

Chords AB and CD meet at P. Angles $\angle PAC$ and $\angle PDB$ subtend the SAME arc, so they are equal; likewise $\angle PCA = \angle PBD$. Hence $\triangle PAC \sim \triangle PDB$, and matching sides gives $\dfrac{PA}{PD} = \dfrac{PC}{PB}$, i.e. $PA \cdot PB = PC \cdot PD$. ■

In plain terms. The inscribed angle theorem hands you two similar triangles, and the product formula is just their ratio cross-multiplied.

Example. Pieces $3, 8$ and $4, x$: $3 \cdot 8 = 4x$ gives $x = 6$.

(diagram in the online version)

7.4 AM–GM, Proved

Proof: $\dfrac{a+b}{2} \geq \sqrt{ab}$ for $a, b \geq 0$

Squares are never negative: $\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0$. Expand: $a - 2\sqrt{ab} + b \geq 0$, i.e. $a + b \geq 2\sqrt{ab}$ — with equality exactly when $\sqrt{a} = \sqrt{b}$, i.e. $a = b$. ■

In plain terms. The whole inequality is one squared bracket refusing to be negative. Equality means the bracket was zero: the numbers were equal.

Example. For $a + b = 20$, the product ab is largest at $a = b = 10$: $ab \leq \left(\frac{20}{2}\right)^2 = 100$.

Using the equality case on purpose

AM–GM problems are usually about WHERE equality happens: to optimize, arrange the terms so they can all be equal, then read off the extreme value.

In plain terms. The maximum or minimum almost always happens at "all parts equal."

Example. Minimize $x + \dfrac{9}{x}$ for $x > 0$: equality of x and $\frac{9}{x}$ at $x = 3$ gives minimum 6.

8 Complete Syllabus & Gap Fillers

Everything the AMC 10 tests, with the pieces that were missing

8.1 What the AMC 10 Tests

The checklist

Everything on the AMC 8, plus: quadratics and Vieta; polynomial remainder/factor theorems; systems of equations; exponents and radicals; basic logarithms (rarely) and no trigonometry; modular arithmetic, bases, divisor counting; counting with inclusion–exclusion, stars and bars, casework and complements; probability with conditional reasoning and expected value basics; geometry with similar triangles, power of a point, inscribed angles, area ratios, coordinate and 3D geometry (volumes, surface areas); sequences (arithmetic, geometric, telescoping). Twenty-five questions, seventy-five minutes.

8.2 Gap Fillers

3D geometry

Cube diagonals ($s\sqrt{3}$), cone and pyramid volumes ($\frac{1}{3}$ base $\times$ height), sphere ($\frac{4}{3}\pi r^3$, surface $4\pi r^2$), and "unfold the surface" for shortest paths on boxes. A cube with surface area 150 has edge 5 and volume 125.

Expected value, the fast way

Linearity: $E[X+Y] = E[X] + E[Y]$ with NO independence needed. The expected number of fixed points of a random permutation is 1; the expected sum of two dice is 7; "expected number of steps" problems set up first-step equations $E = 1 + \sum p_i E_i$.

Circles: the four angle facts

Inscribed angle is half its arc; angles from the same arc are equal; a tangent makes a right angle with the radius; the angle between a tangent and a chord equals the inscribed angle on the other side. Power of a point ($PA \cdot PB = PT^2$) converts lengths on secants and tangents.

9 Formulas & Methods

The AMC 10 formula sheet with the methods behind each entry

9.1 Algebra

Quadratics and Vieta

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$; roots sum to $-\frac{b}{a}$, multiply to $\frac{c}{a}$; vertex at $x = -\frac{b}{2a}$. For a cubic x^3+px^2+qx+r: roots sum to $-p$, pairwise products q, product $-r$. Difference of squares, sum/difference of cubes, $(a+b+c)^2 = \sum a^2 + 2\sum ab$.

Use Vieta whenever a problem asks about roots without asking for them. Complete the square for vertex/extremum questions. Factor by grouping or by testing rational roots.

Exponents, radicals, logs (light)

$a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, $a^{-n} = \frac{1}{a^n}$, $a^{1/n} = \sqrt[n]{a}$. Rationalize with conjugates. $\log_b(xy) = \log_b x + \log_b y$, $\log_b x^n = n \log_b x$, change of base $\log_b x = \frac{\ln x}{\ln b}$.

Rewrite everything in one base before comparing. Substitute $u = 2^x$ (or similar) to turn exponential equations into polynomials.

Sequences and telescoping

Arithmetic: $S_n = \frac{n(a_1+a_n)}{2}$. Geometric: $S_n = a\frac{1-r^n}{1-r}$, infinite $\frac{a}{1-r}$ for $|r| < 1$. $\sum k^2 = \frac{n(n+1)(2n+1)}{6}$, $\sum k^3 = \left(\frac{n(n+1)}{2}\right)^2$. Telescope: $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$.

Write out the first three and last term of any sum. Products of fractions often telescope after factoring numerators and denominators.

9.2 Counting, Probability & Number Theory

Counting

Permutations, combinations, $\binom{n}{k} = \binom{n}{n-k}$, Pascal $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. Stars and bars: $\binom{n+k-1}{k-1}$ nonnegative solutions of $x_1 + \cdots + x_k = n$. Inclusion–exclusion for two or three sets. Circular arrangements $(n-1)!$.

Decide ordered vs unordered first. Complement for "at least." Casework on a stated criterion. Bijection to a simpler set when the direct count is messy.

Probability and expected value

Conditional $P(A \mid B) = \frac{P(A\cap B)}{P(B)}$; independent events multiply. Expected value $\sum x\, P(x)$; linearity: $E[X+Y] = E[X] + E[Y]$ always. Geometric probability: area ratios.

Draw the sample space; for continuous problems draw the region. Use indicators for expected counts.

Number theory

Modular arithmetic; Fermat $a^{p-1} \equiv 1$; Euler $a^{\varphi(n)} \equiv 1$; $\varphi(n) = n\prod(1 - \frac{1}{p})$. Divisor count and sum. Base conversion. Legendre: $v_p(n!) = \sum \lfloor n/p^k \rfloor$. CRT for simultaneous congruences.

Reduce mod small numbers to kill cases. Factor N before counting divisors. For trailing zeros count factors of 5.

9.3 Geometry

Triangles

Heron: $\sqrt{s(s-a)(s-b)(s-c)}$. Area $= rs = \frac{abc}{4R} = \frac{1}{2}ab\sin C$. Law of cosines $c^2 = a^2 + b^2 - 2ab\cos C$. Angle bisector theorem $\frac{BD}{DC} = \frac{AB}{AC}$. Similar triangles: ratios of corresponding sides; areas scale by the square.

Look for similar triangles created by parallel lines. Compute area two ways to find an unknown height or inradius. Special triangles 30-60-90 and 45-45-90 by sight.

Circles and polygons

Inscribed angle $= \frac{1}{2}$ arc; tangent $\perp$ radius; power of a point $PA \cdot PB = PC \cdot PD = PT^2$; chord length $2\sqrt{r^2 - d^2}$. Regular n-gon: interior angle $\frac{(n-2)180^\circ}{n}$, area $\frac{1}{2}$ perimeter $\times$ apothem. Sector area $\frac{\theta}{360}\pi r^2$.

Connect centers to tangent points and to chord midpoints (right angles appear). For two circles, the line of centers, the radii, and the tangent form right triangles.

Coordinates and 3D

Distance, midpoint, slope; perpendicular slopes multiply to -1; point-to-line $\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$; shoelace for polygon area. Solids: prism/cylinder Bh, pyramid/cone $\frac{1}{3}Bh$, sphere $\frac{4}{3}\pi r^3$ and $4\pi r^2$; cube diagonal $s\sqrt{3}$.

Place the figure so that a vertex is the origin and a side lies on an axis. For 3D shortest paths, unfold the surface into a plane.

The universal problem-solving loop

Read twice. Restate the goal in your own words. List the givens. Pick a representation. Try small cases. Look for symmetry, an invariant, or an extremal object. Compute, then CHECK against a second method or a sanity bound.

At AMC 10/12 and AIME level, the second method is what separates a 4-second guess from a 12-second certainty: compute a probability two ways (complement and direct), a length via two theorems, a count via a recurrence and a formula. Casework must be organized by a stated criterion so nothing is double-counted; algebra must be checked by substituting back.