MathCastle

Algebra 2

Complete course booklet • 13 topics

Contents

1 Core Ideas in Plain Terms

The next layer, in everyday language

1.1 Functions and Their Graphs

A function is an input-output machine

$f(x)$ means "the output when the input is x." Small changes to the formula move the graph in predictable ways: $f(x)+k$ shifts it up, $f(x-h)$ shifts it right, and a minus sign flips it. Learn the handful of "parent" shapes and everything else is a shifted, stretched, or flipped version.

Shift a parabola

$y = (x-2)^2 + 3$ is $y = x^2$ moved right 2 and up 3, so its lowest point is $(2, 3)$.

(diagram in the online version)

1.2 Complex Numbers

i is the square root of 1

Some equations have no real solution because you'd need the square root of a negative. Mathematicians define $i = \sqrt{-1}$, so $i^2 = -1$. A **complex number** $a + bi$ just has a real part and an "imaginary" part; you add and multiply them like binomials, using $i^2 = -1$.

Multiply complex numbers

$(2 + i)(3 - i) = 6 - 2i + 3i - i^2 = 6 + i + 1 = 7 + i$.

1.3 Exponentials and Logarithms

Logs undo exponents

$\log_b x = y$ is just another way to write $b^y = x$ — it asks "what power turns b into x?" That is why logs are the tool for solving equations where the unknown is stuck in an exponent, and why they turn multiplication into addition.

Solve $2^x = 8$

$\log_2 8 = 3$ because $2^3 = 8$.

1.4 Going Deeper: Polynomials and Rational Functions

The remainder theorem in one line

Plugging a into a polynomial gives the remainder when dividing by $(x-a)$: $P(a)$ *is* the remainder. If $P(a)=0$, the division is exact and $(x-a)$ is a factor — that is how you hunt for roots.

Check it

$P(x)=x^3-2x^2+x-2$ at $x=2$: $8-8+2-2=0$. So $(x-2)$ is a factor, and dividing gives $P(x)=(x-2)(x^2+1)$ — the other two roots are the complex pair $\pm i$.

Asymptotes, plainly

A rational function blows up where its denominator hits 0 (vertical asymptotes), and for huge x it flattens toward the ratio of the leading terms (horizontal asymptote). $y=\dfrac{2x+1}{x-3}$: vertical at $x=3$, horizontal at $y=2$.

A line meets a parabola (multi-step)

Find the intersections of $y=x^2-2x-3$ and $y=x+1$. Step 1 — set the formulas equal: $x^2-2x-3=x+1$. Step 2 — bring everything to one side: $x^2-3x-4=0$, which factors as $(x-4)(x+1)=0$. Step 3 — **zero-product property**: $x=4$ or $x=-1$, so the points are $(4,5)$ and $(-1,0)$.

(diagram in the online version)

1.5 Problem-Solving Playbook

Substitute to simplify

A scary equation is often a familiar one in disguise. $x^4-13x^2+36=0$ is a **quadratic in** $u=x^2$. Spot the pattern, substitute, solve the easy version, then translate back.

Worked: quadratic in disguise

$x^4-13x^2+36=0$. Let $u=x^2$: $u^2-13u+36=(u-4)(u-9)=0$, so $u=4$ or $u=9$. Back-substitute: $x^2=4$ or $x^2=9$, giving $x=\pm2,\pm3$.

2 Conic Sections

Circles, parabolas, ellipses & hyperbolas — with worked examples and labeled graphs

2.1 The Four Conics & Review Formulas

Big idea

A **conic section** is a curve you get by slicing a double cone with a flat plane. Tilting the plane a little more each time produces, in order, a **circle**, an **ellipse**, a **parabola**, and a **hyperbola**. Every one of them can be written as a single second-degree equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

and the whole topic is really one question asked four ways: *what shape does this equation draw, and where are its key points?*

Two tools you already own (review)

For points (x_1, y_1) and (x_2, y_2):

Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, **Midpoint:** $M = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$.

The distance formula *is* the circle equation in disguise (a circle is all points a fixed distance from the center), so keep it close.

Example: distance and midpoint

For $(-3, -2)$ and $(5, 4)$:

$$d = \sqrt{(5 - (-3))^2 + (4 - (-2))^2} = \sqrt{8^2 + 6^2} = \sqrt{100} = 10,$$

$$M = \left(\tfrac{-3+5}{2}, \tfrac{-2+4}{2}\right) = (1, 1).$$

Tip: A circle is just a special ellipse where both radii are equal. If you understand the ellipse deeply, the circle is free.

2.2 Circles

Standard equation of a circle

A circle with **center** (h, k) and **radius** r is the set of all points a distance r from the center:

$$(x - h)^2 + (y - k)^2 = r^2.$$

Read the center straight off the equation as the *opposites* of the numbers inside the parentheses, and take $r = \sqrt{r^2}$.

Example: center and radius

For $(x-3)^2 + (y+2)^2 = 25$ the center is $(3, -2)$ (note $y + 2 = y - (-2)$) and the radius is $r = \sqrt{25} = 5$.

Example: completing the square (general form $\rightarrow$ standard form)

Convert $x^2 + y^2 - 6x + 4y - 12 = 0$ into standard form.

Group the x's and y's and move the constant:

$$(x^2 - 6x) + (y^2 + 4y) = 12.$$

Complete each square by adding $\left(\frac{-6}{2}\right)^2 = 9$ and $\left(\frac{4}{2}\right)^2 = 4$ to *both* sides:

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 12 + 9 + 4,$$

$$(x-3)^2 + (y+2)^2 = 25.$$

So the center is $(3, -2)$ and the radius is $r = 5$.

Tip: When you complete the square, whatever you add on the left you must add on the right. A common slip is adding 9 and 4 only on one side.

2.3 Parabolas as Conics

Vertex form and direction of opening

A parabola is every point equidistant from a fixed **focus** and a fixed line called the **directrix**. In **vertex form**:

$$y = a(x-h)^2 + k \quad \text{opens up } (a > 0) \text{ or down } (a < 0),$$

$$x = a(y-k)^2 + h \quad \text{opens right } (a > 0) \text{ or left } (a < 0).$$

Either way the **vertex** is (h, k): it is the turning point, halfway between the focus and the directrix.

Focus and directrix (intro)

Write an upward parabola as $y = \dfrac{1}{4p}(x-h)^2 + k$. Then p is the distance from the vertex to the focus (and to the directrix):

$$\textbf{focus } (h,\, k+p), \qquad \textbf{directrix } y = k - p.$$

The focus sits *inside* the curve; the directrix is the same distance out the other side.

Example: vertex, focus, directrix

For $y = \frac{1}{4}x^2$ we have $\frac{1}{4p} = \frac{1}{4}$, so $4p = 4$ and $p = 1$. The vertex is $(0, 0)$, the parabola opens up, the focus is $(0, 1)$, and the directrix is the line $y = -1$.

Tip: An x^2 term (and no y^2) means the parabola opens up or down; a y^2 term (and no x^2) means it opens left or right. A parabola is the only conic with just *one* squared variable.

2.4 Ellipses

Standard equation of an ellipse (center at origin)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0.$$

The larger denominator sits under the axis holding the **major axis** (the long one). Let a be the larger of the two, b the smaller. Then:

- **Vertices**: endpoints of the major axis, distance a from center.
- **Co-vertices**: endpoints of the minor axis, distance b from center.
- **Foci**: distance c from center *along the major axis*, where $c^2 = a^2 - b^2$.

A shifted ellipse uses $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ with center (h, k).

Example: full feature analysis

Analyze $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

The larger denominator (25) is under x^2, so the major axis is **horizontal** with $a = \sqrt{25} = 5$ and $b = \sqrt{9} = 3$. Then $c^2 = a^2 - b^2 = 25 - 9 = 16$, so $c = 4$.

- Center $(0, 0)$; Vertices $(\pm 5, 0)$; Co-vertices $(0, \pm 3)$; Foci $(\pm 4, 0)$.

Tip: For an ellipse $a > b$ and the foci hug the *long* axis, so $c < a$ always. If you ever get $c^2 < 0$, you mislabeled a and b.

2.5 Hyperbolas

Standard equation of a hyperbola (center at origin)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (opens left–right)}, \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \text{ (opens up–down)}.$$

The **positive** term tells you which way it opens; a is always the number under the positive term.

- **Vertices**: distance a from center, on the opening axis.
- **Foci**: distance c from center, where $c^2 = a^2 + b^2$ (note the +!).
- **Asymptotes**: for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ they are $y = \pm\frac{b}{a}x$; for the up–down form they are $y = \pm\frac{a}{b}x$.

Example: full feature analysis

Analyze $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

The x^2 term is positive, so it opens left–right with $a = \sqrt{9} = 3$ and $b = \sqrt{16} = 4$. Then $c^2 = a^2 + b^2 = 9 + 16 = 25$, so $c = 5$.

- Center $(0, 0)$; Vertices $(\pm 3, 0)$; Foci $(\pm 5, 0)$; Asymptotes $y = \pm\frac{4}{3}x$.

Tip: Ellipse uses $c^2 = a^2 - b^2$; hyperbola uses $c^2 = a^2 + b^2$. For a hyperbola the foci are *farther* out than the vertices, so $c > a$.

2.6 Identifying a Conic from General Form

The classification test

Given $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (no xy term), look only at A and C:

- **Circle**: $A = C$ (same coefficient on x^2 and y^2).
- **Parabola**: exactly one of A, C is 0 (only one squared term).
- **Ellipse**: $A \neq C$ but A and C have the *same* sign.

- **Hyperbola**: A and C have *opposite* signs.

Example: classify each

- $x^2 + y^2 - 4x = 1$: $A = C = 1 \Rightarrow$ **circle**.
- $y = 3x^2 - x$ (only x^2): **parabola**.
- $4x^2 + 9y^2 = 36$: $A = 4$, $C = 9$ same sign, unequal $\Rightarrow$ **ellipse**.
- $9x^2 - 4y^2 = 36$: $A = 9$, $C = -4$ opposite signs $\Rightarrow$ **hyperbola**.

Tip (how to tell them apart at a glance): *One* squared term $\Rightarrow$ parabola. *Two* squared terms: same number $\Rightarrow$ circle, added $\Rightarrow$ ellipse, subtracted $\Rightarrow$ hyperbola.

2.7 Applications (light)

Where conics show up

- **Ellipses — orbits:** planets and satellites travel in ellipses with the larger body at one *focus* (Kepler's First Law).
- **Ellipses — whispering galleries:** a sound starting at one focus reflects off the elliptical ceiling and gathers at the other focus, so a whisper carries across the room.
- **Parabolas — dishes and headlights:** a satellite dish or reflector collects incoming rays at its focus (and a bulb at the focus sends out a straight beam).
- **Hyperbolas — navigation and comets:** some comets swing past the sun on an open hyperbolic path and never return.

Tip: The *focus* is the star of every application. Ellipses and hyperbolas have two foci; a parabola has one. That reflecting property is exactly why the focus matters.

2.8 Going Deeper: Advanced Conics

Building a conic from conditions

So far you have *read* a conic off its equation. The reverse skill — **constructing** the equation from geometric conditions — is what real problems ask for. Each unknown coefficient needs one condition:

- **Through given points:** plug each point into the general form and solve the resulting linear system for the coefficients.

- **Given foci and eccentricity:** the foci fix the center and c; then $e = \dfrac{c}{a}$ gives a, and b comes from $b^2 = a^2 - c^2$ (ellipse) or $b^2 = c^2 - a^2$ (hyperbola).
- **Given a tangent line:** impose the tangency condition (next box) as one more equation.

Eccentricity e measures how "stretched" a conic is: $e = 0$ circle, $0 < e < 1$ ellipse, $e = 1$ parabola, $e > 1$ hyperbola.

Example: ellipse from foci and eccentricity

Find the ellipse centered at the origin with foci $(\pm 3, 0)$ and eccentricity $e = \dfrac{3}{5}$.

The foci are on the x-axis at distance $c = 3$, so the major axis is horizontal. From $e = \dfrac{c}{a}$,

$$a = \frac{c}{e} = \frac{3}{3/5} = 5, \qquad a^2 = 25.$$

Recompute b using $c^2 = a^2 - b^2$:

$$b^2 = a^2 - c^2 = 25 - 9 = 16.$$

So the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{16} = 1,$$

with vertices $(\pm 5, 0)$, co-vertices $(0, \pm 4)$, and foci $(\pm 3, 0)$ as required.

The line–conic tangency condition (discriminant $= 0$)

To find where a line meets a conic, substitute the line into the conic. You get a *quadratic* in one variable, so the discriminant $\Delta = B^2 - 4AC$ of *that* quadratic tells you everything:

$$\Delta > 0 \;\Rightarrow\; 2 \text{ points (secant)}, \quad \Delta = 0 \;\Rightarrow\; 1 \text{ point (tangent)}, \quad \Delta < 0 \;\Rightarrow\; \text{no intersection}.$$

So "the line is tangent to the conic" translates into the single algebraic equation $\Delta = 0$. For the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ this collapses to the tidy rule: the line $y = mx + k$ is tangent exactly when

$$k^2 = a^2 m^2 + b^2.$$

Example: tangent lines by the discriminant

For which values of k is the line $y = x + k$ tangent to $\dfrac{x^2}{4} + y^2 = 1$?

Substitute $y = x + k$ and clear the fraction (multiply by 4):

$$\frac{x^2}{4} + (x+k)^2 = 1 \implies x^2 + 4(x+k)^2 = 4.$$

Expand into a quadratic in x:

$$5x^2 + 8kx + (4k^2 - 4) = 0.$$

Set its discriminant to zero:

$$\Delta = (8k)^2 - 4(5)(4k^2 - 4) = 64k^2 - 80k^2 + 80 = -16k^2 + 80 = 0,$$

so $k^2 = 5$ and $k = \pm\sqrt{5}$. Checking against the shortcut with $a^2 = 4$, $b^2 = 1$, $m = 1$: $k^2 = a^2m^2 + b^2 = 4 + 1 = 5$. ✓

Latus rectum, focal chords, and the reflection property

A **focal chord** is any chord through a focus. The special focal chord *perpendicular* to the major (or focal) axis is the **latus rectum**, and its length measures how "wide" the curve is at the focus:

$$\text{ellipse / hyperbola: } \frac{2b^2}{a}, \qquad \text{parabola } y = \tfrac{1}{4p}x^2 : \ 4p.$$

The **reflection property** explains every application from Section 7: a ray aimed at one focus of an ellipse bounces to the *other* focus; a ray parallel to a parabola's axis reflects through its *single* focus; and a ray toward one focus of a hyperbola reflects away along the line to the *other* focus.

Example: latus rectum of an ellipse

For $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ we have $a = 5$, $b = 4$, so each latus rectum has length

$$\frac{2b^2}{a} = \frac{2(16)}{5} = \frac{32}{5} = 6.4.$$

Since the foci are $(\pm 3, 0)$, the latus-rectum endpoints above and below the right focus are $\left(3, \pm\dfrac{16}{5}\right) = (3, \pm 3.2)$.

Classifying the full second-degree equation via $B^2 - 4AC$

When an xy term is present the conic is *rotated*, and reading A vs. C is no longer enough. The general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is classified by the **invariant** $B^2 - 4AC$:

$$B^2 - 4AC < 0 \;\Rightarrow\; \textbf{ellipse} \text{ (circle if } A = C,\; B = 0),$$

$$B^2 - 4AC = 0 \;\Rightarrow\; \textbf{parabola}, \qquad B^2 - 4AC > 0 \;\Rightarrow\; \textbf{hyperbola}.$$

This quantity is unchanged by any rotation of the axes, which is why it works even when $B \neq 0$. (When $B = 0$ it agrees with the sign test of Section 6.)

Example: classify a rotated conic

Classify $2x^2 + 3xy + 2y^2 - 5 = 0$. Here $A = 2$, $B = 3$, $C = 2$, so

$$B^2 - 4AC = 3^2 - 4(2)(2) = 9 - 16 = -7 < 0,$$

an **ellipse** (tilted, since $B \neq 0$). By contrast $x^2 - 4xy + 4y^2 + x = 0$ gives $B^2 - 4AC = 16 - 4(1)(4) = 0$, a **parabola**.

Degenerate conics

A "conic" equation does not always draw a smooth curve. When the slicing plane passes through the *apex* of the cone you get a **degenerate** conic:

- **A single point** — e.g. $x^2 + y^2 = 0$ is only $(0, 0)$ (a "circle of radius 0").
- **One line / two parallel lines** — e.g. $x^2 = 4$ gives $x = \pm 2$ (a degenerate parabola).
- **Two intersecting lines** — e.g. $x^2 - y^2 = 0$ gives $y = \pm x$ (a degenerate hyperbola).
- **Empty set (no points)** — e.g. $x^2 + y^2 = -1$ has no real solutions.

Watch for these after completing the square: a right-hand side of 0 or a negative number is the tell.

Shortest distance from a point to a conic

The closest point Q on a conic to an external point P has a clean geometric signature: the segment PQ is **perpendicular to the tangent line** at Q (it lies along the *normal*). For a circle this makes the answer immediate — the nearest point lies on the line through P and the center:

$$\text{distance from } P \text{ to circle} = \left|\, \overline{PC} - r \,\right|,$$

where C is the center and r the radius. For other conics you either minimize the squared distance d^2 with calculus or impose the normal condition and solve.

Example: point to a circle

Find the shortest distance from $P = (8, 0)$ to the circle $(x-2)^2 + y^2 = 9$.

The center is $C = (2, 0)$ and $r = 3$. The distance from P to the center is

$$\overline{PC} = \sqrt{(8-2)^2 + (0-0)^2} = \sqrt{36} = 6.$$

Since P is outside the circle $(6 > 3)$, the shortest distance is

$$\left|\overline{PC} - r\right| = |6-3| = 3,$$

reached at the point $(5, 0)$ where the segment PC crosses the circle.

Tip: Discriminants run this whole section. $B^2 - 4AC$ (from the *equation*) tells you *which* conic; the discriminant of the substituted quadratic (from a line-plus-conic) tells you *how a line meets it*. Same tool, two jobs.

3 Probability & Statistics

Counting, probability, and the shape of data — one desk reference

3.1 The Fundamental Counting Principle

Multiply the Choices

If a task is made of several independent stages, and stage 1 can happen in m ways, stage 2 in n ways, and so on, then the whole task can happen in

$$m \times n \times \cdots \text{ ways.}$$

This is the engine behind every counting formula in this topic: whenever choices are made **in sequence**, you **multiply**.

Worked Example: Building an Outfit

A wardrobe has 4 shirts, 3 pairs of pants, and 2 pairs of shoes. How many outfits (one of each) are possible?

$$4 \times 3 \times 2 = 24 \text{ outfits.}$$

If a password uses 3 digits and *repeats are allowed*, there are $10 \times 10 \times 10 = 1000$ passwords.

Repeats allowed or not? If items *can* repeat, each stage keeps the full count ($10 \times 10 \times 10$). If they *cannot* repeat, the pool shrinks by one each time ($10 \times 9 \times 8$).

3.2 Permutations and Combinations

Order Matters vs. Order Doesn't

Both count ways to choose r items from n distinct items.

- **Permutation** — order *matters* (arrangements, rankings, seating):

$$P(n, r) = {}_nP_r = \frac{n!}{(n-r)!}.$$

- **Combination** — order does *not* matter (committees, teams, hands):

$$C(n, r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}.$$

Because a combination ignores the $r!$ orderings of each group, $\binom{n}{r} = \dfrac{P(n,r)}{r!}$.

Worked Example: Same Numbers, Different Question

From 5 students, form a line of 3 (permutation) versus a team of 3 (combination).

$$P(5,3) = \frac{5!}{2!} = 5 \times 4 \times 3 = 60 \quad \text{(order matters)}$$
$$C(5,3) = \binom{5}{3} = \frac{5 \times 4 \times 3}{3!} = \frac{60}{6} = 10 \quad \text{(order doesn't)}$$

There are exactly $3! = 6$ times as many lines as teams, because each team of 3 can be lined up in 6 ways.

Decision test: Ask "If I swap two chosen items, is it a different outcome?" **Yes** $\Rightarrow$ permutation (order matters). **No** $\Rightarrow$ combination.

3.3 Basic Probability

Theoretical, Experimental, Complement

For equally likely outcomes,

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}, \qquad 0 \le P \le 1.$$

Theoretical probability comes from the model (a fair die gives $\frac{1}{6}$). **Experimental** probability comes from data: $\dfrac{\text{times it happened}}{\text{trials}}$. The **complement** rule saves work:

$$P(\text{not } A) = 1 - P(A).$$

Worked Example: Marbles and a Complement

A bag holds 3 red and 5 blue marbles (8 total).

$$P(\text{red}) = \frac{3}{8}, \qquad P(\text{not red}) = 1 - \frac{3}{8} = \frac{5}{8}.$$

Over many draws, the experimental fraction of reds should hover near $\frac{3}{8} = 0.375$ (the Law of Large Numbers).

"At least one" trick: It is almost always easier to compute $1 - P(\text{none})$ than to add up every "one or more" case.

3.4 Independent & Dependent Events; Conditional Probability

The Multiplication Rule

For the probability that **both** A *and* B happen:

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A).$$

- **Independent** — one outcome doesn't affect the other, so $P(B \mid A) = P(B)$ and the rule becomes $P(A) \cdot P(B)$. (Coin flips, drawing *with* replacement.)
- **Dependent** — the first outcome changes the second (drawing *without* replacement).

Conditional probability isolates $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)}$.

Worked Example: Dependent Draw (No Replacement)

A deck of 52 cards: draw two, without replacement. Find $P(\text{both hearts})$. There are 13 hearts.

$$P(\text{both hearts}) = \frac{13}{52} \cdot \frac{12}{51} = \frac{1}{4} \cdot \frac{12}{51} = \frac{12}{204} = \frac{1}{17} \approx 0.059.$$

The second fraction uses 12 and 51 because one heart is already gone.

Worked Example: Conditional Probability

In a class, 60% play a sport and 25% play a sport *and* an instrument. Given a student plays a sport, the chance they also play an instrument is

$$P(\text{instrument} \mid \text{sport}) = \frac{P(\text{both})}{P(\text{sport})} = \frac{0.25}{0.60} = \frac{5}{12} \approx 0.417.$$

With vs. without replacement. "With replacement" ⇒ independent (denominators stay the same). "Without replacement" ⇒ dependent (denominator drops by 1 each draw).

3.5 Mutually Exclusive & Overlapping Events (Addition Rule)

The Addition Rule

For the probability that A *or* B happens:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

- **Mutually exclusive** (disjoint) — can't both happen, so $P(A \text{ and } B) = 0$ and the rule is just $P(A) + P(B)$.
- **Overlapping** — can happen together, so subtract the overlap once (it was counted twice).

Worked Example: Overlap in a Deck

Draw one card. Find $P(\text{king or heart})$. There are 4 kings, 13 hearts, and the king of hearts is in both.

$$P(\text{king or heart}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

If we had *added* without subtracting, the king of hearts would be double-counted.

Spot the overlap. If two categories can describe the *same* outcome, they overlap — always subtract $P(A \text{ and } B)$. If they never coincide, the overlap is 0.

3.6 Binomial Probability (Introduction)

Counting Successes in Repeated Trials

A **binomial** experiment has: (1) a fixed number of trials n, (2) two outcomes per trial — success or failure, (3) a constant success probability p, and (4) independent trials. The probability of **exactly** k successes is

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Here $\binom{n}{k}$ counts *which* trials succeed, p^k is their successes, and $(1-p)^{n-k}$ the remaining failures.

Worked Example: Exactly 3 Heads in 5 Flips

A fair coin, so $p = \frac{1}{2}$, $n = 5$, $k = 3$:

$$P(3) = \binom{5}{3} \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^2 = 10 \cdot \frac{1}{32} = \frac{10}{32} = \frac{5}{16} = 0.3125.$$

"At least" means add up cases: $P(k \geq 1) = 1 - P(0)$ is usually the fastest route, using the complement again.

3.7 Measures of Center and Spread

Center, Then Spread

Center: the **mean** $\bar{x} = \dfrac{\sum x}{n}$, the **median** (middle value when ordered), and the **mode** (most frequent). **Spread:** the **variance** σ^2 is the average squared distance from the mean, and the **standard deviation** σ is its square root:

$$\sigma^2 = \frac{\sum (x - \bar{x})^2}{n}, \qquad \sigma = \sqrt{\sigma^2}.$$

Standard deviation is in the same units as the data, so it measures "typical distance from the mean."

One outlier drags the mean but not the median.

(diagram in the online version)

Worked Example: Standard Deviation Step by Step

Data: 2, 4, 4, 4, 5, 5, 7, 9 $(n = 8)$.

$$\bar{x} = \frac{2+4+4+4+5+5+7+9}{8} = \frac{40}{8} = 5$$

$$\sum(x-\bar{x})^2 = (-3)^2 + (-1)^2 + (-1)^2 + (-1)^2 + 0^2 + 0^2 + 2^2 + 4^2$$

$$= 9+1+1+1+0+0+4+16 = 32$$

$$\sigma^2 = \frac{32}{8} = 4, \qquad \sigma = \sqrt{4} = 2.$$

So a typical value sits about 2 units from the mean of 5.

Population vs. sample. Dividing by n gives the **population** standard deviation (used here). Dividing by $n-1$ gives the **sample** standard deviation, used when the data is a sample of a larger group.

3.8 The Normal Distribution, the Empirical Rule & z-Scores

The Bell Curve

Many natural measurements (heights, test scores, errors) follow a **normal distribution**: a symmetric, bell-shaped curve centered at the mean μ, with spread set by σ. For *any* normal distribution, the **Empirical Rule (68–95–99.7)** says:

- about 68% of data lies within 1σ of the mean,
- about 95% within 2σ,
- about 99.7% within 3σ.

One outlier drags the mean but not the median.

(diagram in the online version)

z-Scores: How Many Standard Deviations Out

A **z-score** rewrites any value x as its distance from the mean, measured in standard deviations:

$$z = \frac{x-\mu}{\sigma}.$$

A positive z is above the mean, negative is below, and $z = 0$ is exactly average. z-scores let you compare values from different distributions on one common scale.

Worked Example: Empirical Rule & a z-Score

IQ scores are normal with $\mu = 100$, $\sigma = 15$.

- Between 85 and 115 (that is $\mu \pm 1\sigma$): about 68% of people.
- An IQ of 130 has $z = \dfrac{130 - 100}{15} = 2$, so it sits at $\mu + 2\sigma$. Only about 2.5% of people score higher.

Symmetry shortcut. Because the curve is symmetric, 50% lies below the mean and 50% above. To find a one-sided tail, take half of what's left: above 2σ is $\frac{100\% - 95\%}{2} = 2.5\%$.

3.9 Going Deeper: Advanced Probability & Statistics

Counting with Restrictions: Circular & Identical Items

Not every arrangement is a plain permutation — watch for two twists.

- **Circular arrangements.** Seating n people around a round table has no "first" seat, so every arrangement can be rotated n ways to look the same. Divide the overcount away:

$$\text{circular arrangements} = \frac{n!}{n} = (n-1)!.$$

- **Identical (repeated) items.** Arranging n objects where one kind repeats n_1 times, another n_2 times, and so on:

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}.$$

  We divide by each $n_i!$ because swapping identical items produces no new word.

One outlier drags the mean but not the median.

(diagram in the online version)

Worked Example: Arranging the Letters of MISSISSIPPI

The word MISSISSIPPI has 11 letters: M (1), I (4), S (4), P (2). Distinct arrangements:

$$\frac{11!}{1!\,4!\,4!\,2!} = \frac{39916800}{1 \times 24 \times 24 \times 2} = \frac{39916800}{1152} = 34650.$$

Without dividing out the repeats we would count $11! = 39{,}916{,}800$ — overcounting every rearrangement of identical letters.

Overcounting checklist. Whenever a symmetry makes different-looking arrangements "the same" (rotations of a circle, swaps of identical items, or reflections of a bracelet), *divide* the raw count by the number of ways that symmetry acts.

Complementary Counting and "At Least One"

When a direct count splits into many cases, count the *opposite* and subtract from the total. For probabilities this is the complement rule; for raw counts it is the same idea:

$$(\text{ways with at least one } A) = (\text{total ways}) - (\text{ways with no } A).$$

The phrase "at least one" is the signal: the negation "none" is almost always a single, clean computation.

Worked Example: At Least One Six in Four Rolls

Roll a fair die four times. Find $P(\text{at least one } 6)$. The complement "no sixes" means each roll lands on one of five faces:

$$P(\text{no } 6) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296}, \qquad P(\text{at least one } 6) = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518.$$

Adding the "exactly one, exactly two, three, four" cases separately would take four binomial terms; the complement needs just one.

Bayes' Theorem: Reversing the Condition

Conditional probabilities are *not* symmetric: $P(A \mid B) \neq P(B \mid A)$ in general. Bayes' theorem flips a known conditional into the one you want:

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}, \qquad P(B) = P(B \mid A)\,P(A) + P(B \mid A^c)\,P(A^c).$$

The denominator is the **law of total probability**: it rebuilds $P(B)$ by splitting across A and its complement A^c.

Worked Example: A Medical Test (Base Rates Matter)

A disease affects 1% of people. A test is 99% accurate both ways: $P(+ \mid \text{sick}) = 0.99$ and $P(- \mid \text{healthy}) = 0.99$ (so $P(+ \mid \text{healthy}) = 0.01$). A patient tests positive — what is

$P(\text{sick} \mid +)$?

$$P(+) = P(+ \mid \text{sick})P(\text{sick}) + P(+ \mid \text{healthy})P(\text{healthy})$$
$$= (0.99)(0.01) + (0.01)(0.99) = 0.0099 + 0.0099 = 0.0198$$
$$P(\text{sick} \mid +) = \frac{(0.99)(0.01)}{0.0198} = \frac{0.0099}{0.0198} = \frac{1}{2} = 50\%.$$

Despite a "99% accurate" test, a positive result is only a coin flip — because the disease is rare, false positives from the huge healthy group roughly equal the true positives.

Don't ignore the base rate. A test's accuracy is $P(+ \mid \text{sick})$, but a patient cares about $P(\text{sick} \mid +)$. When the condition is rare, that reversal can be dramatically smaller. Always run it through Bayes.

Expected Value: The Long-Run Average

The **expected value** of a numerical outcome X weights each value by its probability:

$$E(X) = \sum x_i \, P(x_i).$$

It is the average you would approach over many repetitions — not necessarily a possible single outcome. A game is **fair** when $E(\text{net gain}) = 0$.

Worked Example: Is the Lottery Ticket Worth It?

A \$2 ticket pays \$100 with probability $\frac{1}{200}$ and nothing otherwise. Find the expected net gain.

$$E(\text{net}) = \underbrace{(100-2)\cdot\frac{1}{200}}_{\text{win}} + \underbrace{(-2)\cdot\frac{199}{200}}_{\text{lose}} = \frac{98}{200} - \frac{398}{200} = -\frac{300}{200} = -\$1.50.$$

On average you lose \$1.50 per ticket, so the game is *not* fair (it favors the seller).

Binomial Threshold Probabilities ("At Least k")

For a binomial with n trials and success probability p, a threshold question sums several exact-k terms:

$$P(X \ge k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}.$$

When k is small, the complement is faster: $P(X \ge 1) = 1 - P(X = 0)$, and $P(X \ge 2) = 1 - P(X = 0) - P(X = 1)$.

Worked Example: At Least 2 Defects in 6 Parts

Each part is defective with probability $p = 0.1$, independently, $n = 6$. Find $P(X \geq 2)$ using the complement:

$$\begin{aligned} P(X=0) &= \binom{6}{0}(0.1)^0(0.9)^6 = (0.9)^6 \approx 0.5314 \\ P(X=1) &= \binom{6}{1}(0.1)^1(0.9)^5 = 6(0.1)(0.9)^5 \approx 0.3543 \\ P(X \geq 2) &= 1 - 0.5314 - 0.3543 \approx 0.1143. \end{aligned}$$

So there is about an 11% chance of two or more defects — found with two terms instead of five.

Linear Transformations of Data: $y = ax + b$

Shifting and scaling every data value changes center and spread in predictable ways. If $y = ax + b$:

- **Mean** shifts and scales: $\bar{y} = a\bar{x} + b$.
- **Standard deviation** scales but does *not* shift: $\sigma_y = |a|\,\sigma_x$.
- **Variance** scales by the square: $\sigma_y^2 = a^2\,\sigma_x^2$.

Adding a constant b slides all data together, so spread is unchanged; multiplying by a stretches the gaps by $|a|$.

Worked Example: Converting Celsius to Fahrenheit

A week of temperatures in Celsius has mean $\bar{x} = 20°$ and standard deviation $\sigma_x = 3°$. Convert with $F = 1.8\,C + 32$ ($a = 1.8$, $b = 32$):

$$\begin{aligned} \bar{F} &= 1.8(20) + 32 = 68° \\ \sigma_F &= |1.8| \cdot 3 = 5.4° \quad \text{(the } +32 \text{ has no effect on spread).} \end{aligned}$$

Standardizing to a z-score is the special case $z = \dfrac{x - \mu}{\sigma}$, i.e. $a = \frac{1}{\sigma}$, $b = -\frac{\mu}{\sigma}$, which forces mean 0 and standard deviation 1.

Add vs. multiply. Adding a constant moves the center but leaves the spread alone. Multiplying rescales both. This is why σ never changes when you merely shift data.

Combining Two Groups' Means and Variances

To pool two groups of sizes n_1, n_2, the combined mean is the **weighted** average — never the plain average of the two means unless the groups are the same size:

$$\bar{x}_{\text{combined}} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}.$$

Variance must be pooled about the *combined* mean. Using each group's sum of squared deviations $S_i = n_i\sigma_i^2$ and combined mean $\bar{x}_c$:

$$\sigma^2_{\text{combined}} = \frac{n_1(\sigma_1^2 + (\bar{x}_1 - \bar{x}_c)^2) + n_2(\sigma_2^2 + (\bar{x}_2 - \bar{x}_c)^2)}{n_1 + n_2}.$$

The extra $(\bar{x}_i - \bar{x}_c)^2$ terms account for how far each group's center sits from the shared center.

Worked Example: Pooling Two Classes

Class A: $n_1 = 20$, $\bar{x}_1 = 80$, $\sigma_1^2 = 25$. Class B: $n_2 = 30$, $\bar{x}_2 = 90$, $\sigma_2^2 = 16$. Combined:

$$\begin{aligned}
\bar{x}_c &= \frac{20(80) + 30(90)}{50} = \frac{1600 + 2700}{50} = \frac{4300}{50} = 86 \\
\sigma_c^2 &= \frac{20\big(25 + (80 - 86)^2\big) + 30\big(16 + (90 - 86)^2\big)}{50} \\
&= \frac{20(25 + 36) + 30(16 + 16)}{50} = \frac{20(61) + 30(32)}{50} \\
&= \frac{1220 + 960}{50} = \frac{2180}{50} = 43.6.
\end{aligned}$$

So $\sigma_c = \sqrt{43.6} \approx 6.60$ — larger than either group's own spread, because the two centers (80 and 90) are pulled apart.

Weighted, not averaged. A combined mean weights by group size, and combined variance adds a "spread between the groups" correction. Never just average the two means or the two variances unless $n_1 = n_2$ and the means coincide.

4 Equations, Inequalities & Absolute Value

Everything for Topic 1 on one desk reference

4.1 Solving Multi-Step Linear Equations

The Big Idea: Keep it Balanced

An equation is a balance scale. Whatever you do to one side, you must do to the other. Your goal is to **isolate the variable**. A reliable order of moves:

1. Clear parentheses by **distributing**.
2. Clear fractions by multiplying every term by the **LCD**.
3. Collect the variable terms on **one** side, constants on the other.
4. Combine like terms, then divide by the coefficient.

Every step should leave the equation *equivalent* (same solution set).

Worked Example: Variables on Both Sides

Solve $3(x-4) = 5x + 2$.

$$\begin{aligned} 3x - 12 &= 5x + 2 && \text{distribute} \\ -12 - 2 &= 5x - 3x && \text{gather } x \text{ on the right, constants on the left} \\ -14 &= 2x \\ x &= -7 && \text{divide by 2} \end{aligned}$$

Check: $3(-7-4) = 3(-11) = -33$ and $5(-7) + 2 = -33$. ✓

Worked Example: Clearing Fractions

Solve $\dfrac{x}{2} + \dfrac{x-1}{3} = 4$. The LCD of 2 and 3 is 6.

$$\begin{aligned} 6 \cdot \frac{x}{2} + 6 \cdot \frac{x-1}{3} &= 6 \cdot 4 && \text{multiply every term by 6} \\ 3x + 2(x-1) &= 24 \\ 3x + 2x - 2 &= 24 \\ 5x &= 26 \\ x &= \tfrac{26}{5} \end{aligned}$$

Watch for special cases. If the variable cancels and you are left with a *true* statement like $6 = 6$, the equation is an **identity** (all real numbers). If you get a *false* statement like $3 = 7$, there is **no solution**.

4.2 Literal Equations & Formulas

Solving for a Variable

A **literal equation** contains several letters. Solving one for a chosen variable uses the exact same inverse-operation moves as a numeric equation — just treat the other letters as constants. Undo operations in **reverse order**: get the target variable alone.

Worked Example: Solve for a Variable

Solve $A = \frac{1}{2}(b_1 + b_2)h$ (area of a trapezoid) for b_1.

$$\begin{aligned} 2A &= (b_1 + b_2)h && \text{multiply both sides by 2} \\ \frac{2A}{h} &= b_1 + b_2 && \text{divide both sides by } h \\ b_1 &= \frac{2A}{h} - b_2 && \text{subtract } b_2 \end{aligned}$$

Tip: If the target variable appears in more than one term, **factor it out** first, then divide. Example: from $ax + bx = c$, factor to $x(a + b) = c$, so $x = \dfrac{c}{a + b}$.

4.3 Linear Inequalities & Interval Notation

Solving Inequalities

Solve an inequality just like an equation, with **one crucial rule**: when you **multiply or divide by a negative number, flip the inequality sign**. A solution is a whole *range* of numbers, best shown on a number line and written in **interval notation**.

An open circle means the endpoint is not included.

(diagram in the online version)

Always use a **parenthesis** next to ∞ or $-\infty$ — infinity is never included.

Worked Example: Flipping the Sign

Solve $-2x + 5 < 11$ and graph the solution.

$$\begin{aligned} -2x &< 6 && \text{subtract 5} \\ x &> -3 && \text{divide by } -2\text{, \textbf{flip} the sign} \end{aligned}$$

Solution: $x > -3$, or in interval notation $(-3,\ \infty)$.

4.4 Compound Inequalities (AND / OR)

Two Statements Joined

A **compound inequality** joins two inequalities.

- **AND** (intersection): both must hold. Graph is the *overlap* — a segment *between* two values. Written $a < x < b$, interval (a, b).
- **OR** (union): at least one holds. Graph is *two rays* pointing outward, interval $(-\infty, a) \cup (b, \infty)$.

Worked Example: AND

Solve $-1 \le 2x + 3 < 7$.

$$\begin{aligned} -1 - 3 \le 2x &< 7 - 3 && \text{subtract 3 from all three parts} \\ -4 \le 2x &< 4 \\ -2 \le x &< 2 && \text{divide all parts by 2} \end{aligned}$$

Interval notation: $[-2,\ 2)$.

Worked Example: OR

Solve $x - 1 \le -3$ **or** $2x > 6$.

$$x \le -2 \quad \text{or} \quad x > 3.$$

Interval notation: $(-\infty,\ -2] \cup (3,\ \infty)$.

4.5 Absolute Value Equations

Distance from Zero

$|x|$ is the **distance** of x from 0, so it is never negative. To solve $|ax + b| = c$ with $c > 0$, split into **two cases**:

$$ax + b = c \qquad \text{or} \qquad ax + b = -c.$$

Isolate the absolute value first, before splitting.

Worked Example: Two Solutions

Solve $|2x - 3| = 7$.

$$2x - 3 = 7 \quad \text{or} \quad 2x - 3 = -7$$

$$2x = 10 \Rightarrow x = 5 \qquad \text{or} \qquad 2x = -4 \Rightarrow x = -2.$$

Solutions: $x = 5$ or $x = -2$.

No-solution case. An absolute value can never equal a negative number. So $|x+4| = -3$ has **no solution**. Always isolate first: $|x| + 5 = 2$ becomes $|x| = -3$, also no solution.

4.6 Absolute Value Inequalities

"Less thAND" and "greatOR"

After isolating the absolute value (with $c > 0$):

- $|ax+b| < c$ means $-c < ax+b < c$ — an **AND** (a segment). *Less thAND.*
- $|ax+b| > c$ means $ax+b < -c$ **or** $ax+b > c$ — an **OR** (two rays). *GreatOR.*

Worked Example: AND type

Solve $|x-1| \leq 3$.

$$-3 \leq x-1 \leq 3 \;\Rightarrow\; -2 \leq x \leq 4.$$

Interval notation: $[-2,\ 4]$.

Worked Example: OR type

Solve $|2x+1| > 5$.

$$2x+1 < -5 \quad \text{or} \quad 2x+1 > 5$$

$$2x < -6 \Rightarrow x < -3 \qquad \text{or} \qquad 2x > 4 \Rightarrow x > 2.$$

Interval notation: $(-\infty,\ -3) \cup (2,\ \infty)$.

Special cases. $|x| < -2$ has **no solution** (distance can't be negative). $|x| > -2$ is **all real numbers** (distance is always $\geq 0 > -2$).

4.7 Applications & Word Problems

Translating Words

Turn the words into an equation or inequality, solve, then **interpret** the answer in context.

- "at least" $\to \geq$, "at most" $\to \leq$, "more than" $\to >$, "fewer than" $\to <$.
- **Tolerance** problems use absolute value: $|\text{measured} - \text{target}| \leq \text{tolerance}$.

Worked Example: Tolerance

A bolt must be 8 mm wide with a tolerance of 0.05 mm. Write and solve an inequality for acceptable widths w.

$$|w-8| \le 0.05 \;\Rightarrow\; -0.05 \le w-8 \le 0.05 \;\Rightarrow\; 7.95 \le w \le 8.05.$$

Acceptable widths: from 7.95 mm to 8.05 mm, i.e. $[7.95,\ 8.05]$.

Worked Example: Range / Budget

A taxi charges a \$3 base fee plus \$2 per mile. With \$20 to spend, how many miles m can you afford?

$$3+2m \le 20 \;\Rightarrow\; 2m \le 17 \;\Rightarrow\; m \le 8.5.$$

You can ride at most 8.5 miles.

Mixture idea. If you mix x liters of a 20% solution with 5 liters of a 50% solution, the acid amount is $0.20x + 0.50(5)$, and the total volume is $x+5$. Set the concentration equal to your target to solve.

4.8 Going Deeper: Advanced Ideas

Nested Absolute Values: Peel One Layer at a Time

An equation like $\big||x-2|-3\big| = 4$ has an absolute value *inside* another. Treat the outer bars as a single unit and split first, then split again inside each branch. Each split can double the number of equations, so an outer/inner pair can yield up to **four** candidate solutions. Always solve from the **outside in**, and **check every candidate** back in the original — some branches collapse or produce values that fail.

Worked Example: Nested Absolute Value

Solve $\big||x-2|-3\big| = 4$. Let $u = |x-2|$, so $|u-3| = 4$.

$$\begin{array}{cccc} u-3=4 & \text{or} & u-3=-4 & \text{split the outer bars} \\ u=7 & & u=-1 & \end{array}$$

Since $u = |x-2| \ge 0$, the branch $u=-1$ is **rejected**. Keep $u=7$:

$$|x-2| = 7 \;\Rightarrow\; x-2=7 \;\text{ or }\; x-2=-7 \;\Rightarrow\; x=9 \;\text{ or }\; x=-5.$$

Check: $\big||9-2|-3\big| = |7-3| = 4$ ✓ and $\big||-5-2|-3\big| = |7-3| = 4$ ✓. Solutions: $x=9$ or $x=-5$.

Summed Absolute Values: Full Casework by Critical Points

To solve $|x-a|+|x-b|=c$ you cannot square or split into two — instead find the **critical points** where each expression changes sign ($x=a$ and $x=b$). These break the number line into **three regions**. On each region every bar opens with a *fixed* sign, so the equation becomes ordinary linear. Solve on each region and **keep only the roots that land inside that region**.

Worked Example: Summed Absolute Values by Regions

Solve $|x-1|+|x+2|=7$. Critical points: $x=1$ and $x=-2$.

Region I: $x<-2$. Both bars flip: $-(x-1)-(x+2)=7$.

$$-x+1-x-2=7 \qquad \Rightarrow \qquad -2x-1=7 \qquad \Rightarrow \qquad x=-4.$$

Since $-4<-2$, **keep** $x=-4$.

Region II: $-2\le x\le 1$. Then $|x-1|=-(x-1)$, $|x+2|=x+2$:

$$-(x-1)+(x+2)=3\neq 7.$$

The variable cancels to a false statement, so **no solution** here (the sum is constant at 3 between the points — the minimum distance).

Region III: $x>1$. Both bars open positively: $(x-1)+(x+2)=7$.

$$2x+1=7 \;\Rightarrow\; x=3.$$

Since $3>1$, **keep** $x=3$. Solutions: $x=-4$ or $x=3$.

The Distance Interpretation of $|x-a|+|x-b|$

Read $|x-a|$ as "the distance from x to a." Then $|x-a|+|x-b|$ is the **total distance** from x to the two fixed points a and b. This gives instant intuition:

- The sum is **minimized** for any x *between* a and b, where it equals $|a-b|$, the gap itself.
- $|x-a|+|x-b|=c$ has **no solution** if $c<|a-b|$, **infinitely many** (the whole segment) if $c=|a-b|$, and **exactly two** symmetric solutions if $c>|a-b|$.
- $|x-a|\le|x-b|$ means "x is at least as close to a" — the solution is the half-line on a's side of the midpoint $\frac{a+b}{2}$.

Quick check with the gap. In the example above $a=1$, $b=-2$, so the gap is $|a-b|=3$. Because we wanted the sum to equal $7>3$, the distance rule promises *exactly two* solutions before any algebra — matching $x=-4$ and $x=3$. If the right side had been 2 (less than 3), you could stop immediately: **no solution**.

Systems of Absolute-Value Inequalities Define Regions

A single absolute-value inequality in x and y carves the plane into a shape; a **system** intersects several such shapes.

- $|x| \le p$ is the vertical strip $-p \le x \le p$; $|y| \le q$ is a horizontal strip. Together they form a **rectangle**.
- $|x| + |y| \le r$ is a **diamond** (square rotated 45°) with vertices at $(\pm r, 0)$ and $(0, \pm r)$.
- $|y| \ge |x|$ is the pair of "bowtie" wedges opening up and down.

Sketch each boundary, decide inside vs. outside with a **test point** (the origin is easiest when it is not on a boundary), then shade the **overlap**.

Worked Example: A Region from Two Absolute-Value Inequalities

Describe the region satisfying both $|x| + |y| \le 4$ and $|x| \ge 1$.

The first inequality is the diamond with vertices $(\pm 4, 0)$ and $(0, \pm 4)$, *including* the interior. The second, $|x| \ge 1$, means $x \le -1$ **or** $x \ge 1$ — everything *except* the vertical strip $-1 < x < 1$.

Test the origin $(0, 0)$: it satisfies $|0| + |0| = 0 \le 4$ (inside the diamond) but fails $|0| \ge 1$. So the origin is **excluded**, confirming we remove the central band. The solution is the diamond with a vertical slot of width 2 cut out of its middle — two symmetric "wing" pieces, one with $x \ge 1$ and one with $x \le -1$, each bounded by the diamond's edges.

Parameter Conditions for a Solution Set

Sometimes the unknown is a **parameter** that controls whether — and how many — solutions exist. Isolate the absolute value and compare with the structure rules:

- $|ax + b| = k$ has **two** solutions if $k > 0$, **one** if $k = 0$, **none** if $k < 0$.
- $|ax + b| < k$ has solutions **only if** $k > 0$ (if $k \le 0$ the solution set is empty).
- $|ax + b| > k$ is **all reals** when $k < 0$, and all reals except one point when $k = 0$.

Translate the requested behavior into an inequality on the parameter, then solve *that*.

Worked Example: Choosing a Parameter

For which values of k does $|3x - 6| = k - 1$ have **exactly two** solutions?

The left side is a genuine absolute value, so it produces two solutions precisely when the right side is **strictly positive**:

$$k - 1 > 0 \;\Rightarrow\; k > 1.$$

For $k = 1$ the equation is $|3x - 6| = 0$, giving the *single* solution $x = 2$; for $k < 1$ the right side is negative and there is **no solution**. Answer: $k > 1$, i.e. $k \in (1, \infty)$.

Inequalities Solved by Substitution

When an inequality contains a repeated chunk, let a new variable stand for it. Solving $(x^2 - 1)^2 - 5(x^2 - 1) + 4 \le 0$ is hopeless head-on, but with $u = x^2 - 1$ it becomes $u^2 - 5u + 4 \le 0$, i.e. $(u-1)(u-4) \le 0$, giving $1 \le u \le 4$. **Back-substitute** and solve the resulting compound inequality — often itself an absolute-value or interval statement. The same trick tames $|x|^2 - 3|x| \ge 0$ via $u = |x| \ge 0$.

Worked Example: Substitution in an Absolute-Value Inequality

Solve $|x|^2 - 3|x| - 4 \le 0$. Let $u = |x|$, where $u \ge 0$.

$$\begin{aligned} u^2 - 3u - 4 &\le 0 && \text{substitute} \\ (u-4)(u+1) &\le 0 && \text{factor} \\ -1 \le u &\le 4 && \text{sign chart between the roots} \end{aligned}$$

But $u = |x| \ge 0$ forces $0 \le u \le 4$, i.e. $|x| \le 4$. Therefore

$$-4 \le x \le 4, \qquad \text{interval notation } [-4,\ 4].$$

Interval-notation subtleties.

- An **empty** solution set is written $\varnothing$, *not* (a, a) loosely — although (a, a) does denote the empty set, prefer $\varnothing$ for clarity.
- A single isolated point $\{a\}$ is **not** an interval; write it with set braces, e.g. $|x - 3| \le 0$ gives $\{3\}$.
- A union must list pieces **left to right** and stay disjoint: $(-\infty, -2) \cup (2, \infty)$, never overlapping or out of order.
- "All real numbers" is $(-\infty,\ \infty)$; a punctured line like $x \ne 5$ is $(-\infty, 5) \cup (5, \infty)$.
- Always pair ∞ and $-\infty$ with a **parenthesis**, never a bracket — they are directions, not attainable endpoints.

5 Functions, Relations & Transformations

Everything you need for Topic 2, with worked examples and graphs

5.1 Relations, Functions & the Vertical Line Test

Big idea

A **relation** is any set of ordered pairs (x, y). A **function** is a special relation in which *every input x is paired with exactly one output y*. In other words: no x-value is allowed to repeat with two different y-values.

- The set of all allowed inputs is the **domain**.
- The set of all resulting outputs is the **range**.

Example: is it a function?

Consider $R = \{(-2, 3), (0, 1), (1, 4), (0, 5)\}$.

The input 0 appears twice, paired with both 1 and 5. Since one input gives two outputs, R is **not** a function.

Now $S = \{(-2, 3), (0, 1), (1, 4), (3, 1)\}$ *is* a function: every input is used once. (It is fine for two inputs to share an output, as 0 and 3 both give 1.)

The Vertical Line Test

A graph represents a function if and only if **no vertical line crosses it more than once**. A vertical line is a set of points with the same x; hitting the graph twice would mean one x has two y's.

Tip: The dashed vertical line on the right hits the sideways parabola *twice*, so that curve fails the test. A tilted straight line always passes.

5.2 Function Notation, Evaluating, Discrete vs. Continuous

Function notation

We write $f(x)$ (read "f of x") for the output of function f at input x. The letter x is a *placeholder*: to **evaluate**, substitute the given value everywhere x appears and simplify.

Example: evaluating

Let $f(x) = 2x^2 - 3x + 1$. Then

$$f(3) = 2(3)^2 - 3(3) + 1 = 18 - 9 + 1 = 10, \qquad f(-1) = 2(-1)^2 - 3(-1) + 1 = 2 + 3 + 1 = 6.$$

You can also evaluate at an expression: $f(a+1) = 2(a+1)^2 - 3(a+1) + 1 = 2a^2 + a$.

Discrete vs. continuous

A **discrete** function is defined only at separated points (draw dots), e.g. "number of students" vs. "number of pizzas." A **continuous** function is defined on an unbroken interval (draw a solid curve), e.g. "temperature" vs. "time."

Tip: $f(x)$ does *not* mean f times x. It is a single output value.

5.3 Parent Functions and Their Graphs

The six parents

Each family of functions is built from a simple "parent." Memorize their shapes, domains, and ranges.

Tip: $y = x^2$ and $y = |x|$ both open upward with a minimum at the origin, but the parabola is curved while the absolute value is two straight rays meeting in a sharp corner.

5.4 Transformations: $y = a\,f(x-h) + k$

Reading the transformation

Starting from a parent $f(x)$, the graph of $y = a\,f(x-h) + k$ is transformed as follows:

- k: vertical shift — up if $k > 0$, down if $k < 0$.
- h: horizontal shift — right if $h > 0$, left if $h < 0$ (it moves *opposite* the sign inside).
- a: vertical stretch if $|a| > 1$, vertical compression if $0 < |a| < 1$.
- $a < 0$: reflection over the x-axis. Replacing x with $-x$ reflects over the y-axis.

Example: describe the transformation

Compare $g(x) = -2(x-1)^2 + 3$ to the parent $f(x) = x^2$. Here $a = -2$, $h = 1$, $k = 3$, so g is the parabola **stretched** vertically by 2, **reflected** over the x-axis, shifted **right** 1 and **up** 3. Its vertex is $(1, 3)$ and it opens downward.

Tip (order of operations for graphs): apply horizontal shift and stretch/reflection first, then the vertical shift last. The vertex of $y = a(x-h)^2 + k$ is always (h, k).

5.5 Operations on Functions

Combining functions

For two functions f and g:

$$(f+g)(x) = f(x) + g(x), \quad (f-g)(x) = f(x) - g(x),$$

$$(fg)(x) = f(x) \cdot g(x), \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad (g(x) \neq 0).$$

The domain of the result is where *both* pieces are defined, and for the quotient we also exclude any x with $g(x) = 0$.

Example

Let $f(x) = x^2 + 1$ and $g(x) = x - 2$. Then

$$(f+g)(x) = x^2 + x - 1, \qquad (fg)(x) = (x^2+1)(x-2) = x^3 - 2x^2 + x - 2,$$

$$\left(\frac{f}{g}\right)(x) = \frac{x^2+1}{x-2} \quad (x \neq 2).$$

Tip: Operations happen at the *same* input. Only the quotient forces you to worry about a zero denominator.

5.6 Composition of Functions

Composition

The **composition** $(f \circ g)(x) = f\big(g(x)\big)$ means: run g first, then feed its output into f. Composition is usually *not* commutative: $(f \circ g)(x) \neq (g \circ f)(x)$ in general.

Example

Let $f(x) = x^2 + 1$ and $g(x) = x - 2$.

$$(f \circ g)(x) = f(x-2) = (x-2)^2 + 1 = x^2 - 4x + 5,$$

$$(g \circ f)(x) = g(x^2+1) = (x^2+1) - 2 = x^2 - 1.$$

And numerically $(f \circ g)(5) = f(3) = 10$.

Tip: Work from the *inside out*. Substitute the whole expression for $g(x)$ into every x of f.

5.7 Inverse Functions

What an inverse does

The **inverse** f^{-1} undoes f: it swaps inputs and outputs, so if $f(a) = b$ then $f^{-1}(b) = a$. To find it: replace $f(x)$ with y, **swap** x **and** y, then solve for y.

Example: find and verify

Let $f(x) = 2x - 6$. Swap and solve: $x = 2y - 6 \Rightarrow y = \dfrac{x+6}{2}$, so $f^{-1}(x) = \dfrac{x+6}{2}$.

Verify by composition:

$$f\big(f^{-1}(x)\big) = 2\big(\tfrac{x+6}{2}\big) - 6 = x, \qquad f^{-1}\big(f(x)\big) = \tfrac{(2x-6)+6}{2} = x. \ \checkmark$$

Both compositions give x, confirming the inverse.

Tip: The graph of f^{-1} is the *mirror image* of f across the line $y = x$. If f passes the vertical line test and its inverse must too, f must be one-to-one (pass the *horizontal* line test).

5.8 Piecewise Functions

Reading a piecewise rule

A **piecewise** function uses different formulas on different parts of the domain. To evaluate, first decide *which interval* the input falls in, then use that piece.

$$f(x) = \begin{cases} x+3, & x<0 \\ x^2, & 0 \le x \le 2 \\ 4, & x>2 \end{cases}$$

Example: evaluate the piecewise function above

$f(-4) = -4+3 = -1$ (use $x+3$), $f(1) = 1^2 = 1$ (use x^2), $f(5) = 4$ (use the constant).

Watch the endpoints: at $x = 0$ we use x^2 because that piece includes 0 ($0 \le x$).

Tip: A filled dot means the endpoint *is* included ($\le$ or $\ge$); an open dot means it is *not* ($<$ or $>$). Only one piece may own each x.

5.9 Going Deeper: Advanced Function Ideas

Functional equations: solve for the *rule*, not a number

A **functional equation** constrains an unknown function f instead of an unknown number. The classic trick: substitute a cleverly chosen input to generate a *second* equation, then treat $f(x)$ and $f(\text{other})$ as two unknowns and eliminate. For equations mixing $f(x)$ with $f(1/x)$, the magic substitution is $x \mapsto 1/x$, which swaps the two terms.

Amplitude is the height; period is one full cycle.

(diagram in the online version)

Worked example: solve $f(x) + 2f(1/x) = 3x$

Write the given equation, then replace every x with $1/x$ (note $1/(1/x) = x$):

$$f(x) + 2f\left(\tfrac{1}{x}\right) = 3x, \tag{1}$$
$$f\left(\tfrac{1}{x}\right) + 2f(x) = \tfrac{3}{x}. \tag{2}$$

Now eliminate $f(1/x)$: multiply (2) by 2 and subtract (1):

$$\left(2f(\tfrac{1}{x}) + 4f(x)\right) - \left(f(x) + 2f(\tfrac{1}{x})\right) = \tfrac{6}{x} - 3x \implies 3f(x) = \tfrac{6}{x} - 3x.$$

So $f(x) = \dfrac{2}{x} - x$. **Check:** $f(x) + 2f(1/x) = \left(\frac{2}{x} - x\right) + 2\left(2x - \frac{1}{x}\right) = \frac{2}{x} - x + 4x - \frac{2}{x} = 3x$. ✓

Iterated composition and involutions

Composing f with itself is written $f^{[n]} = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ copies}}$, so $f^{[2]}(x) = f(f(x))$. Beware: this is *iteration*, not the power $\big(f(x)\big)^n$ and not the inverse f^{-1}.

- If $f^{[n]} = \mathrm{id}$ (the identity $x \mapsto x$) for some n, then f is **periodic** under iteration with period dividing n, and $f^{[m]}$ depends only on m mod n.
- An **involution** is the special case $n = 2$: $f(f(x)) = x$, so f is its *own inverse*, $f^{-1} = f$. Its graph is symmetric across $y = x$.

Worked example: a period-3 map and a self-inverse map

(a) Iteration. Let $f(x) = \dfrac{1}{1-x}$. Then

$$f^{[2]}(x) = f\Big(\tfrac{1}{1-x}\Big) = \frac{1}{1-\frac{1}{1-x}} = \frac{1-x}{-x} = \frac{x-1}{x}, \qquad f^{[3]}(x) = f\big(\tfrac{x-1}{x}\big) = \frac{1}{1-\frac{x-1}{x}} = x.$$

So $f^{[3]} = \mathrm{id}$: the map has period 3. To find $f^{[100]}$, reduce the exponent: $100 \equiv 1 \pmod 3$, hence $f^{[100]} = f^{[1]} = f$.

(b) Involution. Let $g(x) = \dfrac{2x-3}{x-2}$. Composing g with itself,

$$g(g(x)) = \frac{2 \cdot \frac{2x-3}{x-2} - 3}{\frac{2x-3}{x-2} - 2} = \frac{\frac{x}{x-2}}{\frac{1}{x-2}} = x,$$

so g is an involution: $g^{-1} = g$. Any $g(x) = \dfrac{ax+b}{cx-a}$ is self-inverse for the same reason.

Worked example: the domain of a nasty composition

Let $f(x) = \dfrac{1}{x-1}$ and $g(x) = \sqrt{4-x^2}$, and form $h(x) = (f \circ g)(x) = \dfrac{1}{\sqrt{4-x^2}-1}$. The domain needs *two* conditions, checked in order:

- x must be legal for the inner g: $4 - x^2 \ge 0 \Rightarrow -2 \le x \le 2$.
- The output $g(x)$ must be legal for the outer f, i.e. $g(x) \ne 1$: $\sqrt{4-x^2} \ne 1 \Rightarrow 4 - x^2 \ne 1 \Rightarrow x \ne \pm\sqrt{3}$.

Domain: $[-2, 2]$ with the two holes $x = \pm\sqrt{3}$ removed. **Moral:** the domain of $f \circ g$ is not just the domain of g; you must also delete inputs whose output lands in a forbidden spot for f.

Invertibility via monotonicity

A function is invertible exactly when it is **one-to-one** (passes the horizontal line test). You do *not* need an explicit formula for f^{-1} to know it exists:

If f is strictly increasing on its domain (or strictly decreasing), then f is one-to-one, so f^{-1} exists.

For example, $f(x) = x^5+x+1$ is a sum of the strictly increasing pieces x^5, x, and a constant, so it is strictly increasing and therefore invertible on all of $\mathbb{R}$ — even though solving $y = x^5+x+1$ for x by hand is hopeless. Monotonicity is also how you *restrict* a non-injective parent (like $y = x^2$) to make it invertible: choose a branch, e.g. $x \geq 0$.

Exploiting even and odd symmetry

A function is **even** if $f(-x) = f(x)$ (graph symmetric across the y-axis) and **odd** if $f(-x) = -f(x)$ (symmetric through the origin). Symmetry halves your work: knowing f on $x \geq 0$ determines it everywhere. Two facts worth memorizing:

- Products behave like parity signs: even $\times$ even $=$ even, odd $\times$ odd $=$ even, even $\times$ odd $=$ odd.
- **Every** function splits uniquely into an even part plus an odd part:

$$f(x) = \underbrace{\tfrac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\tfrac{f(x)-f(-x)}{2}}_{\text{odd}}.$$

Tip: If f is odd then $f(0) = 0$ (set $x = 0$ in $f(-x) = -f(x)$). And if a problem asks for $f(-a)$ and you already know $f(a)$, don't recompute — just flip the sign for odd, copy for even.

Worked example: piecewise continuity with a parameter

Choose the constant a so that

$$f(x) = \begin{cases} x^2 + a, & x \leq 1 \\ 3x - 1, & x > 1 \end{cases}$$

is continuous. Continuity at the seam $x = 1$ requires the two pieces to *agree* there. The left piece gives $f(1) = 1^2 + a = 1 + a$; the right piece approaches $3(1) - 1 = 2$ as $x \to 1^+$. Set them equal:

$$1 + a = 2 \implies a = 1.$$

With $a = 1$ the graph joins with no jump. For any other a there is a gap of height $|1 + a - 2| = |a - 1|$ at $x = 1$.

Composing several transformations into one mapping

A chain of transformations can always be collapsed into a single rule $y = a\,f\big(b(x-h)\big) + k$. Track what happens to a generic point, applying *inner* (horizontal) changes to x and *outer* (vertical) changes to the whole expression. Order matters, because a horizontal shift done before a horizontal stretch is not the same as after.

Example. Starting from f, reflect over the y-axis, then shift left 2, then stretch vertically by 3, then shift up 1. Building inside-out, the horizontal reflect-and-shift turns the input into $-(x+2)$, and the outer stretch-and-shift wraps it:

$$y = 3\,f\big(-(x+2)\big) + 1.$$

A single expression now encodes all four moves — feed in one x, get the fully transformed y.

6 Linear Systems & Matrices

Systems in two and three variables, inequalities, matrices, determinants, and Cramer's Rule

6.1 Solving Two-Variable Systems (Review)

The three methods

A **system** of two linear equations asks for the point (x, y) that satisfies *both* equations at once. There are three standard methods:

- **Graphing:** plot both lines; the solution is where they cross.
- **Substitution:** solve one equation for a variable, then substitute into the other.
- **Elimination:** add or subtract multiples of the equations to cancel one variable.

Slope is rise over run.

(diagram in the online version)

How many solutions?

Compare the two lines:

- Different slopes $\Rightarrow$ lines cross once $\Rightarrow$ **one solution** (consistent, independent).
- Same slope, different intercepts $\Rightarrow$ parallel lines $\Rightarrow$ **no solution** (inconsistent).
- Same slope *and* same intercept $\Rightarrow$ same line $\Rightarrow$ **infinitely many solutions** (dependent).

Worked example: elimination

Solve $\begin{cases} 2x + 3y = 12 \\ 2x - y = 4 \end{cases}$

The x-terms already match. Subtract the second equation from the first:

$$(2x + 3y) - (2x - y) = 12 - 4 \quad \Rightarrow \quad 4y = 8 \quad \Rightarrow \quad y = 2.$$

Substitute into $2x - y = 4$: $2x - 2 = 4 \Rightarrow 2x = 6 \Rightarrow x = 3$.

Solution: $(3, 2)$. Check: $2(3) + 3(2) = 12$ ✓ and $2(3) - 2 = 4$ ✓.

Worked example: graphing

Solve $\begin{cases} y = x + 1 \\ y = -x + 3 \end{cases}$ by graphing. The lines meet where $x + 1 = -x + 3$, i.e. $2x = 2$, so $x = 1$ and $y = 2$.

Tip: Elimination is fastest when a variable already has matching or opposite coefficients. Substitution shines when one equation is already solved for a variable (like $y = \dots$).

6.2 Systems of Linear Inequalities

The solution is a region, not a point

For a system of linear *inequalities*, the solution set is the region where all shaded half-planes overlap.

- Graph each boundary line: use a **solid** line for $\leq$ or $\geq$, a **dashed** line for $<$ or $>$.
- Shade the side that makes each inequality true (use a **test point** such as $(0, 0)$ if it is not on the line).
- The overlap of all shadings is the solution region.

Worked example: describe the region

Describe the solution of $\begin{cases} y \leq x + 1 \\ y > -2 \end{cases}$

The boundary $y = x + 1$ is *solid* (from $\leq$); shade *below/on* it. The boundary $y = -2$ is *dashed* (from $>$); shade *above* it. The solution is the wedge that is on or below the slanted line and strictly above the horizontal line $y = -2$.

Test point $(0,0)$: is $0 \le 0+1$? Yes. Is $0 > -2$? Yes. So $(0,0)$ lies in the region.

Tip: A point is a solution of the system only if it satisfies *every* inequality. Test each one separately.

6.3 Solving Three-Variable Systems by Elimination

The strategy

A system in x, y, z needs three equations. To solve:

1. Pick a variable to eliminate. Combine two equations to remove it, then combine a *different* pair to remove the *same* variable. This leaves a 2×2 system.
2. Solve that 2×2 system for the remaining two variables.
3. Back-substitute to find the third variable.
4. **Always** check the solution in all three original equations.

Worked example: full 3-variable elimination

Solve $\begin{cases} x+y+z=6 \\ 2x-y+z=3 \\ x+2y-z=2 \end{cases}$

Step 1 — eliminate z. Add equation (1) and equation (3):

$$(x+y+z)+(x+2y-z)=6+2 \;\Rightarrow\; 2x+3y=8. \quad \text{(A)}$$

Add equation (2) and equation (3):

$$(2x-y+z)+(x+2y-z)=3+2 \;\Rightarrow\; 3x+y=5. \quad \text{(B)}$$

Step 2 — solve the 2×2 system. From (B), $y=5-3x$. Substitute into (A):

$$2x+3(5-3x)=8 \;\Rightarrow\; 2x+15-9x=8 \;\Rightarrow\; -7x=-7 \;\Rightarrow\; x=1.$$

Then $y=5-3(1)=2$.

Step 3 — back-substitute. From (1): $z=6-x-y=6-1-2=3$.

Solution: $(1,2,3)$. Check: $1+2+3=6$ ✓; $2-2+3=3$ ✓; $1+4-3=2$ ✓.

Tip: Eliminate the *same* variable both times. If you accidentally remove z once and x once, you cannot combine the two results.

6.4 Matrix Basics

Dimensions, equality, and the simple operations

A **matrix** is a rectangular array of numbers. Its **order** (dimension) is rows × columns; the entry in row i, column j is a_{ij}.

- **Equal matrices:** same order *and* every matching entry is equal.
- **Addition/subtraction:** only for matrices of the *same* order; add or subtract entry by entry.
- **Scalar multiplication:** multiply *every* entry by the scalar k.

Worked example: combining matrices

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ (both 2×2).

$$A + B = \begin{bmatrix} 2+1 & 1+(-2) \\ 0+4 & 3+5 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & 8 \end{bmatrix}, \qquad 3A = \begin{bmatrix} 3(2) & 3(1) \\ 3(0) & 3(3) \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 0 & 9 \end{bmatrix}.$$

Equal matrices: if $\begin{bmatrix} x & 3 \\ 1 & y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$, then $x = 2$ and $y = -5$.

Tip: You can only add or subtract matrices of *identical* order. A 2×3 and a 3×2 cannot be added.

6.5 Matrix Multiplication

When is it defined, and what size is the product?

The product AB is defined only when the number of **columns of** A equals the number of **rows of** B:

$$(m \times \mathbf{n}) \cdot (\mathbf{n} \times p) = (m \times p).$$

Each entry of AB is a **row of** A **dotted with a column of** B: multiply matching entries and add. Matrix multiplication is *not* commutative: usually $AB \neq BA$.

Worked example: full row-by-column product

Compute AB for $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. Both are 2×2, so AB is 2×2.

$$AB = \begin{bmatrix} (1)(5)+(2)(7) & (1)(6)+(2)(8) \\ (3)(5)+(4)(7) & (3)(6)+(4)(8) \end{bmatrix} = \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$

Read it as: (row 1 of A)·(column 1 of B) gives the top-left entry, and so on.

Tip: Before multiplying, write the orders side by side. If the two inner numbers match, the product exists; the two outer numbers give its size.

6.6 Determinants: 2×2 and 3×3

The determinant of a square matrix

For a 2×2 matrix, the determinant is

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

For a 3×3 matrix, expand along the first row (note the alternating $+ - +$ signs):

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\,(ei - fh) - b\,(di - fg) + c\,(dh - eg).$$

Worked example: a 3×3 determinant

Compute $\det \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & 5 \end{bmatrix}$ by expanding along the top row:

$$\begin{aligned} &= 2\,(0 \cdot 5 - 2 \cdot 1) \; - \; 1\,(1 \cdot 5 - 2 \cdot 4) \; + \; 3\,(1 \cdot 1 - 0 \cdot 4) \\ &= 2(0-2) - 1(5-8) + 3(1-0) \\ &= 2(-2) - 1(-3) + 3(1) = -4 + 3 + 3 = 2. \end{aligned}$$

Tip: Expanding along a row or column that contains zeros saves work, because those terms vanish. Remember the checkerboard sign pattern $\begin{smallmatrix} + & - & + \\ - & + & - \\ + & - & + \end{smallmatrix}$.

6.7 Solving Systems with Determinants: Cramer's Rule

Cramer's Rule for a 2×2 system

For $\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$ form the coefficient determinant

$$D = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}.$$

Replace the x-column with the constants to get D_x, and the y-column to get D_y:

$$D_x = \det \begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}, \qquad D_y = \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}.$$

Then, provided $D \neq 0$, $\quad x = \dfrac{D_x}{D}, \quad y = \dfrac{D_y}{D}$. If $D = 0$, Cramer's Rule does not apply (no unique solution).

Worked example: Cramer's Rule

Solve $\begin{cases} 2x + 3y = 12 \\ x - y = 1 \end{cases}$

$$D = \det \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = (2)(-1) - (1)(3) = -5, \quad D_x = \det \begin{bmatrix} 12 & 3 \\ 1 & -1 \end{bmatrix} = -12 - 3 = -15,$$

$$D_y = \det \begin{bmatrix} 2 & 12 \\ 1 & 1 \end{bmatrix} = 2 - 12 = -10.$$

So $x = \dfrac{-15}{-5} = 3$ and $y = \dfrac{-10}{-5} = 2$. **Solution:** $(3, 2)$. Check: $2(3) + 3(2) = 12$ ✓, $3 - 2 = 1$ ✓.

Tip: Keep the columns in order. D_x swaps only the x-column for the constants; everything else stays put.

6.8 Applications: Setting Up Systems

From words to equations

To model a word problem:

1. Name each unknown with a variable (one per unknown quantity).
2. Translate each sentence of information into one equation. You need as many equations as unknowns.

3. Solve by any method, then interpret the answer in context and check it against the words.

Worked example: a ticket problem

A theater sells 100 tickets for a total of \$680. Adult tickets cost \$8 and child tickets cost \$5. How many of each were sold?

Let a = adult tickets, c = child tickets. Then

$$\begin{cases} a + c = 100 \\ 8a + 5c = 680 \end{cases}$$

From the first equation $c = 100 - a$. Substitute: $8a + 5(100 - a) = 680 \Rightarrow 8a + 500 - 5a = 680 \Rightarrow 3a = 180 \Rightarrow a = 60$. Then $c = 40$.

Answer: 60 adult and 40 child tickets. Check: $60+40 = 100$ ✓ and $8(60)+5(40) = 480+200 = 680$ ✓.

Tip: Count your unknowns first. Two unknowns need two independent facts; three unknowns (say nickels, dimes, quarters) need three.

6.9 Going Deeper: Advanced Systems & Matrices

Symmetric systems and elementary symmetric functions

A system is **symmetric** if swapping $x \leftrightarrow y$ leaves it unchanged, e.g. $\begin{cases} x + y = s \\ xy = p \end{cases}$. The two quantities

$$s = x + y \quad \text{(the } \textit{sum}\text{)}, \qquad p = xy \quad \text{(the } \textit{product}\text{)}$$

are the **elementary symmetric functions**. Any symmetric expression rewrites in terms of s and p; the key identities are

$$x^2 + y^2 = s^2 - 2p, \qquad x^2 - xy + y^2 = s^2 - 3p, \qquad x^3 + y^3 = s^3 - 3ps.$$

Once s and p are known, x and y are the roots of the single quadratic

$$t^2 - s\,t + p = 0.$$

Worked example: solving a symmetric system

Solve $\begin{cases} x + y = 5 \\ x^2 + y^2 = 13 \end{cases}$

Step 1 — get s and p. Here $s = x + y = 5$. Use $x^2 + y^2 = s^2 - 2p$:

$$13 = 5^2 - 2p \;\Rightarrow\; 13 = 25 - 2p \;\Rightarrow\; 2p = 12 \;\Rightarrow\; p = 6.$$

Step 2 — build the quadratic. With $s = 5$, $p = 6$, the values x, y are the roots of

$$t^2 - 5t + 6 = 0 \;\Rightarrow\; (t-2)(t-3) = 0 \;\Rightarrow\; t = 2 \text{ or } t = 3.$$

Solution: $(x, y) = (2, 3)$ or $(3, 2)$. Check: $2 + 3 = 5$ ✓ and $2^2 + 3^2 = 4 + 9 = 13$ ✓.

Nonlinear systems: line–conic and conic–conic

When at least one equation is a conic (circle, parabola, ellipse, hyperbola), count solutions geometrically:

- **Line meets a conic:** substitute the line into the conic to get one quadratic. The discriminant tells you 2 points (secant), 1 point (tangent), or 0 points (miss).
- **Two conics:** up to 4 intersection points. *Subtract* the two equations to cancel the quadratic terms when possible — this often leaves a line (the **radical axis** for two circles), reducing the problem to a line–conic case.

Worked example: two-conic system by subtraction

Solve $\begin{cases} x^2 + y^2 = 25 \\ x^2 + y^2 - 6x - 8y = -15 \end{cases}$

Step 1 — subtract to kill the squares. Equation (1) minus equation (2):

$$(x^2 + y^2) - (x^2 + y^2 - 6x - 8y) = 25 - (-15) \;\Rightarrow\; 6x + 8y = 40 \;\Rightarrow\; 3x + 4y = 20.$$

This line is the radical axis. Solve it for x: $x = \dfrac{20 - 4y}{3}$.

Step 2 — substitute into the circle. Better: from $3x + 4y = 20$ take $x = \frac{20-4y}{3}$ and plug into $x^2 + y^2 = 25$. Multiplying through by 9,

$$(20 - 4y)^2 + 9y^2 = 225 \;\Rightarrow\; 400 - 160y + 16y^2 + 9y^2 = 225 \;\Rightarrow\; 25y^2 - 160y + 175 = 0.$$

Divide by 5: $5y^2 - 32y + 35 = 0 \Rightarrow (5y - 7)(y - 5) = 0 \Rightarrow y = \frac{7}{5}$ or $y = 5$.

Step 3 — back-substitute. If $y = 5$: $x = \frac{20-20}{3} = 0$, giving $(0, 5)$. If $y = \frac{7}{5}$: $x = \frac{20-28/5}{3} = \frac{72/5}{3} = \frac{24}{5}$, giving $\left(\frac{24}{5}, \frac{7}{5}\right)$.

Solutions: $(0, 5)$ and $\left(\frac{24}{5}, \frac{7}{5}\right)$. Check $(0, 5)$: $0 + 25 = 25$ ✓.

The inverse of a 3×3 matrix

For a square matrix A, the **inverse** A^{-1} satisfies $AA^{-1} = A^{-1}A = I$. It exists precisely when $\det A \neq 0$. The adjugate formula is

$$A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A),$$

where $\mathrm{adj}(A)$ is the **transpose of the cofactor matrix**. Building it takes three steps:

1. **Cofactors:** $C_{ij} = (-1)^{i+j} M_{ij}$, where the minor M_{ij} is the determinant of A with row i and column j deleted.
2. **Adjugate:** transpose the matrix of cofactors ($\mathrm{adj}(A)_{ij} = C_{ji}$).
3. **Scale:** divide every entry by $\det A$.

Once A^{-1} is known, the system $A\mathbf{x} = \mathbf{b}$ has the one-line solution $\mathbf{x} = A^{-1}\mathbf{b}$.

Worked example: invert a 3×3 and solve

Solve $\begin{cases} x + 2y + 3z = 5 \\ \quad 2y + 3z = 4 \\ \quad\quad 3z = 3 \end{cases}$ using the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

Step 1 — determinant. A is upper triangular, so $\det A = 1 \cdot 2 \cdot 3 = 6 \neq 0$; the inverse exists.

Step 2 — cofactors and adjugate. Computing cofactors and transposing gives

$$\mathrm{adj}(A) = \begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix}, \qquad A^{-1} = \frac{1}{6}\begin{bmatrix} 6 & -6 & 0 \\ 0 & 3 & -3 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Step 3 — multiply by $\mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$.

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5-4 \\ 2-\frac{3}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Solution: $x = 1,\ y = \frac{1}{2},\ z = 1$. Check the first equation: $1 + 2(\frac{1}{2}) + 3(1) = 1 + 1 + 3 = 5$ ✓.

Determinant properties and special identities

These rules let you evaluate determinants without brute force:

- **Row/column operations:** adding a multiple of one row to another does *not* change det; swapping two rows multiplies det by -1; scaling a row by k scales det by k.
- **Zero determinant:** a repeated row, a zero row, or one row a multiple of another all force $\det = 0$.
- **Products & transposes:** $\det(AB) = \det A \det B$, $\det(A^\mathsf{T}) = \det A$, $\det(A^{-1}) = 1/\det A$, $\det(kA) = k^n \det A$ for an $n \times n$ matrix.
- **Vandermonde:** $\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.
- **Rank-one update:** $\det(I + aJ)$, where J is the $n \times n$ all-ones matrix, equals $1 + na$ (since J has eigenvalues n and 0).

Tip: The Vandermonde determinant is zero exactly when two of a, b, c are equal — which is why n distinct data points determine a unique degree-$(n-1)$ interpolating polynomial.

Singular matrices: when there is no inverse

A square matrix A is **singular** (non-invertible) exactly when $\det A = 0$. For the system $A\mathbf{x} = \mathbf{b}$ this means *no unique solution* — either no solution or infinitely many. Equivalent warning signs of singularity:

- the rows (or columns) are **linearly dependent** — one is a combination of the others;
- a row reduces to all zeros during elimination;
- Cramer's Rule breaks because $D = \det A = 0$.

To find the parameter that makes a matrix singular, *set its determinant equal to zero and solve.*

Worked example: for which k is the matrix singular?

Find all k for which $A = \begin{bmatrix} k & 2 \\ 8 & k \end{bmatrix}$ has *no* inverse, and describe the system $A\mathbf{x} = \mathbf{0}$ there.

Singular means $\det A = 0$:

$$\det A = k \cdot k - (2)(8) = k^2 - 16 = 0 \;\Rightarrow\; k = 4 \text{ or } k = -4.$$

At $k = 4$: $A = \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$, whose second row is twice the first — the rows are linearly dependent, confirming $\det = 0$. The homogeneous system then reduces to the single equation $4x + 2y = 0$, i.e. $y = -2x$, giving **infinitely many** solutions along that line rather than only $\mathbf{x} = \mathbf{0}$.

Matrix powers A^n and recurrences

Repeated multiplication $A^n = A \cdot A \cdots A$ (n factors) models step-by-step processes (Markov chains, Fibonacci-type recurrences). Two shortcuts:

- **Diagonal matrices** power entrywise: $\operatorname{diag}(a, b)^n = \operatorname{diag}(a^n, b^n)$.
- **Cayley–Hamilton:** every matrix satisfies its own characteristic equation. For a 2×2 matrix, $A^2 = (\operatorname{tr} A)\, A - (\det A)\, I$, which lets you reduce *any* power A^n to the form $\alpha A + \beta I$.

The Fibonacci matrix illustrates the recurrence link:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

Tip: To compute a high power like A^{10}, use repeated *squaring*: A^2, $A^4 = (A^2)^2$, $A^8 = (A^4)^2$, then $A^{10} = A^8 A^2$. That is 4 multiplications instead of 9.

Find $x + y + z$ without solving the whole system

Sometimes a problem asks only for a *combination* of the unknowns — not each one. Look for a shortcut before grinding out a full solution:

- **Add all equations.** If the coefficients of x, y, z each sum to the same value c, then adding gives $c(x + y + z) = (\text{sum of constants})$, so $x + y + z$ pops out directly.
- **Take a linear combination** of the equations designed to produce exactly the target combination.

Worked example: get $x + y + z$ directly

Given $\begin{cases} x + 2y + 3z = 14 \\ 3x + y + 2z = 11 \\ 2x + 3y + z = 11 \end{cases}$, find $x + y + z$ *without* solving for each variable.

Add all three equations. On the left, each variable's coefficients sum to $1 + 3 + 2 = 6$:

$$(1 + 3 + 2)x + (2 + 1 + 3)y + (3 + 2 + 1)z = 14 + 11 + 11 \;\Rightarrow\; 6x + 6y + 6z = 36.$$

Divide by 6:

$$x + y + z = 6.$$

Answer: $x + y + z = 6$, obtained in one step — no elimination needed. (For the record the full solution is $(1, 2, 3)$, and indeed $1 + 2 + 3 = 6$ ✓.)

Tip: Whenever a question asks for a *symmetric* combination like $x + y + z$ or $xy + yz + zx$, first check whether adding or symmetrically combining the equations delivers it immediately. It often does.

7 Quadratic Functions & Complex Numbers

Three forms, graphing, solving, completing the square, the quadratic formula, and complex numbers

7.1 Three Forms of a Quadratic

The same parabola, written three ways

Every quadratic function can be written in three equivalent forms. Each form reveals something different at a glance.

- **Standard form:** $y = ax^2 + bx + c$ (reveals the y-intercept c)
- **Vertex form:** $y = a(x - h)^2 + k$ (reveals the vertex (h, k))
- **Factored form:** $y = a(x - r_1)(x - r_2)$ (reveals the x-intercepts r_1, r_2)

In all three, the number a is the same: it controls direction and width.

The vertex is the turning point.

(diagram in the online version)

One parabola in all three forms

Consider $y = x^2 - 2x - 3$ (standard). Its vertex is at $x = -\dfrac{b}{2a} = \dfrac{2}{2} = 1$, giving $y = 1 - 2 - 3 = -4$, so

$$y = (x - 1)^2 - 4 \quad \text{(vertex form, vertex } (1, -4)\text{)}.$$

Factoring the standard form: $x^2 - 2x - 3 = (x + 1)(x - 3)$, so

$$y = (x + 1)(x - 3) \quad \text{(factored form, roots } x = -1,\ x = 3\text{)}.$$

Tip: To go from standard to vertex form, complete the square. To go from standard to factored form, factor (or use the roots from the quadratic formula). To go back to standard from either, expand and combine like terms.

7.2 Graphing Parabolas

Anatomy of a parabola

For $y = ax^2 + bx + c$:

- **Direction:** opens *up* if $a > 0$ (has a minimum), *down* if $a < 0$ (has a maximum).
- **Axis of symmetry:** the vertical line $x = -\dfrac{b}{2a}$.
- **Vertex:** $\left(-\dfrac{b}{2a},\ f\left(-\dfrac{b}{2a}\right)\right)$; this is the min or max point.
- **y-intercept:** $(0, c)$. **x-intercepts:** solve $ax^2 + bx + c = 0$.

The vertex is the turning point.

(diagram in the online version)

Reading a graph

For $y = -x^2 + 3$: $a = -1 < 0$ so it opens *down* with a *maximum*. The axis of symmetry is $x = 0$ and the vertex is $(0, 3)$.

Transformations of $y = x^2$: $y = a(x - h)^2 + k$ shifts the graph right h, up k, and stretches by a (reflects if $a < 0$). Bigger $|a|$ means a narrower parabola.

7.3 Solving by Factoring and the Square-Root Method

Two quick methods

Factoring uses the Zero Product Property: if $A \cdot B = 0$ then $A = 0$ or $B = 0$. Set the quadratic equal to 0, factor, and solve each factor.

Square-root method works when there is no middle (bx) term, or when one side is a perfect square: isolate the square, then take $\pm$ the square root of both sides.

Factoring

Solve $x^2 - 5x + 6 = 0$. Factor: $(x - 2)(x - 3) = 0$, so $x = 2$ or $x = 3$.

Square-root method

Solve $(x-4)^2 = 25$. Take roots: $x - 4 = \pm 5$, so $x = 9$ or $x = -1$.

Remember the $\pm$! A square root gives two solutions. Dropping the negative root loses half your answers.

7.4 Completing the Square

Building a perfect square

To complete the square on $x^2 + bx$, add $\left(\dfrac{b}{2}\right)^2$. This makes

$$x^2 + bx + \left(\tfrac{b}{2}\right)^2 = \left(x + \tfrac{b}{2}\right)^2.$$

Use this to solve any quadratic or to convert to vertex form.

The vertex is the turning point.

(diagram in the online version)

Completing the square (fully worked)

Solve $x^2 + 6x + 5 = 0$.

$$\begin{aligned} x^2 + 6x &= -5 && \text{move constant right} \\ x^2 + 6x + 9 &= -5 + 9 && \text{add } (6/2)^2 = 9 \text{ to both sides} \\ (x+3)^2 &= 4 && \text{write left as a square} \\ x + 3 &= \pm 2 && \text{take square roots} \\ x &= -3 \pm 2 \;\Rightarrow\; x = -1 \text{ or } x = -5. \end{aligned}$$

Converting to vertex form

Write $y = x^2 - 4x + 7$ in vertex form. Take $(4/2)^2 = 4$:

$$y = (x^2 - 4x + 4) + 7 - 4 = (x-2)^2 + 3, \quad \text{vertex } (2,3).$$

Tip: If $a \neq 1$, factor a out of the x^2 and x terms first, then complete the square inside the parentheses.

7.5 The Quadratic Formula and the Discriminant

The formula that always works

For $ax^2 + bx + c = 0$ with $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity under the radical, $D = b^2 - 4ac$, is the **discriminant**:

- $D > 0$: two distinct real solutions.
- $D = 0$: one repeated real solution.
- $D < 0$: two complex (nonreal) solutions.

Quadratic formula (fully worked)

Solve $2x^2 + 3x - 2 = 0$, so $a = 2,\ b = 3,\ c = -2$.

$$x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}.$$

So $x = \frac{2}{4} = \frac{1}{2}$ or $x = \frac{-8}{4} = -2$. (Here $D = 25 > 0$: two real roots.)

Check first: compute the discriminant before solving. It tells you *how many* and *what kind* of answers to expect.

7.6 Complex Numbers: the Imaginary Unit i

A new number

The imaginary unit is defined by $i = \sqrt{-1}$, so that $i^2 = -1$. A **complex number** has standard form $a + bi$, where a is the real part and b is the imaginary part.

Powers of i cycle with period 4:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \ \ldots$$

Simplifying

$\sqrt{-25} = \sqrt{25} \cdot \sqrt{-1} = 5i. \qquad i^{23} = i^{20} \cdot i^3 = (i^4)^5 \cdot i^3 = 1 \cdot (-i) = -i.$

Powers of i: divide the exponent by 4 and keep the remainder: remainder $0 \to 1$, $1 \to i$, $2 \to -1$, $3 \to -i$.

7.7 Operations with Complex Numbers

Add, subtract, multiply, divide

Add/subtract: combine real parts and imaginary parts separately.

Multiply: use FOIL, then replace i^2 with -1.

Conjugate: the conjugate of $a + bi$ is $a - bi$. Their product $(a + bi)(a - bi) = a^2 + b^2$ is a real number.

Divide: multiply numerator and denominator by the conjugate of the denominator.

Multiplying

$(2 + 3i)(1 - i) = 2 - 2i + 3i - 3i^2 = 2 + i - 3(-1) = 5 + i$.

Dividing (using the conjugate)

$$\frac{3+2i}{1-i} = \frac{3+2i}{1-i} \cdot \frac{1+i}{1+i} = \frac{3+3i+2i+2i^2}{1-i^2} = \frac{1+5i}{2} = \frac{1}{2} + \frac{5}{2}i.$$

Tip: A complex answer is not finished until it is in the form $a + bi$. Real and imaginary parts should be separated.

7.8 Quadratics with Complex Solutions

When the parabola misses the x-axis

If the discriminant $D < 0$, the graph has no x-intercepts and the two solutions are complex conjugates $p \pm qi$. Solve exactly as before—the negative under the radical just introduces i.

The vertex is the turning point.

(diagram in the online version)

Complex roots (fully worked)

Solve $x^2 - 2x + 5 = 0$, so $a = 1,\ b = -2,\ c = 5$.

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

The solutions are $1 + 2i$ and $1 - 2i$, a conjugate pair.

Remember: complex roots always come in conjugate pairs $p + qi$ and $p - qi$ (for real coefficients).

7.9 Modeling with Quadratics

Where parabolas appear

Quadratics model projectile height, areas, and any quantity with a single maximum or minimum.

- **Projectiles:** $h(t) = -16t^2 + v_0 t + h_0$ (feet, seconds). The peak is at the vertex time $t = -\dfrac{v_0}{2(-16)}$; the ground is where $h(t) = 0$.
- **Max/min:** the extreme value always occurs at the vertex.

The vertex is the turning point.

(diagram in the online version)

Projectile

A ball is thrown so $h(t) = -16t^2 + 32t$. Vertex time: $t = -\dfrac{32}{2(-16)} = 1$ s. Max height $h(1) = -16 + 32 = 16$ ft. It lands when $-16t^2 + 32t = 0 \Rightarrow -16t(t - 2) = 0$, at $t = 2$ s.

Tip: "When is it highest?" asks for the vertex time; "how high?" asks for the vertex y-value; "when does it land?" asks for the positive root.

7.10 Going Deeper: Advanced Quadratics & Complex Numbers

Vieta's formulas: roots without solving

For $ax^2 + bx + c = 0$ with roots r_1, r_2, dividing by a gives $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$. Comparing with

$(x-r_1)(x-r_2) = x^2 - (r_1+r_2)x + r_1 r_2$ shows

$$r_1 + r_2 = -\frac{b}{a}, \qquad r_1 r_2 = \frac{c}{a}.$$

These **Vieta's formulas** let you read the sum and product of the roots straight off the coefficients—no solving required. Any *symmetric* function of the roots can then be rebuilt from these two numbers, e.g.

$$r_1^2 + r_2^2 = (r_1+r_2)^2 - 2r_1 r_2, \qquad \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1 r_2}.$$

The vertex is the turning point.

(diagram in the online version)

Symmetric functions of roots (fully worked)

Without solving, evaluate $r_1^2 + r_2^2$ and $\dfrac{1}{r_1} + \dfrac{1}{r_2}$ for $2x^2 - 5x + 1 = 0$.

First read off the symmetric quantities with $a = 2,\ b = -5,\ c = 1$:

$$r_1 + r_2 = -\frac{b}{a} = \frac{5}{2}, \qquad r_1 r_2 = \frac{c}{a} = \frac{1}{2}.$$

Now assemble the answers from these:

$$\begin{aligned} r_1^2 + r_2^2 &= (r_1+r_2)^2 - 2r_1 r_2 = \left(\tfrac{5}{2}\right)^2 - 2 \cdot \tfrac{1}{2} = \tfrac{25}{4} - 1 = \tfrac{21}{4}, \\ \frac{1}{r_1} + \frac{1}{r_2} &= \frac{r_1+r_2}{r_1 r_2} = \frac{5/2}{1/2} = 5. \end{aligned}$$

Test for a symmetric expression: if swapping $r_1 \leftrightarrow r_2$ leaves it unchanged, it can be written using only $r_1 + r_2$ and $r_1 r_2$. Start every such problem by writing down those two values.

Building a quadratic from transformed roots

To construct a monic quadratic with prescribed roots s_1, s_2, use

$$x^2 - (s_1 + s_2)\, x + s_1 s_2 = 0.$$

When the new roots are *transformations* of old roots r_1, r_2, compute the new sum and product from Vieta's formulas of the original equation—you never need the roots themselves.

- Roots $r_1 + k,\ r_2 + k$: replace x by $x - k$ in the original.

- Roots $kr_1,\ kr_2$: new sum $k(r_1+r_2)$, new product $k^2 r_1 r_2$.
- Roots $\dfrac{1}{r_1}, \dfrac{1}{r_2}$: reverse the coefficient list $a, b, c \to c, b, a$.

New quadratic whose roots are the squares (fully worked)

The equation $x^2 - 6x + 4 = 0$ has roots r_1, r_2. Build a quadratic whose roots are r_1^2 and r_2^2.

From Vieta: $r_1 + r_2 = 6$ and $r_1 r_2 = 4$. The new sum and product are

$$\begin{aligned} r_1^2 + r_2^2 &= (r_1+r_2)^2 - 2r_1r_2 = 36 - 8 = 28, \\ r_1^2 r_2^2 &= (r_1r_2)^2 = 16. \end{aligned}$$

So a quadratic with these roots is

$$x^2 - 28x + 16 = 0.$$

Discriminant & parameter conditions

Treat the discriminant $D = b^2 - 4ac$ as a function of an unknown parameter to force a desired root behavior:

- **Two distinct real roots:** $D > 0$.
- **One repeated (double) root:** $D = 0$.
- **No real roots (complex pair):** $D < 0$.

Setting $D = 0$ also gives the **tangency** condition when a line meets a parabola in exactly one point (see below), and the boundary value where the number of real roots changes.

Watch the leading coefficient. If the parameter multiplies x^2, first insist $a \neq 0$ (otherwise the equation is linear, not quadratic) before applying a discriminant condition.

Line tangent to a parabola (fully worked)

For which value of m is the line $y = mx + 2$ tangent to the parabola $y = x^2 + 3x + 6$?

Tangency means the system meets in exactly one point, so set the two expressions equal and require a *double* root:

$$\begin{aligned} x^2 + 3x + 6 &= mx + 2 \\ x^2 + (3-m)x + 4 &= 0. \end{aligned}$$

A double root needs $D = 0$:

$$D = (3-m)^2 - 4(1)(4) = 0$$
$$(3-m)^2 = 16$$
$$3 - m = \pm 4 \;\Rightarrow\; m = -1 \text{ or } m = 7.$$

Both lines touch the parabola at a single point.

Modulus & argument of a complex number

Plot $z = a + bi$ as the point (a, b). Its **modulus** (distance from the origin) and **argument** (angle from the positive real axis) are

$$|z| = \sqrt{a^2 + b^2}, \qquad \theta = \arg z = \operatorname{atan2}(b, a).$$

Key facts: $|z|^2 = z\bar{z}$, moduli multiply ($|z_1 z_2| = |z_1||z_2|$), and multiplying by i rotates a point 90° counterclockwise. This geometric view explains why the powers of i cycle: i has modulus 1 and argument 90°, so i^n walks around the unit circle in 90° steps.

Modulus and argument (fully worked)

Find the modulus and argument of $z = -1 + i\sqrt{3}$.

$$|z| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2,$$
$$\theta = \operatorname{atan2}(\sqrt{3}, -1) = 120°,$$

because $(-1, \sqrt{3})$ sits in the second quadrant. So z has modulus 2 and argument 120°.

Solving $z^2 = \bar{z}$ (fully worked)

Find every complex $z = a + bi$ with $z^2 = \bar{z}$.

Take moduli of both sides: $|z|^2 = |\bar{z}| = |z|$, so $|z|(|z| - 1) = 0$. Hence $|z| = 0$ or $|z| = 1$.

- If $|z| = 0$ then $z = 0$, which checks: $0^2 = 0$.
- If $|z| = 1$, multiply $z^2 = \bar{z}$ by z: $z^3 = z\bar{z} = |z|^2 = 1$. So z is a cube root of unity.

The cube roots of unity are $z = 1$ and $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. Together with $z = 0$ these are the four solutions:

$$z = 0, \quad z = 1, \quad z = -\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, \quad z = -\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i.$$

Conjugate root pairs. A polynomial with *real* coefficients has complex roots only in conjugate pairs: if $p + qi$ is a root, so is $p - qi$. That pair multiplies back to the real quadratic factor

$$\big(x - (p + qi)\big)\big(x - (p - qi)\big) = x^2 - 2px + (p^2 + q^2),$$

whose sum of roots $2p$ and product $p^2 + q^2 = |p + qi|^2$ are exactly Vieta's formulas again.

8 Polynomial Functions

Vocabulary, end behavior, division, zeros, and graphs

8.1 Polynomial Vocabulary

The language of polynomials

A **polynomial in one variable** is a sum of terms of the form $a_n x^n$, where each exponent is a whole number.

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

- **Degree:** the greatest exponent on the variable.
- **Leading coefficient:** the coefficient a_n of the highest-degree term.
- **Standard form:** terms written from highest degree to lowest.
- **Constant term:** a_0, the term with no variable.

Classify by degree: 0 constant, 1 linear, 2 quadratic, 3 cubic, 4 quartic, 5 quintic.

Reading off the parts

Consider $g(x) = 7 - 2x^3 + 4x$.
Write in standard form: $g(x) = -2x^3 + 4x + 7$.
Degree $= 3$ (cubic), leading coefficient $= -2$, constant term $= 7$.

Tip: A polynomial must have only whole-number exponents. Expressions like x^{-1}, $\sqrt{x} = x^{1/2}$, or $\dfrac{1}{x}$ are *not* polynomial terms.

8.2 End Behavior

What the "arms" do

The **end behavior** of a graph describes what happens to $f(x)$ as $x \to +\infty$ and $x \to -\infty$. It depends only on the **degree parity** and the **sign of the leading coefficient**.

degree	leading coeff.	end behavior
even	$+$	up-left, up-right
even	$-$	down-left, down-right
odd	$+$	down-left, up-right
odd	$-$	up-left, down-right

Predicting the ends

For $f(x) = -2x^3 + 5x - 1$: degree 3 is odd and the leading coefficient -2 is negative, so the graph goes **up on the left and down on the right**. In symbols: as $x \to -\infty$, $f(x) \to +\infty$, and as $x \to +\infty$, $f(x) \to -\infty$.

Remember: Only the leading term controls end behavior. Every other term becomes negligible for very large $|x|$.

8.3 Adding, Subtracting, and Multiplying Polynomials

Combining polynomials

Add / subtract: combine like terms. When subtracting, distribute the $-$ sign to every term first.
Multiply: distribute each term of one factor across the other, then combine like terms. The degrees add: $(\deg m) \cdot (\deg n)$ product has degree $m + n$.

Three operations

Sum: $(3x^2 - 2x + 1) + (x^2 + 5x - 4) = 4x^2 + 3x - 3$.
Difference: $(5x^3 - 2x + 4) - (2x^3 + x - 1) = 3x^3 - 3x + 5$.
Product: $(x + 3)(x^2 - 2x + 5)$

$$= x^3 - 2x^2 + 5x + 3x^2 - 6x + 15 = x^3 + x^2 - x + 15.$$

Tip: Line up like terms in columns for addition and subtraction, just like whole numbers. It prevents dropped terms.

8.4 Long Division and Synthetic Division

Two ways to divide

Long division works for any divisor. **Synthetic division** is a fast shortcut that works only when dividing by a linear factor $x - c$. The Division Algorithm says

$$f(x) = (\text{divisor})(\text{quotient}) + \text{remainder}.$$

Long division: $(2x^3 + 3x^2 - x + 5) \div (x + 2)$

$$\begin{array}{r} 2x^2 - x + 1 \\ x + 2 \,\overline{\smash{)}\, 2x^3 + 3x^2 - x + 5} \end{array}$$

Step by step the quotient is $2x^2 - x + 1$ with remainder 3, so

$$\frac{2x^3 + 3x^2 - x + 5}{x + 2} = 2x^2 - x + 1 + \frac{3}{x + 2}.$$

Synthetic division: $(x^3 - 4x^2 + 5x - 2) \div (x - 1)$

Use $c = 1$ with the coefficients $1,\ -4,\ 5,\ -2$:

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 5 & -2 \\ & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & 0 \end{array}$$

The bottom row gives quotient $x^2 - 3x + 2$ and remainder 0. Since the remainder is 0, $(x - 1)$ is a factor:

$$x^3 - 4x^2 + 5x - 2 = (x - 1)(x^2 - 3x + 2) = (x - 1)^2(x - 2).$$

Watch out: Include a 0 placeholder for every missing power before you divide. For $x^3 - 7$ use coefficients $1,\ 0,\ 0,\ -7$.

8.5 The Remainder and Factor Theorems

Evaluating by dividing

Remainder Theorem: When $f(x)$ is divided by $x - c$, the remainder equals $f(c)$.
Factor Theorem: $(x - c)$ is a factor of $f(x)$ *if and only if* $f(c) = 0$ (that is, c is a zero).

Testing a factor

Is $(x-2)$ a factor of $f(x) = x^3 - 4x^2 + 5x - 2$? Evaluate:

$$f(2) = 8 - 16 + 10 - 2 = 0.$$

Because $f(2) = 0$, yes—$(x-2)$ is a factor, and $x = 2$ is a zero.

Tip: Synthetic division and the Remainder Theorem agree: the last number in a synthetic-division row *is* $f(c)$.

8.6 Factoring Higher-Degree Polynomials

A factoring toolkit

- **GCF first:** always pull out the greatest common factor.
- **Grouping:** for four terms, group in pairs and factor each.
- **Sum of cubes:** $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.
- **Difference of cubes:** $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$.
- **Quadratic form:** treat $x^4 - 5x^2 + 4$ like $u^2 - 5u + 4$ with $u = x^2$.

Four factoring moves

GCF: $6x^4 - 9x^3 + 3x^2 = 3x^2(2x^2 - 3x + 1) = 3x^2(2x-1)(x-1)$.
Grouping: $x^3 + 2x^2 - 9x - 18 = x^2(x+2) - 9(x+2) = (x+2)(x^2-9) = (x+2)(x-3)(x+3)$.
Difference of cubes: $8x^3 - 1 = (2x-1)(4x^2 + 2x + 1)$.
Quadratic form: $x^4 - 5x^2 + 4 = (x^2-1)(x^2-4) = (x-1)(x+1)(x-2)(x+2)$.

Remember: "Factor completely" means keep going until every factor is prime over the integers (or as far as the problem allows).

8.7 Finding All Zeros

Rational Root Theorem, multiplicity, and depressing

Rational Root Theorem: any rational zero of a polynomial with integer coefficients has the form $\frac{p}{q}$, where p divides the constant term and q divides the leading coefficient.
Strategy: test candidates until one works, use synthetic division to *depress* the polynomial to a lower degree, then repeat or factor the result.
Multiplicity: the number of times a factor repeats. A zero of multiplicity 2 makes the graph

touch (bounce off) the x-axis; odd multiplicity makes it *cross*.

Find all zeros of $f(x) = x^3 - 2x^2 - 5x + 6$

Candidates: $\pm1, \pm2, \pm3, \pm6$. Test $x = 1$: $1 - 2 - 5 + 6 = 0$, so $(x - 1)$ is a factor. Depress:

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

The quotient is $x^2 - x - 6 = (x - 3)(x + 2)$. Therefore

$$f(x) = (x - 1)(x - 3)(x + 2), \qquad \text{zeros: } x = 1,\ 3,\ -2.$$

A case with complex zeros: $f(x) = x^3 - 2x^2 + x - 2$

Test $x = 2$: $8 - 8 + 2 - 2 = 0$. Depress with $c = 2$:

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 1 & -2 \\ & & 2 & 0 & 2 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

The quotient is $x^2 + 1$, which has zeros $x = \pm i$. So the zeros are $x = 2,\ i,\ -i$.

Tip: Once you depress down to a quadratic, stop dividing—finish with factoring or the quadratic formula.

8.8 Fundamental Theorem of Algebra

Counting zeros

Fundamental Theorem of Algebra: every polynomial of degree $n \geq 1$ has exactly n zeros in the complex numbers, counted with multiplicity.

Complex Conjugate Theorem: if the coefficients are real and $a + bi$ is a zero, then its conjugate $a - bi$ is also a zero. Complex zeros always come in pairs, so a polynomial of *odd* degree with real coefficients has at least one real zero.

Writing a polynomial from its zeros

(a) Zeros $2,\ -3,\ 5$:

$$(x - 2)(x + 3)(x - 5) = (x^2 + x - 6)(x - 5) = x^3 - 4x^2 - 11x + 30.$$

(b) Zeros $3,\ 2i,\ -2i$ (the pair is required for real coefficients):

$$(x-3)(x^2+4) = x^3 - 3x^2 + 4x - 12.$$

Tip: A pair of conjugate zeros $a \pm bi$ multiplies to the real quadratic $x^2 - 2ax + (a^2 + b^2)$—no imaginary numbers left behind.

8.9 Graphing Polynomial Functions

Putting it all together

To sketch $y = f(x)$:

1. Find the **end behavior** from degree and leading coefficient.
2. Find the **zeros** (x-intercepts) and their **multiplicities**: cross at odd, touch at even.
3. Plot the **y-intercept** $f(0)$.
4. A degree-n polynomial has at most $n-1$ **turning points**.
5. Connect smoothly—polynomial graphs have no breaks or sharp corners.

Graphing $f(x) = x^3 - 4x = x(x-2)(x+2)$

Odd degree, positive leading coefficient: down-left, up-right. Zeros $x = -2, 0, 2$, each multiplicity 1 (the graph crosses at each). The curve below shows all three crossings.

A repeated zero: $f(x) = (x+1)^2(x-2) = x^3 - 3x - 2$

Zeros $x = -1$ (multiplicity 2, the graph *touches* and turns) and $x = 2$ (multiplicity 1, the graph *crosses*). Odd degree, positive lead: down-left, up-right.

An even-degree graph: $f(x) = x^4 - 5x^2 + 4$

Even degree, positive lead: both ends up. Four real zeros at $x = -2, -1, 1, 2$ (each crossing).

Remember: The number of real zeros can be less than the degree—some zeros may be complex, appearing as "missing" x-intercepts even though the Fundamental Theorem still counts them.

8.10 Going Deeper: Advanced Polynomial Ideas

Vieta's formulas: roots to coefficients

For a monic polynomial with roots $r_1, \ldots, r_n$ (repeated by multiplicity), expanding $\prod(x - r_i)$ shows that the **elementary symmetric functions** of the roots are the coefficients, with alternating signs. For a cubic $x^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3:

$$r_1 + r_2 + r_3 = -b, \quad r_1r_2 + r_1r_3 + r_2r_3 = c, \quad r_1r_2r_3 = -d.$$

For a quartic $x^4 + bx^3 + cx^2 + dx + e$ with roots r_1, r_2, r_3, r_4:

$$\sum r_i = -b, \quad \sum_{i<j} r_ir_j = c, \quad \sum_{i<j<k} r_ir_jr_k = -d, \quad r_1r_2r_3r_4 = e.$$

Non-monic: for leading coefficient a_n, divide through first—each symmetric function is $(-1)^k a_{n-k}/a_n$.

Using Vieta without finding the roots

The roots of $x^3 - 6x^2 + 11x - 6$ satisfy $r_1 + r_2 + r_3 = 6$, $r_1r_2 + r_1r_3 + r_2r_3 = 11$, and $r_1r_2r_3 = 6$. Find $r_1^2 + r_2^2 + r_3^2$ *without* solving. Use the identity

$$r_1^2 + r_2^2 + r_3^2 = \left(\sum r_i\right)^2 - 2\textstyle\sum_{i<j} r_ir_j = 6^2 - 2(11) = 14.$$

(Check: the roots are $1, 2, 3$, and $1+4+9 = 14$.) Also $\dfrac{1}{r_1} + \dfrac{1}{r_2} + \dfrac{1}{r_3} = \dfrac{r_1r_2 + r_1r_3 + r_2r_3}{r_1r_2r_3} = \dfrac{11}{6}$.

Power sums and Newton's identities

Let $p_k = r_1^k + \cdots + r_n^k$ be the kth **power sum** of the roots and let e_k be the kth elementary symmetric function (so $e_1 = \sum r_i$, $e_2 = \sum_{i<j} r_ir_j$, ...). **Newton's identities** convert between them:

$$p_1 = e_1, \quad p_2 = e_1p_1 - 2e_2, \quad p_3 = e_1p_2 - e_2p_1 + 3e_3,$$

$$p_4 = e_1p_3 - e_2p_2 + e_3p_1 - 4e_4.$$

These let you compute $\sum r_i^k$ from the coefficients alone—no roots needed.

Power sums via Newton's identities

For $x^3 - 6x^2 + 11x - 6$ we read off $e_1 = 6$, $e_2 = 11$, $e_3 = 6$. Then

$$\begin{aligned} p_1 &= e_1 = 6, \\ p_2 &= e_1p_1 - 2e_2 = 6 \cdot 6 - 2 \cdot 11 = 14, \\ p_3 &= e_1p_2 - e_2p_1 + 3e_3 = 6 \cdot 14 - 11 \cdot 6 + 3 \cdot 6 = 36. \end{aligned}$$

So $r_1^3 + r_2^3 + r_3^3 = 36$. (Check with roots $1, 2, 3$: $1 + 8 + 27 = 36$.)

Remainder of $P(x)$ modulo $(x-a)(x-b)$

Dividing $P(x)$ by a *quadratic* leaves a remainder of degree at most 1, say $R(x) = mx + k$. Since $P(x) = (x-a)(x-b)Q(x) + mx + k$, substituting the two roots kills the quotient term:

$$P(a) = ma + k, \qquad P(b) = mb + k.$$

Solve this 2×2 linear system for m and k. The remainder is then

$$R(x) = \frac{P(a) - P(b)}{a - b}\,(x - a) + P(a),$$

a form of linear interpolation through the points $(a, P(a))$ and $(b, P(b))$.

Remainder when dividing by $(x-1)(x-2)$

Find the remainder of $P(x) = x^{50} - 3x + 7$ upon division by $(x-1)(x-2) = x^2 - 3x + 2$. Evaluate at the two roots:

$$P(1) = 1 - 3 + 7 = 5, \qquad P(2) = 2^{50} - 6 + 7 = 2^{50} + 1.$$

Write $R(x) = mx + k$. Then $m + k = 5$ and $2m + k = 2^{50} + 1$. Subtracting,

$$m = 2^{50} - 4, \qquad k = 5 - m = 9 - 2^{50}.$$

So the remainder is $R(x) = (2^{50} - 4)x + (9 - 2^{50})$.

Building a polynomial from finite differences

Tabulate a degree-n polynomial at equally spaced inputs and take successive **differences** of the outputs. The nth differences are *constant* and equal $a_n \cdot n!$ (for step size 1), while the $(n+1)$th differences are 0. This both *detects* the degree and *reconstructs* the leading coefficient.

x	0	1	2	3	4
f	f_0	f_1	f_2	f_3	f_4
Δ		$f_1 - f_0$	$f_2 - f_1$	$\cdots$	
Δ^2			const if deg 2		

More generally, $n+1$ data points determine a unique polynomial of degree $\leq n$ (interpolation).

Recovering a polynomial from a table

A cubic f gives the values $f(0), \ldots, f(4) = 1,\ 2,\ 11,\ 34,\ 77$. Build the difference table:

$$\begin{array}{c|ccccc} f & 1 & 2 & 11 & 34 & 77 \\ \Delta & & 1 & 9 & 23 & 43 \\ \Delta^2 & & & 8 & 14 & 20 \\ \Delta^3 & & & & 6 & 6 \end{array}$$

Third differences are constant (6), confirming degree 3 with leading coefficient $6/3! = 1$. Now fit $f(x) = x^3 + bx^2 + cx + d$: from $f(0) = 1$, $d = 1$; from $f(1) = 2$ and $f(2) = 11$,

$$1 + b + c + 1 = 2, \qquad 8 + 4b + 2c + 1 = 11 \ \Rightarrow\ b + c = 0,\ 4b + 2c = 2,$$

giving $b = 1,\ c = -1$. So $f(x) = x^3 + x^2 - x + 1$. (Check $f(4) = 64 + 16 - 4 + 1 = 77$.)

Roots in arithmetic or geometric progression

When a problem promises the roots follow a pattern, choose **symmetric variables** so Vieta collapses nicely.

- **Arithmetic (three roots):** write them $r - d,\ r,\ r + d$. Their sum is $3r$, so r equals $-\frac{1}{3}$(coeff. of x^2) immediately—the middle root is forced by the sum.
- **Geometric (three roots):** write them $\dfrac{r}{s},\ r,\ rs$. Their product is r^3, so r is the cube root of the product of the roots.

Cubic with roots in arithmetic progression

The roots of $x^3 - 12x^2 + 39x - 28$ are in arithmetic progression. Let them be $r - d,\ r,\ r + d$. By Vieta the sum is $3r = 12$, so $r = 4$: one root is exactly 4. Check: $f(4) = 64 - 192 + 156 - 28 = 0$. Depress by $(x - 4)$ to get $x^2 - 8x + 7 = (x - 1)(x - 7)$, so the roots are $1, 4, 7$—indeed an arithmetic progression with common difference 3.

Rational Root Theorem with bounds and Descartes' Rule

Three tools sharply narrow the candidate list before you test anything.

- **Rational Root Theorem:** every rational root is $\pm\dfrac{p}{q}$ with $p \mid a_0,\ q \mid a_n$.
- **Upper/lower bounds (synthetic division):** if you divide by $x - c$ with $c > 0$ and the bottom row is *all nonnegative*, then c is an **upper bound**—no root exceeds it. If you divide by $x - c$ with $c < 0$ and the bottom row *alternates in sign*, then c is a **lower bound**.
- **Descartes' Rule of Signs:** the number of positive real roots equals the number of sign changes in $P(x)$, or less by an even number. Apply to $P(-x)$ for the count of negative real roots.

Pruning candidates with Descartes and bounds

For $P(x) = x^3 - 4x^2 + x + 6$: the sign pattern of the coefficients $+, -, +, +$ has 2 changes, so there are 2 or 0 positive roots. Then $P(-x) = -x^3 - 4x^2 - x + 6$ has pattern $-, -, -, +$ with 1 change, so exactly 1 negative root. Rational candidates are $\pm 1, \pm 2, \pm 3, \pm 6$. Testing $x = -1$: $-1-4-1+6 = 0$—the guaranteed negative root. Depressing gives $x^2-5x+6 = (x-2)(x-3)$, so the two positive roots are 2 and 3, matching Descartes exactly. Dividing by $x - 4$ leaves all-nonnegative entries, confirming 4 is an upper bound.

The "$(a-b) \mid P(a)-P(b)$" idea: for a polynomial P with *integer* coefficients and any integers a, b,

$$(a - b) \bigm| P(a) - P(b).$$

This is because each $a^k - b^k$ is divisible by $a - b$. It is a powerful screening tool: e.g. if P has integer coefficients and $P(0) = 3$, $P(4) = 6$, no integer n with $P(n) = 0$ can exist between them that violates $n \mid 3$ and $(n - 4) \mid 3$. It also proves that a polynomial taking the value 1 at three distinct integers can never equal 0 at an integer.

9 Rational Expressions & Functions

Everything for Topic 6 on one desk reference

9.1 Simplifying Rational Expressions & Domain

What is a Rational Expression?

A **rational expression** is a ratio of two polynomials, $\dfrac{P(x)}{Q(x)}$. It is undefined wherever the **denominator equals zero** — those inputs are the **excluded values**. The **domain** is every real number *except* the excluded values.

To **simplify**: factor the numerator and denominator completely, then cancel any **common factors**. Always list excluded values *from the original denominator*.

Worked Example: Factor, then Cancel

Simplify $\dfrac{x^2 - 9}{x^2 - x - 6}$ and state the domain.

$$\begin{aligned} \frac{x^2 - 9}{x^2 - x - 6} &= \frac{(x-3)(x+3)}{(x-3)(x+2)} && \text{factor both} \\ &= \frac{x+3}{x+2} && \text{cancel the common factor } (x-3) \end{aligned}$$

Excluded values come from the *original* denominator $(x-3)(x+2)=0$, so $x \neq 3$ and $x \neq -2$. Domain: all reals except -2 and 3.

Cancel factors, never terms. You may cancel $\dfrac{(x-3)}{(x-3)}$ because $x-3$ is a *factor*. You may **not** cancel the x's in $\dfrac{x+3}{x+2}$ — those are *terms* joined by $+$, not factors.

9.2 Multiplying & Dividing

Multiply Straight Across; Divide by Flipping

Multiply: factor everything, cancel any factor in a numerator against a matching factor in a denominator, then write what remains.

Divide: multiply by the **reciprocal** of the second fraction — "*keep, change, flip*" — then proceed as a multiplication.

Worked Example: Multiply

Simplify $\dfrac{x^2-4}{x^2+6x+9} \cdot \dfrac{x+3}{x-2}$.

$$\begin{aligned} &= \frac{(x-2)(x+2)}{(x+3)^2} \cdot \frac{x+3}{x-2} && \text{factor} \\ &= \frac{x+2}{x+3} && \text{cancel } (x-2) \text{ and one } (x+3) \end{aligned}$$

Worked Example: Divide

Simplify $\dfrac{x^2-1}{x+2} \div \dfrac{x-1}{x^2-4}$.

$$\begin{aligned} &= \frac{(x-1)(x+1)}{x+2} \cdot \frac{(x-2)(x+2)}{x-1} && \text{flip and factor} \\ &= (x+1)(x-2) && \text{cancel } (x-1) \text{ and } (x+2) \end{aligned}$$

9.3 Adding & Subtracting (LCD)

Common Denominators

You can only combine fractions over a **common denominator**. Find the **LCD**: factor each denominator, then take each distinct factor to its *highest* power that appears. Rewrite each fraction with the LCD, combine the numerators, then simplify.

Worked Example: Add with Unlike Denominators

Simplify $\frac{3}{x+2} + \frac{2}{x-1}$. The LCD is $(x+2)(x-1)$.

$$\begin{aligned} &= \frac{3(x-1)}{(x+2)(x-1)} + \frac{2(x+2)}{(x+2)(x-1)} \\ &= \frac{3x-3+2x+4}{(x+2)(x-1)} = \frac{5x+1}{(x+2)(x-1)} \end{aligned}$$

Worked Example: Subtract (watch the sign!)

Simplify $\frac{x}{x-3} - \frac{2}{x+1}$. The LCD is $(x-3)(x+1)$.

$$\begin{aligned} &= \frac{x(x+1)-2(x-3)}{(x-3)(x+1)} && \text{distribute the minus to } \textit{both} \text{ terms} \\ &= \frac{x^2+x-2x+6}{(x-3)(x+1)} = \frac{x^2-x+6}{(x-3)(x+1)} \end{aligned}$$

Subtraction trap. The minus sign applies to the *entire* numerator of the second fraction. Wrap it in parentheses first: $-(2)(x-3) = -2x+6$, not $-2x-6$.

9.4 Complex Fractions

A Fraction Inside a Fraction

A **complex fraction** has fractions in its numerator, denominator, or both. The fastest method: find the LCD of *all* the little fractions and **multiply the top and bottom by that LCD**. This clears every small denominator at once.

Worked Example: Clear the Small Denominators

Simplify $\dfrac{1-\frac{4}{x^2}}{1-\frac{2}{x}}$. The LCD of x^2 and x is x^2.

$$= \frac{x^2\left(1-\frac{4}{x^2}\right)}{x^2\left(1-\frac{2}{x}\right)} = \frac{x^2-4}{x^2-2x}$$
$$= \frac{(x-2)(x+2)}{x(x-2)} = \frac{x+2}{x}$$

9.5 Solving Rational Equations & Extraneous Solutions

Clear the Denominators

To **solve** (not simplify) a rational equation, multiply *every term* by the LCD to clear all fractions, then solve the resulting polynomial equation. Because multiplying by a variable expression can introduce false roots, you **must check** each answer.

Extraneous solutions. Any answer that makes an *original* denominator zero is **extraneous** — discard it. Always list the excluded values before you start, then reject any solution that lands on one.

Worked Example: Extraneous Root

Solve $\dfrac{x}{x-4} = \dfrac{4}{x-4} + 3$. Excluded: $x \neq 4$.

$$\begin{aligned} x &= 4 + 3(x-4) && \text{multiply every term by } (x-4) \\ x &= 4 + 3x - 12 \\ -2x &= -8 \;\Rightarrow\; x = 4 \end{aligned}$$

But $x = 4$ is **excluded**, so it is extraneous. **No solution.**

Worked Example: Quadratic Result

Solve $x + \dfrac{6}{x} = 5$. Excluded: $x \neq 0$.

$$\begin{aligned} x^2 + 6 &= 5x && \text{multiply every term by } x \\ x^2 - 5x + 6 &= 0 \\ (x-2)(x-3) &= 0 \;\Rightarrow\; x = 2 \text{ or } x = 3 \end{aligned}$$

Neither is excluded, so both check: $x = 2,\ 3$.

9.6 Graphing Rational Functions

The Full Checklist

For $f(x) = \dfrac{P(x)}{Q(x)}$ in *lowest terms*:

- **Holes:** a factor that cancels from top and bottom gives a hole at that x-value.
- **Vertical asymptotes (VA):** set the *remaining* denominator equal to 0.
- **Horizontal asymptote (HA):** compare degrees of P and Q.
- **Intercepts:** x-intercepts where $P(x) = 0$; y-intercept at $f(0)$.

How to find the HA. Let $n = \deg P$, $m = \deg Q$.

- $n < m$: HA is $y = 0$ (the x-axis).
- $n = m$: HA is $y = \dfrac{\text{leading coeff. of } P}{\text{leading coeff. of } Q}$.
- $n > m$: **no** HA (there is a slant/oblique asymptote instead).

Worked Example: Full Analysis

Analyze $f(x) = \dfrac{x+1}{x-2}$.

- **Domain / VA:** denominator $x - 2 = 0 \Rightarrow$ VA at $x = 2$; domain $x \neq 2$.
- **HA:** degrees equal $(1 = 1)$, so $y = \frac{1}{1} = 1$.
- **x-intercept:** $x + 1 = 0 \Rightarrow (-1, 0)$.
- **y-intercept:** $f(0) = \frac{1}{-2} = -\frac{1}{2} \Rightarrow \left(0, -\frac{1}{2}\right)$.
- **Holes:** no common factor, so none.

Worked Example: A Hole

Analyze $g(x) = \dfrac{x^2-4}{x-2} = \dfrac{(x-2)(x+2)}{x-2} = x + 2$, with $x \neq 2$. The factor $(x - 2)$ cancels, so instead of a vertical asymptote there is a **hole** at $x = 2$. Its height is g's reduced form at $x = 2$: $2 + 2 = 4$, so the hole is at $(2, 4)$. The graph is the line $y = x + 2$ with a single point punched out.

9.7 Direct, Inverse & Joint Variation

Three Variation Models

Every variation problem hides a constant k. Use one data point to find k, then answer the question.

- **Direct:** $y = kx$ (as x grows, y grows proportionally).
- **Inverse:** $y = \frac{k}{x}$ (as x grows, y shrinks).
- **Joint:** $y = kxz$ (y varies with the product of two quantities).

Worked Example: Inverse Variation

y varies inversely with x, and $y = 8$ when $x = 3$. Find y when $x = 6$.

$$y = \frac{k}{x}: \quad 8 = \frac{k}{3} \Rightarrow k = 24$$
$$\text{So } y = \frac{24}{x}; \quad \text{at } x = 6, \quad y = \frac{24}{6} = 4$$

9.8 Applications: Work/Rate & Mixture

Rates Add

If one worker finishes a job in a hours, their rate is $\frac{1}{a}$ job per hour. When several work together, **rates add**. For a shared time t:

$$\frac{t}{a} + \frac{t}{b} = 1 \qquad \text{(one whole job completed).}$$

For **mixture** problems, track the *amount* of pure substance and set the concentration equal to your target.

Worked Example: Working Together

Pipe A fills a tank in 4 hours, pipe B in 6 hours. How long together?

$$\begin{aligned} \frac{t}{4} + \frac{t}{6} &= 1 && \text{LCD 12} \\ 3t + 2t &= 12 \\ 5t = 12 \Rightarrow t &= \frac{12}{5} = 2.4 \text{ hours (2 h 24 min).} \end{aligned}$$

Worked Example: Dilution

How much water must be added to 10 L of 40% acid to weaken it to 25%? The pure acid stays at $0.40(10) = 4$ L. Let $x =$ liters of water added:

$$\frac{4}{10+x} = 0.25 \;\Rightarrow\; 4 = 2.5 + 0.25x \;\Rightarrow\; x = 6 \text{ L}.$$

Sanity check. A "together" time must be *shorter* than either worker alone. If your answer is longer than the fastest worker, recheck the setup.

9.9 Going Deeper: Advanced Rational Functions

Holes vs. Vertical Asymptotes — Told Precisely

Both come from a zero of the *original* denominator, but they behave very differently. Factor top and bottom completely and compare the multiplicity of each root:

- **Vertical asymptote** at $x = a$: the factor $(x-a)$ survives in the denominator of the *reduced* form. The function blows up ($\to \pm\infty$) there.
- **Hole** at $x = a$: the factor $(x-a)$ cancels completely. If it appears to power p on top and q on the bottom, the outcome depends on which is larger.

Compare powers: with $(x-a)^p$ on top and $(x-a)^q$ on the bottom, the net factor is $(x-a)^{p-q}$. If $p \geq q$ the point is a **hole** (removable); if $p < q$ it is a **vertical asymptote** of order $q - p$.

Worked Example: Same Root, Two Fates

Analyze $f(x) = \dfrac{(x-3)^2(x+1)}{(x-3)(x-3)(x-3)} = \dfrac{(x-3)^2(x+1)}{(x-3)^3}$ near $x = 3$.

$$\begin{aligned} f(x) &= \frac{(x-3)^2(x+1)}{(x-3)^3} && \text{count multiplicities: } p = 2,\; q = 3 \\ &= \frac{x+1}{x-3} && \text{net factor } (x-3)^{2-3} = (x-3)^{-1} \end{aligned}$$

Since $p < q$, one factor of $(x-3)$ remains *below*: there is a **vertical asymptote** at $x = 3$, not a hole. Had the top been $(x-3)^3$ instead, all three would cancel and $x = 3$ would be a hole.

Slant (Oblique) Asymptotes via Long Division

When $\deg P = \deg Q + 1$ (numerator exactly one degree higher), there is no horizontal asymptote — instead the graph follows a **slant line** $y = mx + b$. Find it by **polynomial long division**:

$$\frac{P(x)}{Q(x)} = \underbrace{(mx+b)}_{\text{slant asymptote}} + \frac{\text{remainder}}{Q(x)}.$$

As $x \to \pm\infty$ the remainder term $\to 0$, so the curve hugs the line $y = mx + b$. The remainder is what you *discard* for the asymptote — but its *sign* tells you whether the curve sits above or below the line.

Worked Example: Find a Slant Asymptote

Find the slant asymptote of $f(x) = \dfrac{x^2 + 3x + 5}{x + 1}$.

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1} \qquad \text{long division: quotient } x + 2, \text{ remainder } 3$$

The slant asymptote is $y = x + 2$. Because the remainder $\dfrac{3}{x+1} > 0$ for $x > -1$, the curve lies *above* the line to the right and below it to the left. (Degrees: $\deg P = 2 = \deg Q + 1$, confirming a slant, not horizontal, asymptote.)

When does a graph cross its horizontal asymptote? A horizontal asymptote only describes *end behavior* ($x \to \pm\infty$); it may be crossed in the middle. Set $f(x) = L$ (the HA value) and solve. If there is a real solution, the graph *crosses* its HA there. A graph can **never** cross a vertical asymptote, but crossing a horizontal or slant asymptote is perfectly legal.

Worked Example: Crossing the Horizontal Asymptote

Does $f(x) = \dfrac{x^2 - 2x}{x^2 + 1}$ cross its horizontal asymptote? Degrees are equal, so HA $= y = \frac{1}{1} = 1$. Set $f(x) = 1$:

$$\begin{aligned} \frac{x^2 - 2x}{x^2 + 1} &= 1 \\ x^2 - 2x &= x^2 + 1 && \text{cross-multiply (denominator never 0)} \\ -2x &= 1 \;\Rightarrow\; x = -\tfrac{1}{2} \end{aligned}$$

Yes — the graph crosses $y = 1$ at $\left(-\frac{1}{2}, 1\right)$. Beyond that point it approaches $y = 1$ without touching it again.

Partial-Fraction Decomposition

The reverse of adding fractions: split one complicated rational expression into a *sum* of simple ones. First make sure the fraction is **proper** ($\deg P < \deg Q$); if not, divide first. Then factor Q and assign an unknown numerator to each factor type:

- **Distinct linear** $(x - a)$: term $\dfrac{A}{x - a}$.

- **Repeated linear** $(x-a)^k$: one term *per power*, $\dfrac{A_1}{x-a} + \dfrac{A_2}{(x-a)^2} + \cdots + \dfrac{A_k}{(x-a)^k}$.
- **Irreducible quadratic** (x^2+bx+c): term $\dfrac{Bx+C}{x^2+bx+c}$ (a *linear* numerator).

Clear denominators and solve for the constants — either by substituting convenient x-values or by matching coefficients.

Worked Example: Distinct Linear Factors

Decompose $\dfrac{5x+1}{(x+2)(x-1)}$. Write $\dfrac{5x+1}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1}$ and clear denominators:

$$\begin{aligned} 5x+1 &= A(x-1) + B(x+2) \\ x = 1: \quad 6 &= 3B \;\Rightarrow\; B = 2 \\ x = -2: \quad -9 &= -3A \;\Rightarrow\; A = 3 \end{aligned}$$

So $\dfrac{5x+1}{(x+2)(x-1)} = \dfrac{3}{x+2} + \dfrac{2}{x-1}$. (This is exactly the "add" example from Section 3, run backwards.)

Worked Example: Repeated and Irreducible-Quadratic Factors

Decompose $\dfrac{x^2+2}{(x-1)^2(x^2+1)}$. Set up *every* required term:

$$\frac{x^2+2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Clearing denominators gives $x^2+2 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$.

$$\begin{aligned} x = 1: &\quad 3 = 2B \;\Rightarrow\; B = \tfrac{3}{2} \\ \text{match } x^3: &\quad 0 = A + C \\ \text{match } x^0: &\quad 2 = -A + B + D \\ \text{match } x^2: &\quad 1 = A + B + C - 2C + D \end{aligned}$$

Solving the system yields $A = \frac{1}{2},\; B = \frac{3}{2},\; C = -\frac{1}{2},\; D = 0$, so

$$\frac{x^2+2}{(x-1)^2(x^2+1)} = \frac{1/2}{x-1} + \frac{3/2}{(x-1)^2} + \frac{-\frac{1}{2}x}{x^2+1}.$$

Note the repeated factor gets *two* terms and the quadratic gets a *linear* numerator $Cx+D$.

Telescoping Sums from Partial Fractions

Partial fractions turn many "impossible" sums into **telescoping** sums, where adjacent terms cancel in a cascade. The key identity is

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Summing it, every interior term is cancelled by its neighbour, leaving only the first and last:

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) = 1 - \frac{1}{N+1}.$$

Worked Example: Telescoping a Finite Sum

Evaluate $\displaystyle\sum_{n=1}^{99} \frac{1}{n(n+1)}$. Decompose the general term, then collapse:

$$\begin{aligned} \frac{1}{n(n+1)} &= \frac{1}{n} - \frac{1}{n+1} && \text{partial fractions} \\ \sum_{n=1}^{99}\left(\frac{1}{n} - \frac{1}{n+1}\right) &= 1 - \frac{1}{100} && \text{interior terms cancel} \\ &= \frac{99}{100} \end{aligned}$$

The whole sum reduces to just the surviving endpoints — no other arithmetic needed.

The $x + \frac{1}{x}$ Family

Many problems become easy once you let $s = x + \dfrac{1}{x}$. Squaring and cubing generate the other symmetric powers:

$$\begin{aligned} \left(x + \frac{1}{x}\right)^2 &= x^2 + 2 + \frac{1}{x^2} && \Rightarrow \quad x^2 + \frac{1}{x^2} = s^2 - 2 \\ \left(x + \frac{1}{x}\right)^3 &= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} && \Rightarrow \quad x^3 + \frac{1}{x^3} = s^3 - 3s \end{aligned}$$

So knowing a single value $s = x + \dfrac{1}{x}$ instantly gives $x^2 + \dfrac{1}{x^2}$ and $x^3 + \dfrac{1}{x^3}$ without ever solving for x.

Worked Example: Powers Without Solving for x

If $x + \dfrac{1}{x} = 3$, find $x^3 + \dfrac{1}{x^3}$. Let $s = 3$ and use the identities:

$$x^2 + \frac{1}{x^2} = s^2 - 2 = 9 - 2 = 7$$
$$x^3 + \frac{1}{x^3} = s^3 - 3s = 27 - 9 = 18$$

Answer: $x^3 + \dfrac{1}{x^3} = 18$. (Check via $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) = 3(7-1) = 18$.)

Continued Fractions

A **continued fraction** nests division inside division:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

Evaluate a *finite* one by working from the **bottom up**. A *periodic infinite* continued fraction satisfies a self-referential equation: if y equals the whole expression and the pattern repeats, substitute y back into itself and solve the resulting (usually quadratic) equation — keeping the positive root.

Worked Example: A Self-Referential Continued Fraction

Evaluate $y = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$, which repeats forever. Because the tail below the first bar is an identical copy of y:

$$y = 1 + \frac{1}{y} \qquad \text{the inner expression } \textit{is } y$$
$$y^2 = y + 1 \qquad \text{multiply by } y$$
$$y^2 - y - 1 = 0 \;\Rightarrow\; y = \frac{1+\sqrt{5}}{2}$$

We take the positive root, so $y = \dfrac{1+\sqrt{5}}{2}$, the **golden ratio** $\varphi \approx 1.618$.

10 Radical Functions & Rational Exponents

Everything for Topic 7 on one desk reference

10.1 nth Roots & Simplifying Radicals

What an nth Root Means

$\sqrt[n]{a} = b$ exactly when $b^n = a$. Here n is the **index** and a is the **radicand**.

- **Odd index** ($\sqrt[3]{\ }$, $\sqrt[5]{\ }$): every real number has exactly one real root, and negatives are fine. $\sqrt[3]{-8} = -2$.
- **Even index** ($\sqrt{\ }$, $\sqrt[4]{\ }$): needs a *nonnegative* radicand, and the principal root is nonnegative. $\sqrt{-9}$ is not real.

To **simplify**, factor out the largest perfect nth power: $\sqrt[n]{a \cdot b} = \sqrt[n]{a}\,\sqrt[n]{b}$.

Worked Example: Numeric Radicals

Simplify $\sqrt{48}$ and $\sqrt[3]{54}$.

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16}\,\sqrt{3} = 4\sqrt{3}$$
$$\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27}\,\sqrt[3]{2} = 3\sqrt[3]{2}$$

Pull out perfect *squares* for $\sqrt{\ }$, perfect *cubes* for $\sqrt[3]{\ }$.

Worked Example: Variables Under the Radical

Simplify $\sqrt{50x^3}$ and $\sqrt[3]{16x^4y^5}$ (assume variables are positive).

$$\sqrt{50x^3} = \sqrt{25x^2 \cdot 2x} = 5x\sqrt{2x}$$
$$\sqrt[3]{16x^4y^5} = \sqrt[3]{8x^3y^3 \cdot 2xy^2} = 2xy\,\sqrt[3]{2xy^2}$$

Trick: divide each exponent by the index. The quotient comes out; the remainder stays in.

Odd vs. even index. With an *even* index, $\sqrt[n]{x^n} = |x|$ (absolute value keeps the root nonnegative). With an *odd* index, $\sqrt[n]{x^n} = x$ (no bars needed). If a problem says "assume variables are positive," you may drop the bars.

10.2 Rational Exponents: Radical $\leftrightarrow$ Exponent Form

The Master Conversion

A fractional exponent is a compact radical. The **denominator is the index**; the **numerator is the power**:

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Special case $a^{1/n} = \sqrt[n]{a}$. A negative exponent means reciprocal: $a^{-m/n} = \dfrac{1}{a^{m/n}}$.

Worked Example: Evaluate

$$\begin{aligned} 8^{2/3} &= \left(\sqrt[3]{8}\right)^2 = 2^2 = 4 \\ 16^{3/4} &= \left(\sqrt[4]{16}\right)^3 = 2^3 = 8 \\ 27^{-1/3} &= \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3} \end{aligned}$$

Take the root first, then apply the power — the numbers stay small.

Worked Example: Convert Both Ways

$$\sqrt[5]{x^3} = x^{3/5}, \qquad x^{2/3} = \sqrt[3]{x^2}, \qquad \frac{1}{\sqrt{x}} = x^{-1/2}.$$

Remember: $a^{m/n} = \sqrt[n]{a^m}$. The *root* (denominator n) lives on the outside as the index; the *power* (numerator m) sits on the radicand.

10.3 Properties of Exponents with Rational Powers

Same Rules, Fractional Exponents

Every exponent law you know still holds — just add, subtract, or multiply the fractions:

$$x^a x^b = x^{a+b}, \quad \frac{x^a}{x^b} = x^{a-b}, \quad (x^a)^b = x^{ab}, \quad (xy)^a = x^a y^a, \quad x^{-a} = \frac{1}{x^a}.$$

Give a final answer with **positive exponents** and **one** copy of each base.

Worked Example: Combine Powers

$$\begin{aligned} x^{1/2} \cdot x^{1/3} &= x^{1/2+1/3} = x^{5/6} \\ \left(x^{2/3}\right)^{3/4} &= x^{(2/3)(3/4)} = x^{1/2} \\ \frac{x^{3/4}}{x^{1/4}} &= x^{3/4-1/4} = x^{1/2} \end{aligned}$$

Worked Example: Distribute Over a Product

Simplify $\left(8x^6y^3\right)^{2/3}$.

$$\left(8x^6y^3\right)^{2/3} = 8^{2/3}\,x^{6\cdot 2/3}\,y^{3\cdot 2/3} = 4x^4y^2.$$

Tip: When a problem mixes radicals and exponents, **convert every radical to a rational exponent first**. Then the exponent laws do all the work, and you convert back only at the very end.

10.4 Operations with Radicals

Add, Subtract, Multiply, Divide

- **Add / subtract:** only *like radicals* (same index *and* same radicand) combine, like adding like terms. Simplify *first* to reveal them.
- **Multiply:** $\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$; multiply the outside numbers together and the inside numbers together.
- **Divide:** $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$, then **rationalize** so no radical is left in a denominator.

Worked Example: Add & Multiply

$$\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3} \qquad \text{simplify, then combine}$$
$$(2\sqrt{3})(4\sqrt{6}) = 8\sqrt{18} = 8\cdot 3\sqrt{2} = 24\sqrt{2}$$

Worked Example: Rationalize the Denominator

$$\frac{6}{\sqrt{2}} = \frac{6}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$$
$$\frac{5}{\sqrt{7}+\sqrt{2}} = \frac{5(\sqrt{7}-\sqrt{2})}{(\sqrt{7}+\sqrt{2})(\sqrt{7}-\sqrt{2})} = \frac{5(\sqrt{7}-\sqrt{2})}{7-2} = \sqrt{7}-\sqrt{2}$$

For a two-term denominator, multiply by the **conjugate** (flip the middle sign).

Conjugate pattern: $(a+\sqrt{b})(a-\sqrt{b}) = a^2 - b$. The radical vanishes because it is a difference of squares — this is the whole reason conjugates work.

10.5 Solving Radical & Rational-Exponent Equations

Isolate, Power, Check

To solve, **isolate the radical**, then **raise both sides to the matching power** to undo it.

1. Get the radical alone on one side.
2. Raise both sides to the index (square a square root, cube a cube root).
3. Solve the resulting equation.
4. **Check every answer in the original equation.**

Squaring can create **extraneous solutions** — values that satisfy the squared equation but *not* the original. For $x^{m/n} = c$, raise both sides to n/m.

Worked Example: A Clean Radical Equation

Solve $\sqrt{x+2} + 3 = 7$.

$$\begin{aligned} \sqrt{x+2} &= 4 && \text{isolate the radical} \\ x+2 &= 16 && \text{square both sides} \\ x &= 14 \end{aligned}$$

Check: $\sqrt{14+2} + 3 = \sqrt{16} + 3 = 4 + 3 = 7$. ✓

Worked Example: Extraneous Solution

Solve $\sqrt{x+5} = x - 1$.

$$\begin{aligned} x+5 = (x-1)^2 &= x^2 - 2x + 1 && \text{square both sides} \\ 0 &= x^2 - 3x - 4 = (x-4)(x+1) \\ x &= 4 \quad \text{or} \quad x = -1 \end{aligned}$$

Check $x = 4$: $\sqrt{9} = 3$ and $4 - 1 = 3$. ✓ **Check** $x = -1$: $\sqrt{4} = 2$ but $-1 - 1 = -2$. **reject** — *extraneous.*
Only solution: $x = 4$.

Worked Example: Rational-Exponent Equation

Solve $x^{3/2} = 8$ (here $x \geq 0$).

$$\left(x^{3/2}\right)^{2/3} = 8^{2/3} \Rightarrow x = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$$

Check: $4^{3/2} = \left(\sqrt{4}\right)^3 = 2^3 = 8.$ ✓

Always check for extraneous solutions whenever you raise both sides to an *even* power (squaring, fourth power). Odd powers (cubing) never introduce extraneous roots, but checking is still smart. When the exponent form is $x^{2/n}$, remember both $\pm$ may work.

10.6 Graphing Square-Root & Cube-Root Functions

Parent Graphs and Their Domains

- $y = \sqrt{x}$: starts at the origin and rises slowly. **Domain** $[0, \infty)$, **range** $[0, \infty)$. Passes through $(0, 0), (1, 1), (4, 2), (9, 3)$.
- $y = \sqrt[3]{x}$: passes through the origin, defined for *all* reals, with an S-shape. **Domain** $(-\infty, \infty)$, **range** $(-\infty, \infty)$. Passes through $(-8, -2), (-1, -1), (0, 0), (1, 1), (8, 2)$.

Transformations of $y = a\sqrt{x - h} + k$: h shifts right, k shifts up, $a < 0$ flips vertically. For a square root, the graph *begins* at the point (h, k).

Worked Example: Read a Transformation

Describe $y = \sqrt{x - 2} + 1$ and state its domain and range.
Start at $(h, k) = (2, 1)$; the graph moves right and up from there. **Domain** $[2, \infty)$, **range** $[1, \infty)$. The radicand must satisfy $x - 2 \geq 0$, confirming $x \geq 2$.

Finding domain fast: for an even-index radical, set the *radicand* ≥ 0 and solve. For an odd-index (cube) root, the domain is *all real numbers* — no restriction.

10.7 Inverses & the Root–Power Connection

Roots Undo Powers

Radical functions are the **inverses** of power functions. Since $\sqrt{x}$ undoes x^2 and $\sqrt[3]{x}$ undoes x^3:

$$f(x) = x^2 \ (x \geq 0) \iff f^{-1}(x) = \sqrt{x}, \qquad f(x) = x^3 \iff f^{-1}(x) = \sqrt[3]{x}.$$

The restriction $x \geq 0$ on x^2 is what makes it one-to-one so an inverse exists. To **find an inverse**: swap x and y, then solve for y.

Worked Example: Find an Inverse

Find the inverse of $f(x) = \sqrt{x-3}$.

$$\begin{aligned} y &= \sqrt{x-3} \\ x &= \sqrt{y-3} && \text{swap } x \text{ and } y \\ x^2 &= y-3 && \text{square both sides} \\ y &= x^2+3 \end{aligned}$$

So $f^{-1}(x) = x^2 + 3$, with domain $x \geq 0$ (the range of the original).

Check an inverse by composing: $f\big(f^{-1}(x)\big) = x$ and $f^{-1}(f(x)) = x$. Graphically, f and f^{-1} are mirror images across the line $y = x$ — which is why a square-root graph is just half of a sideways parabola.

10.8 Going Deeper: Advanced Radicals & Rational Exponents

Denesting a Nested Radical $\sqrt{a+\sqrt{b}}$

Sometimes a radical hidden inside another radical can be "unpacked" into a sum of two simpler square roots. The goal is to write

$$\sqrt{a+\sqrt{b}} \;=\; \sqrt{x}+\sqrt{y}.$$

Squaring the right side gives $x + y + 2\sqrt{xy}$, so matching pieces forces

$$x+y=a \qquad \text{and} \qquad 4xy=b.$$

That is exactly a **sum-and-product** system: x and y are the two roots of $t^2 - a\,t + \frac{b}{4} = 0$. The denesting is *clean* (nice numbers) only when $a^2 - b$ is a perfect square. If it is not, leave the radical nested.

Worked Example: Denest $\sqrt{7+4\sqrt{3}}$

Write $\sqrt{7+4\sqrt{3}}$ in the form $\sqrt{x}+\sqrt{y}$. First rewrite $4\sqrt{3}=\sqrt{48}$, so $a=7$ and $b=48$.

$$\begin{aligned} x+y&=7 && \text{sum condition}\\ 4xy=48\ &\Rightarrow\ xy=12 && \text{product condition}\\ t^2-7t+12=0&=(t-3)(t-4) && \text{solve the system}\\ \{x,y\}&=\{3,4\} \end{aligned}$$

Therefore $\sqrt{7+4\sqrt{3}}=\sqrt{4}+\sqrt{3}=2+\sqrt{3}$.
Check: $\left(2+\sqrt{3}\right)^2=4+4\sqrt{3}+3=7+4\sqrt{3}$. ✓

Quick denesting test. $\sqrt{a+\sqrt{b}}$ denests into nice square roots exactly when a^2-b is a perfect square. Then use $x,y=\dfrac{a\pm\sqrt{a^2-b}}{2}$. For $7+4\sqrt{3}=7+\sqrt{48}$: $a^2-b=49-48=1$, a perfect square, so it denests — and indeed $x,y=\frac{7\pm1}{2}=4,3$.

Rational-Exponent Equations That Hide a Quadratic

An equation like $x^{2/3}-x^{1/3}-6=0$ looks exotic, but the exponents are in a $2:1$ ratio, so a u-**substitution** turns it into a plain quadratic. Let u be the piece with the *smaller* exponent:

$$u=x^{1/3}\implies x^{2/3}=\left(x^{1/3}\right)^2=u^2.$$

The equation becomes $u^2-u-6=0$. Solve for u, then **back-substitute** and undo the exponent by raising to the reciprocal power. Because we eventually cube (an odd power) no extraneous roots appear here — but always confirm the base is allowed (even-index radicals need a nonnegative radicand).

Worked Example: Solve $x^{2/3}-x^{1/3}-6=0$

Let $u=x^{1/3}$, so $x^{2/3}=u^2$.

$$\begin{aligned} u^2-u-6=0&=(u-3)(u+2) && \text{quadratic in } u\\ u=3 \quad &\text{or}\quad u=-2\\ x^{1/3}=3\ &\Rightarrow\ x=3^3=27 && \text{cube both sides}\\ x^{1/3}=-2\ &\Rightarrow\ x=(-2)^3=-8 \end{aligned}$$

Both survive because cube roots accept negatives. **Check** $x=27$: $27^{2/3}-27^{1/3}-6=9-3-6=0$. ✓ **Check** $x=-8$: $(-8)^{2/3}-(-8)^{1/3}-6=4-(-2)-6=0$. ✓

Two Radicals: Isolate, Square, Square Again

When *two* radicals appear, you cannot clear both in one squaring. The strategy:

1. Move one radical to each side (or isolate the messier one).
2. Square both sides — *one* radical survives, sitting in the cross term.
3. Isolate that remaining radical and square a **second** time.
4. Solve, then **check every candidate** in the original equation.

Two squarings make extraneous solutions especially likely, so the final check is non-negotiable.

Worked Example: $\sqrt{x+7} - \sqrt{x} = 1$

Isolate one radical, then square twice.

$$\begin{aligned} \sqrt{x+7} &= 1 + \sqrt{x} && \text{isolate a radical} \\ x + 7 &= 1 + 2\sqrt{x} + x && \text{square both sides} \\ 6 &= 2\sqrt{x} \;\Rightarrow\; \sqrt{x} = 3 && \text{isolate the survivor} \\ x &= 9 && \text{square again} \end{aligned}$$

Check: $\sqrt{9+7} - \sqrt{9} = \sqrt{16} - 3 = 4 - 3 = 1$. ✓ The only solution is $x = 9$.

The Symmetric Family $x^{1/2} + x^{-1/2}$

Expressions built from $\sqrt{x}$ and its reciprocal are **symmetric**: swapping $\sqrt{x} \leftrightarrow \frac{1}{\sqrt{x}}$ leaves them unchanged. Let $s = x^{1/2} + x^{-1/2} = \sqrt{x} + \dfrac{1}{\sqrt{x}}$. Squaring links the sum to the middle term:

$$s^2 = \left(\sqrt{x} + \tfrac{1}{\sqrt{x}}\right)^2 = x + 2 + \frac{1}{x}, \qquad \text{so} \qquad x + \frac{1}{x} = s^2 - 2.$$

The constant 2 appears because the cross term $\sqrt{x} \cdot \frac{1}{\sqrt{x}} = 1$. This trick collapses awkward $\sqrt{x}$ equations into quadratics in s, and it is the same idea behind $x + x^{-1}$ appearing all over precalculus.

Worked Example: A Symmetric Equation

Solve $x + \dfrac{1}{x} = \dfrac{10}{3}$ using $s = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, where $x > 0$.

Since $x + \frac{1}{x} = s^2 - 2$, the equation becomes a quadratic in s:

$$s^2 - 2 = \tfrac{10}{3} \ \Rightarrow\ s^2 = \tfrac{16}{3}$$

$$s = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \qquad \text{take the positive root } (x > 0)$$

To recover x, solve $x + \frac{1}{x} = \frac{10}{3}$ directly: $3x^2 - 10x + 3 = 0 = (3x-1)(x-3)$, giving $x = 3$ or $x = \frac{1}{3}$.
Check $x = 3$: $3 + \frac{1}{3} = \frac{10}{3}$. ✓ These are reciprocals, exactly the symmetry the substitution predicts.

Symmetry shortcut. If an equation is unchanged when you replace x with $\frac{1}{x}$, its solutions come in reciprocal pairs $\left(r, \frac{1}{r}\right)$. Setting $s = \sqrt{x} + \frac{1}{\sqrt{x}}$ or $u = x + \frac{1}{x}$ turns it into a lower-degree equation; recover x at the end.

Rationalizing Multi-Term Denominators

A two-term denominator clears with its conjugate. A **three-term** denominator such as $\sqrt{a} + \sqrt{b} + \sqrt{c}$ needs *grouping*: treat two terms as a block, multiply by the conjugate of that block, and repeat.

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}} \xrightarrow[\text{by } (\sqrt{a}+\sqrt{b})-\sqrt{c}]{\text{multiply}} \frac{(\sqrt{a}+\sqrt{b})-\sqrt{c}}{(a+b-c)+2\sqrt{ab}}.$$

Now the denominator is a two-term expression $(a+b-c)+2\sqrt{ab}$, which you finish off with one more conjugate. The strategy is always: **shrink the number of radical terms by one at each step.**

Worked Example: Rationalize a Three-Term Denominator

Rationalize $\dfrac{1}{1+\sqrt{2}+\sqrt{3}}$. Group $1+\sqrt{2}$ and multiply by its conjugate against $\sqrt{3}$:

$$\begin{aligned}\frac{1}{(1+\sqrt{2})+\sqrt{3}} \cdot \frac{(1+\sqrt{2})-\sqrt{3}}{(1+\sqrt{2})-\sqrt{3}} &= \frac{(1+\sqrt{2})-\sqrt{3}}{(1+\sqrt{2})^2-3} \\ &= \frac{1+\sqrt{2}-\sqrt{3}}{(3+2\sqrt{2})-3} = \frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

Now rationalize the single remaining $\sqrt{2}$ by multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$:

$$\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}+2-\sqrt{6}}{4}.$$

So $\dfrac{1}{1+\sqrt{2}+\sqrt{3}} = \dfrac{2+\sqrt{2}-\sqrt{6}}{4}$.

Domain & Range of Radical Functions and Their Inverses

For a radical function, the domain of the function *is* the range of its inverse, and vice versa — reflecting across $y = x$ swaps the two axes.

- **Even-index radical** $y = \sqrt{g(x)}$: domain requires $g(x) \geq 0$; the output is nonnegative, so range starts at the minimum output.
- **Inverse of a square root** is a *restricted* parabola. The restriction it inherits (e.g. $x \geq k$) comes from the *range* of the original radical.
- **Odd-index radical**: domain and range are both all reals; its inverse (a cubic) is unrestricted too.

Worked Example: Domain, Range, and the Inverse's Domain

Let $f(x) = \sqrt{2x-6}+1$. Find its domain and range, then the domain of f^{-1}.
Domain of f: need $2x - 6 \geq 0$, so $x \geq 3$, i.e. $[3, \infty)$.
Range of f: the root is ≥ 0, so $f(x) \geq 1$; range $[1, \infty)$.
Find the inverse by swapping and solving:

$$\begin{aligned} x &= \sqrt{2y-6}+1 && \text{swap } x \text{ and } y \\ (x-1)^2 &= 2y-6 && \text{isolate, then square} \\ y &= \frac{(x-1)^2+6}{2} \end{aligned}$$

So $f^{-1}(x) = \dfrac{(x-1)^2+6}{2}$ with **domain** $[1, \infty)$ — precisely the *range* of f. The two sets trade places, exactly as the reflection across $y = x$ demands.

Domain/range swap. For f and f^{-1}: Domain(f) = Range(f^{-1}) and Range(f) = Domain(f^{-1}). When a square root's inverse is a parabola, the parabola's domain restriction is *not optional* — it is what keeps the inverse a function.

11 Exponential & Logarithmic Functions

Everything for Topic 8 on one desk reference

11.1 Exponential Functions $y = ab^x$

Anatomy of an Exponential

An **exponential function** has the form

$$y = ab^x, \qquad a \neq 0, \quad b > 0, \quad b \neq 1.$$

The variable lives in the **exponent**. Here a is the **initial value** (the y-intercept, since $b^0 = 1$ gives $y = a$), and b is the **base** or growth factor.

- If $b > 1$: **exponential growth** (curve rises to the right).
- If $0 < b < 1$: **exponential decay** (curve falls to the right).

For $a > 0$: **domain** is all real numbers; **range** is $y > 0$; the line $y = 0$ is a **horizontal asymptote** the curve approaches but never touches.

Growth $y = 2^x$ (red) rises to the right; **decay** $y = \left(\frac{1}{2}\right)^x$ (blue) falls. Both share the asymptote $y = 0$ and pass through $(0, 1)$.

Transformations of $y = b^x$

Starting from the parent $y = b^x$, the function $y = a\,b^{\,x-h} + k$ is:

- shifted **right** h and **up** k;
- stretched by $|a|$ (reflected over the x-axis if $a < 0$);
- its horizontal asymptote moves to $y = k$, and the range becomes $y > k$ (or $y < k$ if $a < 0$).

Worked Example: Reading a Transformation

Describe $y = 2^x - 2$ and give its asymptote and range. It is the parent $y = 2^x$ shifted **down 2**. The asymptote drops to $y = -2$, so the range is $y > -2$. The y-intercept is $2^0 - 2 = -1$.

Tip: To *find* a and b from two points, use the y-intercept for a, then divide a second output by a to get b^x. From $(0, 3)$ and $(1, 12)$: $a = 3$, and $12 = 3b^1$ gives $b = 4$, so $y = 3 \cdot 4^x$.

11.2 The Number e & Continuous Growth

Euler's Number e

The constant $e \approx 2.71828$ is the natural base of growth. It arises as the limit of $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. The function $y = e^x$ is a growth curve just like any b^x with $b > 1$; it models **continuous** change.

Interest Formulas

For principal P, annual rate r (as a decimal), and time t in years:

$$\text{compounded } n \text{ times/year:} \quad A = P\left(1+\frac{r}{n}\right)^{nt}, \qquad \text{compounded continuously:} \quad A = Pe^{rt}.$$

Use $n = 1$ (annually), 4 (quarterly), 12 (monthly), 365 (daily).

Worked Example: Two Kinds of Compounding

Invest \$1000 at 6% for 5 years.

$$\begin{aligned} \text{monthly:}\ A &= 1000\left(1+\tfrac{0.06}{12}\right)^{12\cdot 5} = 1000(1.005)^{60} \approx \$1348.85, \\ \text{continuous:}\ A &= 1000\, e^{0.06\cdot 5} = 1000\, e^{0.3} \approx \$1349.86. \end{aligned}$$

Continuous compounding earns slightly more — it is the limit as $n \to \infty$.

Remember: more frequent compounding always yields more, but the gains shrink and level off at the continuous value Pe^{rt}.

11.3 Logarithms: Definition & Evaluating

A Logarithm is an Exponent

A logarithm answers the question "*to what power?*" The definition is the inverse relationship

$$\log_b y = x \quad \iff \quad b^x = y, \qquad b > 0,\ b \neq 1,\ y > 0.$$

So $\log_b y$ is the exponent you put on b to get y. To evaluate, rewrite in exponential form.

Worked Example: Evaluating

$$\begin{aligned} \log_2 32 &= 5 && \text{because } 2^5 = 32, \\ \log_5 25 &= 2 && \text{because } 5^2 = 25, \\ \log_8 4 &= \tfrac{2}{3} && \text{because } 8^{2/3} = (2^3)^{2/3} = 2^2 = 4, \\ \log_b 1 &= 0 && \text{because } b^0 = 1. \end{aligned}$$

Common and Natural Logs

Two bases are so useful they get their own notation:

- **Common log:** $\log x$ means $\log_{10} x$ (base 10).
- **Natural log:** $\ln x$ means $\log_e x$ (base e).

Inverse facts: $\log 10^k = k$, $\ln e^k = k$, $10^{\log x} = x$, $e^{\ln x} = x$.

Tip: $\log 1000 = 3$ and $\ln e^2 = 2$ take *no* calculator — read them straight from the definition.

11.4 Graphs of Logarithmic Functions

The Inverse of an Exponential

Because $\log_b x$ undoes b^x, the graph of $y = \log_b x$ is the **reflection** of $y = b^x$ across the line $y = x$. Consequences for $y = \log_b x$ (with $b > 1$):

- **Domain** $x > 0$; **range** all real numbers.
- **Vertical asymptote** at $x = 0$ (the y-axis).
- passes through $(1, 0)$ since $\log_b 1 = 0$.

A shift $y = \log_b(x - h)$ moves the asymptote to $x = h$ and the domain to $x > h$.

$y = \log_2 x$ (blue) is the mirror image of $y = 2^x$ (red) across $y = x$. Its vertical asymptote is the y-axis, $x = 0$.

Worked Example: Domain of a Shifted Log

For $y = \log_2(x - 3)$, the inside must be positive: $x - 3 > 0$, so the **domain** is $x > 3$ and the **vertical asymptote** is $x = 3$.

11.5 Properties of Logarithms

The Three Laws (same base b)

$$\begin{aligned} \textbf{Product:} \quad & \log_b(MN) = \log_b M + \log_b N, \\ \textbf{Quotient:} \quad & \log_b\left(\tfrac{M}{N}\right) = \log_b M - \log_b N, \\ \textbf{Power:} \quad & \log_b(M^p) = p \, \log_b M. \end{aligned}$$

These convert products into sums and exponents into multipliers. Reading them left-to-right **expands**; right-to-left **condenses**.

Change of Base

To evaluate a log in any base with a calculator:

$$\log_b M = \frac{\log M}{\log b} = \frac{\ln M}{\ln b}.$$

Worked Example: Expanding

Expand $\log \dfrac{x^3\sqrt{y}}{z^2}$ completely.

$$\begin{aligned}
\log \frac{x^3\sqrt{y}}{z^2} &= \log\left(x^3\sqrt{y}\right) - \log z^2 && \text{quotient} \\
&= \log x^3 + \log y^{1/2} - \log z^2 && \text{product} \\
&= 3\log x + \tfrac{1}{2}\log y - 2\log z && \text{power}
\end{aligned}$$

Worked Example: Condensing

Condense $2\ln x + \ln y - \ln 3$ into a single logarithm.

$$\begin{aligned}
2\ln x + \ln y - \ln 3 &= \ln x^2 + \ln y - \ln 3 && \text{power} \\
&= \ln(x^2 y) - \ln 3 && \text{product} \\
&= \ln \frac{x^2 y}{3} && \text{quotient}
\end{aligned}$$

Careful: there is *no* rule for $\log(M+N)$ — you can only split products and quotients, never sums or differences *inside* the log. And $\dfrac{\log M}{\log N} \neq \log \dfrac{M}{N}$.

11.6 Solving Exponential Equations

Two Strategies

1. **Common base:** rewrite both sides with the *same* base, then set the exponents equal.
2. **Take a log:** if a common base is awkward, isolate the power and apply log or ln to both sides, then use the power rule to bring the exponent down.

Worked Example: Common Base

Solve $4^{x+1} = 32$. Write each side as a power of 2:

$$(2^2)^{x+1} = 2^5 \;\Rightarrow\; 2^{2x+2} = 2^5 \;\Rightarrow\; 2x+2=5 \;\Rightarrow\; x = \tfrac{3}{2}.$$

Worked Example: Taking a Log

Solve $3 \cdot 2^x = 50$. Isolate the power, then take ln of both sides.

$$\begin{aligned}
2^x &= \frac{50}{3} && \\
\ln(2^x) &= \ln\frac{50}{3} && \text{take the natural log} \\
x\ln 2 &= \ln\frac{50}{3} && \text{power rule brings } x \text{ down} \\
x &= \frac{\ln(50/3)}{\ln 2} = \log_2\frac{50}{3} \approx 4.059 && \text{exact, then rounded}
\end{aligned}$$

Leave it exact, then round. An exact answer like $x = \dfrac{\ln 20}{\ln 5}$ is perfectly correct; only round (≈ 1.861) if the problem asks for a decimal.

11.7 Solving Logarithmic Equations

Strategy & the Domain Check

1. Use log laws to **condense** into a single log if needed.
2. Rewrite in **exponential form**, or use "one-to-one" ($\log_b M = \log_b N \Rightarrow M = N$).
3. Solve the resulting equation.
4. **Check every answer:** the input of *any* log must be positive. Discard **extraneous** solutions.

Worked Example: Extraneous Solution

Solve $\log_2 x + \log_2(x-2) = 3$.

$$\begin{aligned}
\log_2\big(x(x-2)\big) &= 3 && \text{product rule} \\
x(x-2) &= 2^3 = 8 && \text{exponential form} \\
x^2 - 2x - 8 = 0 &\;\Rightarrow\; (x-4)(x+2) = 0 &&
\end{aligned}$$

So $x = 4$ or $x = -2$. **Check:** $x = -2$ makes $\log_2(-2)$ undefined — **reject** it. Only $x = 4$ works.

Never skip the check. Condensing can introduce solutions that violate a log's domain. The valid domain here was $x > 2$, which -2 fails.

11.8 Applications

Models You Will See

- **Growth / decay:** $A = A_0(1 \pm r)^t$ or continuous $A = A_0 e^{kt}$.
- **Half-life:** $A = A_0 \left(\frac{1}{2}\right)^{t/h}$, where h is the half-life.
- **pH:** $\mathrm{pH} = -\log[\mathrm{H}^+]$, where $[\mathrm{H}^+]$ is the hydrogen-ion concentration.
- **Richter magnitude:** $M = \log \dfrac{I}{I_0}$; each whole number is a $\times 10$ jump in intensity.

Worked Example: Half-Life

A 200 g sample has a half-life of 8 days. How much remains after 24 days?

$$A = 200 \left(\tfrac{1}{2}\right)^{24/8} = 200 \left(\tfrac{1}{2}\right)^{3} = 200 \cdot \tfrac{1}{8} = 25 \text{ g}.$$

Worked Example: Solving for Time

A town of 20,000 grows continuously at 3% per year. When will it reach 30,000?

$$\begin{aligned} 30000 &= 20000\, e^{0.03t} \\ 1.5 &= e^{0.03t} \\ \ln 1.5 &= 0.03t \;\Rightarrow\; t = \frac{\ln 1.5}{0.03} \approx 13.5 \text{ years.} \end{aligned}$$

Worked Example: pH

A solution has $[\mathrm{H}^+] = 10^{-4}$. Then $\mathrm{pH} = -\log(10^{-4}) = -(-4) = 4$. A pH below 7 is acidic.

Big picture: whenever the unknown is *in the exponent* (“how long?”, “what rate?”), a logarithm is the tool that frees it.

11.9 Going Deeper: Advanced Exponential & Log Ideas

Change-of-Base Chains & Telescoping Products

Change of base can be written in *any* common base c:

$$\log_b M = \frac{\log_c M}{\log_c b}.$$

This is the key to **telescoping** a product of logs. Watch a chain of $\log_k(k+1)$ terms collapse when every log is rewritten over the *same* base c:

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{n-1} n = \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 5}{\log 4} \cdots \frac{\log n}{\log(n-1)}.$$

Each numerator cancels the next denominator, leaving $\dfrac{\log n}{\log 2} = \log_2 n$. The whole chain equals a single log.

Worked Example: A Telescoping Log Product

Evaluate $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

$$\begin{aligned} \prod &= \frac{\log 3}{\log 2} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 5}{\log 4} \cdot \frac{\log 6}{\log 5} \cdot \frac{\log 7}{\log 6} \cdot \frac{\log 8}{\log 7} && \text{change every log to base 10} \\ &= \frac{\log 8}{\log 2} = \log_2 8 = 3 && \text{everything cancels but the ends} \end{aligned}$$

The product of six "ugly" logs is exactly 3.

"Let $\log_2 3 = a$" — Expressing Logs Symbolically

When a problem fixes one log as a letter, express others by **breaking numbers into prime factors** and applying the three laws. If $\log_2 3 = a$, then:

- $\log_2 6 = \log_2(2 \cdot 3) = 1 + a$;
- $\log_2 12 = \log_2(2^2 \cdot 3) = 2 + a$;
- $\log_2 \frac{9}{8} = \log_2 3^2 - \log_2 2^3 = 2a - 3$;
- $\log_3 2 = \dfrac{1}{\log_2 3} = \dfrac{1}{a}$ (reciprocal / change of base).

Worked Example: Express $\log_6 48$ in terms of $a = \log_2 3$

Change to base 2, then factor $48 = 2^4 \cdot 3$ and $6 = 2 \cdot 3$:

$$\begin{aligned}
\log_6 48 &= \frac{\log_2 48}{\log_2 6} && \text{change of base} \\
&= \frac{\log_2(2^4 \cdot 3)}{\log_2(2 \cdot 3)} = \frac{4 + \log_2 3}{1 + \log_2 3} && \text{product + power laws} \\
&= \frac{4+a}{1+a} && \text{substitute } a = \log_2 3
\end{aligned}$$

Exponential Equations Hidden as Quadratics

An equation with terms in 2^x and 2^{2x} (or e^x and e^{2x}) is secretly a **quadratic**. Substitute $u = 2^x > 0$, using $2^{2x} = (2^x)^2 = u^2$, solve the quadratic, then back-substitute. **Reject any** $u \leq 0$, since 2^x is always positive.

Worked Example: Reducible to a Quadratic

Solve $4^x - 5 \cdot 2^x + 4 = 0$. Since $4^x = (2^x)^2$, let $u = 2^x$:

$$\begin{aligned}
u^2 - 5u + 4 &= 0 && \text{substitute } u = 2^x \\
(u-1)(u-4) &= 0 \;\Rightarrow\; u = 1 \text{ or } u = 4 && \text{factor} \\
2^x = 1 \;\Rightarrow\; x = 0, \qquad 2^x &= 4 \;\Rightarrow\; x = 2 && \text{back-substitute}
\end{aligned}$$

Both u-values are positive, so both solutions are valid: $x = 0$ and $x = 2$.

The $x^{\log x}$ Type — Take a Log of a Log-Exponent

When the *variable appears in both the base and the exponent*, as in $x^{\log x}$, take log of both sides first; the power rule turns the exponent into a factor and creates a quadratic in $\log x$. Let $t = \log x$.

Worked Example: Solving $x^{\log x} = 100x$

Take $\log_{10}$ of both sides and let $t = \log x$:

$$\begin{aligned}
\log\left(x^{\log x}\right) &= \log(100x) \\
(\log x)(\log x) &= \log 100 + \log x && \text{power \& product rules} \\
t^2 &= 2 + t \;\Rightarrow\; t^2 - t - 2 = 0 && \text{substitute } t = \log x \\
(t-2)(t+1) &= 0 \;\Rightarrow\; t = 2 \text{ or } t = -1 && \text{factor} \\
\log x = 2 \;\Rightarrow\; x = 100, \qquad \log x &= -1 \;\Rightarrow\; x = \tfrac{1}{10}
\end{aligned}$$

Both are positive, so $x = 100$ and $x = \frac{1}{10}$ both check.

Systems of Logarithmic Equations

A system mixing sums and products of logs is solved by **condensing each equation**, then converting to ordinary algebra. Keep the *same base* throughout, and enforce the domain (every argument > 0) at the end.

Worked Example: A Log System

Solve $\log_2 x + \log_2 y = 5$ and $\log_2 x - \log_2 y = 1$.

$$\begin{aligned} \log_2(xy) = 5 \;&\Rightarrow\; xy = 2^5 = 32 && \text{product rule} \\ \log_2 \tfrac{x}{y} = 1 \;&\Rightarrow\; \tfrac{x}{y} = 2^1 = 2 && \text{quotient rule} \end{aligned}$$

So $x = 2y$, giving $2y^2 = 32$, hence $y = 4$ (reject $y = -4$, out of domain) and $x = 8$. Check: $\log_2 8 + \log_2 4 = 3 + 2 = 5$, as required.

Continuous Compounding & Logistic Growth

Two continuous models worth knowing beyond $A = Pe^{rt}$:

- **Doubling / tripling time:** set $A = 2P$ in Pe^{rt} to get $t = \dfrac{\ln 2}{r}$ (the "rule of 69" in disguise).
- **Logistic growth:** real populations cannot grow forever; they level off at a **carrying capacity** L:
$$P(t) = \frac{L}{1 + Ce^{-kt}}.$$
Early on it looks exponential; as $t \to \infty$, $e^{-kt} \to 0$ so $P \to L$. The curve is S-shaped (sigmoidal) with a horizontal asymptote at $P = L$.

Logistic growth rises like an exponential, then bends over and approaches the carrying capacity L.

Continuous vs. logistic: pure Pe^{rt} has *no* ceiling and grows without bound; logistic growth throttles itself as P nears L. Use logistic whenever a resource limit (space, food, market size) caps the total.

Logarithmic & Exponential Inequalities

Solving an inequality adds a **direction** rule on top of the equation techniques:

- Because $\log_b x$ (with $b > 1$) is **increasing**, applying it *preserves* the inequality: $M < N \Leftrightarrow \log_b M < \log_b N$ for $M, N > 0$.
- If the base satisfies $0 < b < 1$, both b^x and $\log_b x$ are **decreasing**, so the inequality **flips**.
- Always intersect your answer with the **domain** (arguments of every log must be positive).

Worked Example: A Log Inequality

Solve $\log_2(x-1) < 3$. First the domain: $x - 1 > 0$, so $x > 1$. Since base $2 > 1$ is increasing, rewrite $3 = \log_2 8$ and drop the logs:

$$x - 1 < 2^3 = 8 \;\Rightarrow\; x < 9.$$

Intersect with the domain $x > 1$: the solution is $1 < x < 9$.

Two-part discipline for inequalities: (1) solve the associated equation for the boundary, (2) decide the direction from whether the base is > 1 (keep) or < 1 (flip), then (3) trim to the domain. Missing the domain step is the #1 error.

12 Sequences & Series

Every Topic 9 idea, formula, and worked example on one desk reference

12.1 Sequences: Terms, Explicit & Recursive Rules

What is a Sequence?

A **sequence** is an ordered list of numbers called **terms**. We label them with subscripts: a_1 (first term), a_2 (second term), and in general a_n (the nth term). There are two ways to describe the pattern:

- **Explicit rule:** a formula for a_n directly in terms of n. Plug in n to get any term instantly. Example: $a_n = 2n + 1$.
- **Recursive rule:** a starting term plus a formula that builds each term from the *previous* one. Example: $a_1 = 3,\; a_n = a_{n-1} + 2$.

Worked Example: Explicit Rule

Find the first four terms of $a_n = n^2 - 1$.

$$\begin{aligned} a_1 &= 1^2 - 1 = 0 \\ a_2 &= 2^2 - 1 = 3 \\ a_3 &= 3^2 - 1 = 8 \\ a_4 &= 4^2 - 1 = 15 \end{aligned}$$

The sequence is $0,\ 3,\ 8,\ 15,\ \ldots$

Worked Example: Recursive Rule

Find the first four terms of $a_1 = 5,\ a_n = a_{n-1} + 4$.

$$\begin{aligned} a_1 &= 5 && \text{given} \\ a_2 &= a_1 + 4 = 5 + 4 = 9 \\ a_3 &= a_2 + 4 = 9 + 4 = 13 \\ a_4 &= a_3 + 4 = 13 + 4 = 17 \end{aligned}$$

The sequence is $5,\ 9,\ 13,\ 17,\ \ldots$

Tip. An *explicit* rule is a shortcut — you can jump straight to a_{100}. A *recursive* rule must climb one step at a time. When a problem asks for a far-off term, look for an explicit formula.

12.2 Arithmetic Sequences

Common Difference and the nth Term

A sequence is **arithmetic** when each term differs from the one before it by the same amount, the **common difference** d. Test it by subtracting: $d = a_2 - a_1 = a_3 - a_2 = \cdots$

$$\boxed{a_n = a_1 + (n-1)d}$$

Here a_1 is the first term and d is the common difference. Note the $n - 1$: to reach the nth term you take $n - 1$ steps of size d.

Worked Example: Find a Specific Term

For $5,\ 8,\ 11,\ 14,\ \ldots$ find a_{10}.

$$\begin{aligned} d &= 8-5=3, \qquad a_1 = 5 \\ a_{10} &= a_1 + (10-1)d \\ &= 5 + 9(3) \\ &= 5 + 27 = 32 \end{aligned}$$

Remember: arithmetic means ADD. You move from term to term by *adding* d (a negative d means the terms decrease). Do not confuse this with geometric sequences, which *multiply*.

12.3 Arithmetic Series (Partial Sums)

Summing an Arithmetic Sequence

A **series** is the sum of the terms of a sequence. The sum of the first n terms of an arithmetic sequence is

$$\boxed{S_n = \frac{n}{2}\left(a_1 + a_n\right)}$$

Read it as: (number of terms) $\times$ (average of the first and last term). If you do not yet know a_n, find it first with $a_n = a_1 + (n-1)d$.

Worked Example: Sum of the First 10 Terms

Find S_{10} for $5,\ 8,\ 11,\ 14,\ \ldots$ From the previous section, $a_{10} = 32$.

$$\begin{aligned} S_{10} &= \frac{10}{2}\left(a_1 + a_{10}\right) \\ &= 5\left(5+32\right) \\ &= 5(37) = 185 \end{aligned}$$

Worked Example: Sum a Known Range

Find $2+4+6+\cdots+100$ (the first 50 positive even numbers). Here $a_1 = 2$, $a_{50} = 100$, $n = 50$.

$$\begin{aligned} S_{50} &= \frac{50}{2}\left(2+100\right) \\ &= 25\left(102\right) = 2550 \end{aligned}$$

12.4 Geometric Sequences

Common Ratio and the nth Term

A sequence is **geometric** when each term is a fixed multiple of the one before it, the **common ratio** r. Test it by dividing: $r = \dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \cdots$

$$\boxed{a_n = a_1\, r^{\,n-1}}$$

Again note the exponent $n-1$: reaching the nth term takes $n-1$ multiplications by r.

Worked Example: Find a Specific Term

For $3,\ 6,\ 12,\ 24,\ \ldots$ find a_7.

$$\begin{aligned} r &= \frac{6}{3} = 2, \qquad a_1 = 3 \\ a_7 &= a_1\, r^{\,7-1} = 3 \cdot 2^6 \\ &= 3 \cdot 64 = 192 \end{aligned}$$

Remember: geometric means MULTIPLY. Move from term to term by *multiplying* by r. If $|r| > 1$ the terms grow; if $|r| < 1$ they shrink toward 0; a negative r makes the signs alternate.

12.5 Geometric Series (Finite Sums)

Summing a Finite Geometric Sequence

The sum of the first n terms of a geometric sequence (with $r \neq 1$) is

$$\boxed{S_n = a_1\, \frac{1 - r^{\,n}}{1 - r}}$$

Worked Example: Sum of the First 6 Terms

Find S_6 for $3,\ 6,\ 12,\ 24,\ \ldots$ where $a_1 = 3$ and $r = 2$.

$$\begin{aligned} S_6 &= a_1 \frac{1-r^6}{1-r} \\ &= 3 \cdot \frac{1-2^6}{1-2} \\ &= 3 \cdot \frac{1-64}{-1} \\ &= 3 \cdot \frac{-63}{-1} = 3(63) = 189 \end{aligned}$$

Watch the signs. When $r > 1$ both $1 - r^n$ and $1 - r$ are negative, and the negatives cancel to give a positive sum. Compute the exponent r^n before subtracting.

12.6 Infinite Geometric Series

When Does an Infinite Sum Exist?

An infinite geometric series $a_1 + a_1 r + a_1 r^2 + \cdots$ **converges** (adds up to a finite number) exactly when

$$|r| < 1.$$

In that case $r^n \to 0$ as n grows, and the finite formula collapses to

$$\boxed{S = \frac{a_1}{1-r}} \qquad (|r| < 1).$$

If $|r| \geq 1$ the terms do not shrink, the partial sums run off to infinity, and the series **diverges** — there is *no sum*.

Worked Example: A Convergent Sum

Find $8 + 4 + 2 + 1 + \cdots$ Here $a_1 = 8$ and $r = \frac{4}{8} = \frac{1}{2}$. Since $|r| < 1$, the sum exists.

$$\begin{aligned} S &= \frac{a_1}{1-r} = \frac{8}{1-\frac{1}{2}} \\ &= \frac{8}{\frac{1}{2}} = 16 \end{aligned}$$

Worked Example: A Repeating Decimal

Write $0.\overline{7} = 0.7 + 0.07 + 0.007 + \cdots$ as a fraction. This is geometric with $a_1 = 0.7$ and $r = 0.1$.

$$S = \frac{0.7}{1 - 0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

Always check $|r| < 1$ first. An infinite geometric sum only exists when the ratio is between -1 and 1. If $|r| \geq 1$, the correct answer is "diverges (no sum)."

12.7 Summation (Sigma) Notation

Reading and Writing Σ

The Greek capital sigma $\sum$ is shorthand for "add these up."

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

The letter n is the **index**; it starts at the number written below $\sum$ and climbs by 1 up to the number written above. Substitute each index value into the expression and add the results.

Worked Example: Evaluate a Sum

Evaluate $\displaystyle\sum_{n=1}^{4}(2n+1)$.

$$\begin{aligned} &= (2 \cdot 1 + 1) + (2 \cdot 2 + 1) + (2 \cdot 3 + 1) + (2 \cdot 4 + 1) \\ &= 3 + 5 + 7 + 9 = 24 \end{aligned}$$

Worked Example: Write in Sigma Notation

Write $3 + 6 + 9 + 12 + 15$ using sigma notation. Each term is $3n$ for $n = 1$ to 5:

$$3 + 6 + 9 + 12 + 15 = \sum_{n=1}^{5} 3n$$

Tip. If the index starts somewhere other than 1 (say $n = 3$), substitute exactly those values — do not restart at 1. And a constant like $\sum_{k=1}^{5} 3$ just adds that constant once per index value: $5 \times 3 = 15$.

12.8 Applications

Choosing the Right Model

Real-world patterns are often sequences or series. Decide which type first:

- Repeated **adding** (a fixed raise, a fixed monthly deposit) $\Rightarrow$ **arithmetic**.
- Repeated **multiplying** (a percent growth or decay, compound interest, a ball rebounding to a fraction of its height) $\Rightarrow$ **geometric**.
- "How high is the nth bounce?" or "salary in year n?" asks for a single **term**. "Total distance" or "total earned" asks for a **series (sum)**.

Worked Example: Depreciation (Geometric Term)

A machine worth \$50,000 loses 20% of its value each year, so it keeps 80%: $r = 0.8$. Find its value after 5 years, i.e. the term $a_n = 50000\,(0.8)^n$ at $n = 5$.

$$\begin{aligned} a_5 &= 50000\,(0.8)^5 \\ &= 50000\,(0.32768) \\ &= \$16{,}384 \end{aligned}$$

Worked Example: Bouncing Ball (Infinite Series)

A ball dropped from 10 m rebounds to 60% of its previous height. Find the total vertical distance it travels before resting. It falls 10 m, then each rebound goes up *and* back down. The rebound heights $6, 3.6, 2.16, \ldots$ are geometric with $a_1 = 6$, $r = 0.6$.

$$\begin{aligned} \text{rebound sum} &= \frac{6}{1-0.6} = \frac{6}{0.4} = 15 \text{ m} \\ \text{total} &= 10 + 2(15) = 40 \text{ m} \end{aligned}$$

The factor of 2 counts each rebound up and down; the initial 10 m drop is counted once.

Worked Example: Salary Raises (Arithmetic Series)

You start at \$35,000 and receive a \$1,500 raise each year. Total earned over 10 years? Arithmetic with $a_1 = 35000$, $d = 1500$.

$$\begin{aligned} a_{10} &= 35000 + (10-1)(1500) = 35000 + 13500 = 48500 \\ S_{10} &= \frac{10}{2}\,(35000 + 48500) \\ &= 5\,(83500) = \$417{,}500 \end{aligned}$$

12.9 Going Deeper: Advanced Sequences & Series

Telescoping Sums

A sum **telescopes** when each term splits into two pieces that cancel against neighbors, leaving only the very first and very last. The trick is **partial fractions**: rewrite the general term as a *difference* $b_k - b_{k+1}$. Then

$$\sum_{k=1}^{n} \big(b_k - b_{k+1}\big) = b_1 - b_{n+1},$$

because every interior piece appears once with a $+$ and once with a $-$. The classic case:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Worked Example: A Telescoping Sum

Evaluate $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$, then take $n \to \infty$. Split each term with partial fractions:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\ &= \big(1 - \tfrac{1}{2}\big) + \big(\tfrac{1}{2} - \tfrac{1}{3}\big) + \cdots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}.\end{aligned}$$

As $n \to \infty$, $\dfrac{1}{n+1} \to 0$, so the infinite sum equals $\boxed{1}$.

Power Sums and the Cube Identity

Three closed forms let you sum powers of the first n whole numbers without adding term by term:

$$\begin{aligned}\sum_{k=1}^{n} k &= \frac{n(n+1)}{2}, \\ \sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}, \\ \sum_{k=1}^{n} k^3 &= \left(\frac{n(n+1)}{2}\right)^2.\end{aligned}$$

The last line hides a beautiful identity: the sum of the first n cubes is the *square* of the sum of the first n whole numbers,

$$\boxed{\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2}.$$

For example $1 + 8 + 27 + 64 = 100 = (1 + 2 + 3 + 4)^2 = 10^2$.

Break sums apart. A sum like $\sum(3k^2 - 5k + 2)$ splits by linearity: $3\sum k^2 - 5\sum k + \sum 2$. Handle each power sum with its own formula, remembering $\sum_{k=1}^{n} c = cn$ for a constant.

Worked Example: Arithmetic–Geometric Sum $\sum k\, r^k$

Terms like $k\, r^k$ mix an arithmetic factor (k) with a geometric one (r^k). Evaluate $S = \sum_{k=1}^{\infty} k\left(\frac{1}{2}\right)^k$ using the **shift-and-subtract** trick. First check convergence: here $r = \frac{1}{2}$, and since $|r| < 1$ the sum exists. Write the finite sum $S_n = \sum_{k=1}^{n} kr^k$ and subtract rS_n:

$$\begin{aligned}
S_n - rS_n &= \bigl(r + 2r^2 + 3r^3 + \cdots + nr^n\bigr) - \bigl(r^2 + 2r^3 + \cdots + nr^{n+1}\bigr)\\
(1-r)S_n &= r + r^2 + r^3 + \cdots + r^n - nr^{n+1}\\
(1-r)S_n &= \frac{r\bigl(1 - r^n\bigr)}{1-r} - nr^{n+1}.
\end{aligned}$$

With $|r| < 1$, both $r^n \to 0$ and $nr^{n+1} \to 0$ as $n \to \infty$, leaving the tidy formula

$$\sum_{k=1}^{\infty} k\, r^k = \frac{r}{(1-r)^2} \qquad (|r| < 1).$$

Now substitute $r = \frac{1}{2}$:

$$S = \frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2.$$

So $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots = \boxed{2}$.

Worked Example: Recurrence to a Closed Form

Turn the recursive rule $a_1 = 3,\ a_n = 2a_{n-1} + 1$ into an explicit formula. First find the **fixed point** L that the constant term wants to settle at, by solving $L = 2L + 1$, which gives $L = -1$. Add 1 to both sides of the recurrence and the "+1" disappears:

$$a_n + 1 = 2a_{n-1} + 2 = 2\bigl(a_{n-1} + 1\bigr).$$

So the shifted sequence $b_n = a_n + 1$ is *geometric* with ratio 2 and first term $b_1 = a_1 + 1 = 4$.

Therefore

$$b_n = 4 \cdot 2^{n-1} = 2^{n+1},$$
$$a_n = b_n - 1 = 2^{n+1} - 1.$$

Check: $a_1 = 2^2 - 1 = 3$, $a_2 = 2^3 - 1 = 7$, $a_3 = 2^4 - 1 = 15$. Matches $3, 7, 15, 31, \ldots$

Mathematical Induction

Induction proves a statement $P(n)$ is true for *every* whole number $n \geq 1$ using two steps, like knocking over an infinite line of dominoes:

- **Base case:** show $P(1)$ is true (the first domino falls).
- **Inductive step:** *assume* $P(k)$ is true (the inductive hypothesis) and use it to prove $P(k+1)$ (each domino knocks over the next).

Together these force $P(n)$ true for all $n \geq 1$.

Worked Example: A Full Induction Proof

Claim. For every integer $n \geq 1$, $\displaystyle\sum_{k=1}^{n}(2k-1) = n^2$ (the sum of the first n odd numbers).

Base case ($n = 1$). The left side is $2(1) - 1 = 1$; the right side is $1^2 = 1$. They agree, so $P(1)$ holds.

Inductive step. Assume the formula holds for some $n = k$; that is, assume

$$\sum_{i=1}^{k}(2i-1) = k^2. \qquad \text{(inductive hypothesis)}$$

We must show it holds for $n = k+1$, i.e. that the sum equals $(k+1)^2$. Start from the $(k+1)$-term sum and peel off the last term:

$$\begin{aligned}
\sum_{i=1}^{k+1}(2i-1) &= \underbrace{\sum_{i=1}^{k}(2i-1)}_{=k^2 \text{ by hypothesis}} + \big(2(k+1)-1\big) \\
&= k^2 + (2k+1) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2.
\end{aligned}$$

This is exactly $P(k+1)$. By the principle of mathematical induction, the formula holds for all $n \geq 1$. ■

Infinite Geometric Series in Disguise

Many word problems hide a convergent geometric series. The pattern is always the same: identify a_1 and r, *confirm* $|r| < 1$, then apply $S = \dfrac{a_1}{1-r}$.

- **Repeating decimals:** $0.\overline{d}$ is a geometric series with $r = 0.1$ (or $r = 0.01$ for two repeating digits).
- **Bouncing ball:** rebound heights shrink by a fixed fraction r each bounce.
- **Perpetuity / annuity:** a payment made every period, discounted by $\dfrac{1}{1+i}$ per period, has $r = \dfrac{1}{1+i} < 1$ whenever the interest rate $i > 0$.

Worked Example: Present Value of a Perpetuity (Annuity)

An endowment pays \$100 at the end of every year forever, and money is discounted at $i = 5\%$ per year. Its **present value** is the sum of all payments discounted back to today:

$$\mathrm{PV} = \frac{100}{1.05} + \frac{100}{1.05^2} + \frac{100}{1.05^3} + \cdots$$

This is geometric with $a_1 = \dfrac{100}{1.05}$ and $r = \dfrac{1}{1.05}$. Since $i > 0$ we have $r < 1$, so the sum converges:

$$\begin{aligned}\mathrm{PV} = \frac{a_1}{1-r} &= \frac{100/1.05}{1 - 1/1.05} \\ &= \frac{100/1.05}{0.05/1.05} \\ &= \frac{100}{0.05} = \$2000.\end{aligned}$$

A perpetual stream of \$100 per year is worth just \$2000 today — a clean shortcut: $\mathrm{PV} = \dfrac{\text{payment}}{i}$.

Advanced pitfalls. (1) For $\sum k\, r^k$ and other infinite sums, *always verify* $|r| < 1$ *before* writing a finite answer. (2) Partial fractions only telescope if the pieces truly cancel — write out three terms to confirm. (3) An induction proof is incomplete without *both* the base case and an inductive step that explicitly uses the hypothesis.

13 Course Review

Every key definition, rule, and formula across all 11 topics

13.1 Equations, Inequalities & Absolute Value

Core Moves

Equations: isolate the variable by undoing operations in reverse order — distribute, clear fractions with the **LCD**, gather variable terms on one side, divide by the coefficient. Keep every step *equivalent*.

Literal equations: same moves, but treat the other letters as constants. If the target variable appears in two terms, **factor it out** first: $ax + bx = c \Rightarrow x = \dfrac{c}{a+b}$.

Inequalities: solve like equations, but **flip the sign when you multiply or divide by a negative**.

Absolute value: $|x|$ is distance from 0 (never negative). Isolate $|\cdot|$ first, then split.

- $|ax+b| = c$ $(c > 0)$: $ax + b = c$ or $ax + b = -c$.
- $|ax+b| < c$: "less thAND" $-c < ax + b < c$.
- $|ax+b| > c$: "greatOR" $ax + b < -c$ or $ax + b > c$.

Interval Notation

- $x < a$: $(-\infty,\ a)$
- $x \le a$: $(-\infty,\ a]$
- $x > a$: $(a,\ \infty)$
- $x \ge a$: $[a,\ \infty)$
- AND: $a < x < b \to (a, b)$
- OR: $(-\infty, a) \cup (b, \infty)$

Use a parenthesis next to $\pm\infty$ — infinity is never included.

Worked Example: Absolute-Value OR Inequality

Solve $|2x + 1| > 5$.

$$2x + 1 < -5 \quad \text{or} \quad 2x + 1 > 5$$

$$x < -3 \qquad \text{or} \qquad x > 2.$$

Interval notation: $(-\infty,\ -3) \cup (2,\ \infty)$.

Special cases. No solution: $6 = 7$ (contradiction), or $|x| < -2$. All reals: $6 = 6$ (identity), or $|x| > -2$.

13.2 Functions, Relations & Transformations

Functions & Domain/Range

A **relation** pairs inputs with outputs; a **function** assigns exactly one output to each input (passes the **vertical line test**). **Domain** = allowed inputs x; **range** = resulting outputs y.

- Exclude x that make a **denominator zero**.
- Exclude x that make an **even radicand negative** (need radicand ≥ 0).

Parent Functions

- Linear: $y = x$
- Quadratic: $y = x^2$
- Cubic: $y = x^3$
- Absolute value: $y = |x|$
- Square root: $y = \sqrt{x}$
- Reciprocal: $y = \frac{1}{x}$
- Exponential: $y = b^x$
- Logarithmic: $y = \log_b x$

Transformations of $y = a\,f\big(b(x-h)\big) + k$

- h: horizontal shift (right if $h > 0$); k: vertical shift (up if $k > 0$).
- $|a| > 1$ vertical stretch, $0 < |a| < 1$ vertical shrink; $a < 0$ reflects over the x-axis.
- b affects horizontal scaling by factor $\frac{1}{|b|}$; $b < 0$ reflects over the y-axis.

Composition: $(f \circ g)(x) = f\big(g(x)\big)$ — do g first (inside out).

Inverse f^{-1}: swap x and y, then solve for y. Graphs reflect over $y = x$; a one-to-one function (**horizontal line test**) has an inverse. Check: $f\big(f^{-1}(x)\big) = x$.

Worked Example: Composition & Inverse

Let $f(x) = 2x - 3$. $(f \circ f)(x) = 2(2x-3) - 3 = 4x - 9$.

Inverse: $y = 2x - 3 \Rightarrow x = 2y - 3 \Rightarrow y = \dfrac{x+3}{2}$, so $f^{-1}(x) = \dfrac{x+3}{2}$.

Order inside the parentheses matters. $f(x-2) + 5$ shifts right 2 (inside, opposite sign) and up 5. Horizontal moves are counterintuitive; vertical moves are literal.

13.3 Linear Systems & Matrices

Solving Systems

- **Substitution:** solve one equation for a variable, substitute.

- **Elimination:** add multiples of equations to cancel a variable.
- **3-variable:** eliminate one variable twice to reduce to a 2×2 system, then back-substitute.

Outcomes: one solution (lines cross), no solution (parallel, $0 = k$), infinitely many (same line, $0 = 0$).

Matrix Operations

Add/subtract **entrywise** (same dimensions). Scalar c: multiply every entry.

Multiplication AB: needs (columns of A) = (rows of B); entry = row · column dot product. Not commutative.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Determinants, Inverse & Cramer's Rule

2×2: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$. Inverse: $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (exists iff $\det \neq 0$).

Cramer's Rule for $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ with $D = ad - bc \neq 0$:

$$x = \frac{\det \begin{bmatrix} e & b \\ f & d \end{bmatrix}}{D}, \qquad y = \frac{\det \begin{bmatrix} a & e \\ c & f \end{bmatrix}}{D}.$$

Replace the variable's column with the constants.

Worked Example: Cramer's Rule

Solve $\begin{cases} 2x + 3y = 8 \\ x - y = 1 \end{cases}$. $D = 2(-1) - 3(1) = -5$.

$$x = \frac{(8)(-1) - (3)(1)}{-5} = \frac{-11}{-5} = \tfrac{11}{5}, \qquad y = \frac{(2)(1) - (8)(1)}{-5} = \frac{-6}{-5} = \tfrac{6}{5}.$$

If $\det = 0$, there is no unique solution (parallel or coincident) and no inverse exists. Cramer's Rule cannot be used.

13.4 Quadratic Functions & Complex Numbers

Three Forms of a Quadratic

- **Standard:** $y = ax^2 + bx + c$; vertex at $x = -\dfrac{b}{2a}$; y-intercept $(0, c)$.
- **Vertex:** $y = a(x-h)^2 + k$; vertex (h, k); axis $x = h$.
- **Factored:** $y = a(x - r_1)(x - r_2)$; zeros r_1, r_2.

Opens up if $a > 0$, down if $a < 0$.

Solving Methods & Discriminant

- Factoring / Zero Product
- Square roots
- Completing the square
- Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant $D = b^2 - 4ac$: $D > 0$ two real roots; $D = 0$ one repeated real root; $D < 0$ two complex conjugate roots.

Complex Numbers

$i = \sqrt{-1}$, $i^2 = -1$. Powers cycle with period 4:

$$i^1 = i, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1 \;\; \text{(divide exponent by 4, use remainder).}$$

Add/subtract: combine real and imaginary parts. Multiply: FOIL, replace $i^2 = -1$. Divide: multiply by the **conjugate** $a - bi$.

$$(a + bi)(a - bi) = a^2 + b^2.$$

Worked Example: Quadratic Formula with Complex Roots

Solve $x^2 - 4x + 13 = 0$. $D = 16 - 52 = -36$.

$$x = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Simplify negative radicals before the formula: $\sqrt{-36} = 6i$, not ± 6. Complex roots always come in conjugate pairs for real-coefficient equations.

13.5 Polynomial Functions

End Behavior (Leading Term Test)

For degree n with leading coefficient a:

- n even, $a > 0$: both ends up. n even, $a < 0$: both ends down.
- n odd, $a > 0$: down-left, up-right. n odd, $a < 0$: up-left, down-right.

A degree-n polynomial has at most n real zeros and $n - 1$ turning points.

Division & Key Theorems

Synthetic division divides by $(x - c)$: bring down, multiply by c, add.

Remainder Theorem: the remainder of $P(x) \div (x - c)$ equals $P(c)$.

Factor Theorem: $(x - c)$ is a factor $\iff P(c) = 0$.

Rational Root Theorem: any rational zero $= \dfrac{p}{q}$, where $p \mid$ (constant term), $q \mid$ (leading coefficient).

Fundamental Theorem of Algebra: a degree-n polynomial has exactly n complex roots (counting multiplicity). Complex and irrational roots come in conjugate pairs.

Worked Example: Synthetic Division

Divide $x^3 - 2x^2 - 5x + 6$ by $(x - 1)$, i.e. $c = 1$:

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

Quotient $x^2 - x - 6 = (x - 3)(x + 2)$; remainder 0, so zeros are $1, 3, -2$.

A remainder of 0 confirms a factor. Use the depressed quotient to keep factoring until you reach a quadratic you can solve directly.

13.6 Rational Expressions & Functions

Operations

- **Multiply:** factor, cancel common factors, multiply across.
- **Divide:** multiply by the reciprocal.
- **Add/subtract:** get a common denominator (LCD), combine numerators.
- **Complex fractions:** multiply top and bottom by the overall LCD.

Solving: multiply every term by the LCD to clear fractions, then solve. **Check for extraneous roots** (any value making a denominator 0 is rejected).

Asymptotes & Holes

For $f(x) = \dfrac{p(x)}{q(x)}$ in lowest terms:

- **Vertical asymptote:** where $q(x) = 0$ (and $p \neq 0$ there).
- **Hole:** at a factor that cancels from top and bottom.
- **Horizontal asymptote:** compare degrees — deg top $<$ bottom: $y = 0$; equal: $y = \dfrac{\text{lead}}{\text{lead}}$; top $>$ bottom by 1: **slant** asymptote (by division).

Variation

- **Direct:** $y = kx$. **Inverse:** $y = \dfrac{k}{x}$.
- **Joint:** $y = kxz$. **Combined:** $y = \dfrac{kx}{z}$.

Find k from a known data point, then answer the question.

Worked Example: Solve a Rational Equation

Solve $\dfrac{3}{x} + \dfrac{1}{2} = \dfrac{5}{x}$. LCD $= 2x$.

$$2x \cdot \frac{3}{x} + 2x \cdot \frac{1}{2} = 2x \cdot \frac{5}{x} \;\Rightarrow\; 6 + x = 10 \;\Rightarrow\; x = 4.$$

$x = 4$ does not make any denominator 0, so it is valid.

Always factor first. Cancel only *common factors*, never individual terms across $+$ or $-$. Discard solutions that zero a denominator.

13.7 Radical Functions & Rational Exponents

nth Roots & Rational Exponents

$\sqrt[n]{a} = a^{1/n}, \quad a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$

Even index: radicand must be ≥ 0; principal root is nonnegative. **Odd index:** any real radicand.

Exponent rules still apply:

- $a^m a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $a^{-n} = \dfrac{1}{a^n}$

Radical Equations & Inverses

Solve: isolate the radical, raise both sides to the index power, solve, then **check for extraneous roots** (squaring can introduce false solutions).

The inverse of $y = x^2$ $(x \geq 0)$ is $y = \sqrt{x}$; more generally squaring and square-rooting undo each other on the appropriate domain.

Worked Example: Radical Equation

Solve $\sqrt{2x+3} = x$.

$$2x + 3 = x^2 \;\Rightarrow\; x^2 - 2x - 3 = 0 \;\Rightarrow\; (x-3)(x+1) = 0.$$

$x = 3$ or $x = -1$. **Check:** $\sqrt{9} = 3$ ✓; $\sqrt{1} = 1 \neq -1$ (extraneous). Solution: $x = 3$.

Isolate before you power. With two radicals, isolate one, square, isolate the remaining radical, square again. Always check answers in the *original* equation.

13.8 Exponential & Logarithmic Functions

Definitions

$y = b^x$ $(b > 0,\ b \neq 1)$: growth if $b > 1$, decay if $0 < b < 1$; horizontal asymptote $y = 0$. The number $e \approx 2.71828$.

Logarithm is the inverse: $\log_b y = x \iff b^x = y$. $\quad \ln x = \log_e x$. $\quad \log x = \log_{10} x$.

$$\log_b 1 = 0, \qquad \log_b b = 1, \qquad b^{\log_b x} = x.$$

Log Properties & Change of Base

- Product: $\log_b(MN) = \log_b M + \log_b N$
- Quotient: $\log_b \dfrac{M}{N} = \log_b M - \log_b N$
- Power: $\log_b(M^p) = p \log_b M$
- Change of base: $\log_b x = \dfrac{\ln x}{\ln b}$

Solving: exponential equations — take a log of both sides (or equate bases). Log equations

— rewrite in exponential form, then **check the domain** (arguments must be > 0).

Growth, Decay & Interest

- General: $A = A_0(1 \pm r)^t$
- Continuous: $A = A_0 e^{rt}$
- Compound: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
- Continuous interest: $A = Pe^{rt}$

Here r = rate (decimal), n = compounds per year, t = years.

Worked Example: Solve an Exponential Equation

Solve $3^x = 20$.

$$x = \log_3 20 = \frac{\ln 20}{\ln 3} \approx \frac{2.996}{1.099} \approx 2.73.$$

Condense before solving log equations: use the properties to write a single log, convert to exponential form, then reject any solution that makes an argument ≤ 0.

13.9 Sequences & Series

Arithmetic (common difference d)

$$a_n = a_1 + (n-1)d, \qquad S_n = \frac{n}{2}\big(a_1 + a_n\big) = \frac{n}{2}\big(2a_1 + (n-1)d\big).$$

Add a constant d each term.

Geometric (common ratio r)

$$a_n = a_1 r^{\,n-1}, \qquad S_n = a_1 \frac{1 - r^{\,n}}{1 - r} \ (r \neq 1).$$

Infinite geometric sum converges only when $|r| < 1$:

$$S_\infty = \frac{a_1}{1 - r}.$$

Multiply by a constant r each term.

Sigma (Summation) Notation

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

k is the index, the bottom value is the start, the top value is the end. Useful facts:

$$\sum_{k=1}^{n} c = cn, \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Worked Example: Infinite Geometric Series

Evaluate $8 + 4 + 2 + 1 + \cdots$. Here $a_1 = 8$, $r = \frac{1}{2}$, $|r| < 1$:

$$S_\infty = \frac{8}{1 - \frac{1}{2}} = \frac{8}{\frac{1}{2}} = 16.$$

Identify the type first. Constant difference $\to$ arithmetic; constant ratio $\to$ geometric. An infinite geometric series diverges (no finite sum) when $|r| \geq 1$.

13.10 Conic Sections

Circle & Parabola

Circle: center (h, k), radius r:

$$(x - h)^2 + (y - k)^2 = r^2.$$

Parabola (vertex (h, k), $p =$ vertex-to-focus distance):

- Opens up/down: $(x - h)^2 = 4p(y - k)$; focus $(h, k + p)$, directrix $y = k - p$.
- Opens left/right: $(y - k)^2 = 4p(x - h)$; focus $(h + p, k)$, directrix $x = h - p$.

Ellipse & Hyperbola (center (h, k))

Ellipse ($a > b$, $a =$ semi-major): $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, foci on major axis with $c^2 = a^2 - b^2$.

Hyperbola: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (opens left/right), with $c^2 = a^2 + b^2$ and asymptotes $y - k = \pm\dfrac{b}{a}(x - h)$.

Eccentricity $e = \dfrac{c}{a}$: circle $e = 0$; ellipse $0 < e < 1$; parabola $e = 1$; hyperbola $e > 1$.

Identifying a Conic from $Ax^2 + Cy^2 + \cdots = 0$

- Only one squared term $\to$ **parabola**.
- $A = C$ (same sign) $\to$ **circle**.
- $A \neq C$, same sign $\to$ **ellipse**.
- Opposite signs $\to$ **hyperbola**.

Worked Example: Complete the Square (Circle)

Write $x^2 + y^2 - 6x + 4y - 3 = 0$ in standard form.

$$(x^2 - 6x + 9) + (y^2 + 4y + 4) = 3 + 9 + 4$$

$$(x-3)^2 + (y+2)^2 = 16.$$

Center $(3, -2)$, radius 4.

Ellipse vs. hyperbola hinges on the sign between terms: a $+$ gives an ellipse, a $-$ gives a hyperbola. For an ellipse $c^2 = a^2 - b^2$; for a hyperbola $c^2 = a^2 + b^2$.

13.11 Probability & Statistics

Counting: Permutations & Combinations

Fundamental Counting Principle: multiply the choices at each stage.

$${}_nP_r = \frac{n!}{(n-r)!} \text{ (order matters)}, \qquad {}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!} \text{ (order does not)}.$$

Probability Rules

$P(E) = \dfrac{\text{favorable}}{\text{total}}$, with $0 \leq P(E) \leq 1$; complement $P(E') = 1 - P(E)$.

- **Addition:** $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ (subtract 0 if mutually exclusive).
- **Multiplication (independent):** $P(A \text{ and } B) = P(A) \cdot P(B)$.
- **Conditional:** $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)}$.

Binomial Probability

For n independent trials, success probability p, exactly k successes:

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Mean, Standard Deviation & the Normal Curve

$$\text{mean } \bar{x} = \frac{\sum x}{n}, \qquad \text{standard deviation } \sigma = \sqrt{\frac{\sum (x-\bar{x})^2}{n}}.$$

Empirical (68–95–99.7) Rule for a normal distribution: about 68% of data lie within 1σ of the mean, 95% within 2σ, 99.7% within 3σ.

z-score: $z = \dfrac{x-\bar{x}}{\sigma}$ (number of standard deviations from the mean).

Worked Example: Empirical Rule

Test scores are normal with $\bar{x} = 70$, $\sigma = 8$. What percent scored between 62 and 86?

$62 = 70 - 1\sigma$ and $86 = 70 + 2\sigma$. From the mean out to -1σ is 34%; from the mean out to $+2\sigma$ is 47.5%. Total $= 34\% + 47.5\% = 81.5\%$.

Order matters → permutation; order doesn't → combination. "And" usually means multiply, "or" means add (subtract the overlap when events can happen together).