MathCastle

Algebra 1

Complete course booklet • 13 topics

Contents

1 Core Ideas in Plain Terms

The core of algebra, in everyday language

1.1 Lines and Slope

Slope is steepness: rise over run

The **slope** of a line is how much it goes up for each step right — "rise over run." In $y = mx + b$, the m is that slope and b is where the line crosses the y-axis. A bigger m is steeper; a negative m goes downhill.

(diagram in the online version)

Slope through two points

From $(1, 2)$ to $(4, 8)$ the line rises 6 over a run of 3, so the slope is $\frac{6}{3} = 2$.

1.2 Factoring and Quadratics

Factoring is un-multiplying

To **factor** means to rewrite an expression as a product. For $x^2 + 5x + 6$, look for two numbers that multiply to 6 and add to 5 — that is 2 and 3 — so it factors as $(x + 2)(x + 3)$. Factoring reveals the roots through the **zero-product property**: if $ab = 0$ then $a = 0$ or $b = 0$. And factoring itself is the **distributive property** run in reverse — expanding back gives $(x + 2)(x + 3) = x^2 + 3x + 2x + 6$.

Solve x + 5x + 6 = 0

It factors as $(x + 2)(x + 3) = 0$, so by the zero-product property $x = -2$ or $x = -3$.

1.3 Systems of Equations

Two equations, two unknowns

A **system** is two equations that must both be true. Solve by *substitution* (solve one for a variable and plug in) or *elimination* (add or subtract the equations to cancel a variable). The solution is the point where the two lines cross.

(diagram in the online version)

Solve the system

$x + y = 5$ and $x - y = 1$: add them — **elimination**, the y-terms cancel — to get $2x = 6$, so

$x = 3$, and then $y = 2$.

1.4 Going Deeper: The Quadratic Formula and Absolute Value

The quadratic formula, demystified

Completing the square once, in general, on $ax^2 + bx + c = 0$ gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ — so the formula is not magic, it is the same move you already know, done once for every equation. The part under the root, $b^2 - 4ac$, tells you how many real solutions there are: positive → two, zero → one, negative → none.

Use it

$x^2 - 4x - 5 = 0$: $x = \dfrac{4 \pm \sqrt{16 + 20}}{2} = \dfrac{4 \pm 6}{2}$, so $x = 5$ or $x = -1$. (Check: $(x - 5)(x + 1) = x^2 - 4x - 5$.)

Absolute value means distance

$|x - 3| = 7$ says "x is 7 away from 3" — so $x = 10$ or $x = -4$. Reading absolute-value equations as distances turns them from a rule to memorize into a picture on the number line.

A multi-step equation

Solve $3(2x - 4) + 5 = 2(x + 3) - 1$. Step 1 — **distributive property**: $6x - 12 + 5 = 2x + 6 - 1$. Step 2 — collect like terms: $6x - 7 = 2x + 5$, so $4x = 12$. Step 3 — divide: $x = 3$.

Where two graphs cross

Find where $y = 2x + 1$ meets $y = -x + 7$. At an intersection the y-values agree, so set them equal: $2x + 1 = -x + 7$, giving $3x = 6$ and $x = 2$; then $y = 2(2) + 1 = 5$. The graphs cross at $(2, 5)$.

(diagram in the online version)

1.5 Problem-Solving Playbook

Get zero on one side

For anything quadratic, move everything to one side first — the **zero-product property** only works against zero. Then factor (or use the quadratic formula), and **check by substituting back**.

Worked: rearrange, factor, check

Solve $x^2 = 5x - 6$. Step 1: $x^2 - 5x + 6 = 0$. Step 2 — factor: $(x-2)(x-3) = 0$, so $x = 2$ or $x = 3$. Step 3 — check $x = 2$: $4 = 10 - 6$ ✓.

2 Rational Expressions

Excluded values, simplifying, multiplying, dividing, adding, subtracting, complex fractions, equations, and work problems

2.1 What Is a Rational Expression?

Definition

A **rational expression** is a *ratio of two polynomials*—one polynomial on top (the numerator) divided by another polynomial on the bottom (the denominator):

$$\frac{\text{polynomial}}{\text{polynomial}}, \qquad \text{such as } \frac{2x+1}{x-3}, \quad \frac{x^2-9}{x^2+5x+6}, \quad \frac{5}{x}.$$

It is just like a numerical fraction (such as $\frac{3}{4}$), except the top and bottom can contain variables.

Example: Which are rational expressions?

- $\dfrac{2x+1}{x-3}$ **Yes** — a polynomial over a polynomial.
- $\dfrac{5}{x^2}$ **Yes** — the constant 5 and x^2 are both polynomials.
- $\sqrt{x}+1$ **No** — $\sqrt{x}$ is not a polynomial (a variable under a root).

Remember: A constant like 5 is a polynomial, so $\dfrac{5}{x}$ counts. But roots of a variable, or a variable in an exponent, are *not* allowed in a polynomial.

2.2 Excluded Values (Domain Restrictions)

Why the Denominator Matters

Division by zero is undefined. So a rational expression is undefined for any value that makes the **denominator equal to** 0. Those values are called **excluded values**.

Method: Set the denominator equal to 0, solve, and exclude those values.

Example: Find the excluded values of $\dfrac{x+4}{x^2-x-6}$

Set the denominator to 0 and factor:

$$\begin{aligned} x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ x = 3 \quad &\text{or} \quad x = -2. \end{aligned}$$

So the excluded values are $x \neq 3$ and $x \neq -2$.

Tip: Look at the *original* denominator to find excluded values, *before* you cancel anything. Canceling can hide a restriction, but the restriction is still there.

2.3 Simplifying Rational Expressions

Factor, Then Cancel Common Factors

To simplify a rational expression:

1. **Factor** the numerator and denominator completely.
2. **Cancel** any factor that appears in both the top and the bottom.
3. State the excluded values (from the original denominator).

Example: Simplify $\dfrac{x^2-9}{x^2+5x+6}$

Factor top and bottom:

$$\frac{x^2-9}{x^2+5x+6} = \frac{(x-3)(x+3)}{(x+2)(x+3)}.$$

Cancel the common factor $(x+3)$:

$$= \frac{x-3}{x+2}, \qquad x \neq -3,\ x \neq -2.$$

Example: Simplify $\dfrac{x^2-x}{x}$

Factor the top: $x^2 - x = x(x-1)$. Then cancel the x:

$$\frac{x(x-1)}{x} = x - 1, \qquad x \neq 0.$$

You may only cancel FACTORS, never terms! In $\dfrac{x+2}{x}$ you *cannot* cancel the x's, because $x+2$ is a *sum*, not a product. Only cancel things that are multiplied. For instance $\dfrac{x+3}{x+2}$ is already simplified—nothing cancels.

2.4 Multiplying Rational Expressions

Factor, Cancel, Multiply

To multiply, **factor everything**, **cancel** any factor common to a top and a bottom, then multiply the remaining tops together and the remaining bottoms together. (You do *not* need a common denominator to multiply.)

Example: $\dfrac{x^2-4}{x^2+6x+9} \cdot \dfrac{x+3}{x-2}$

Factor each part:

$$\frac{(x-2)(x+2)}{(x+3)(x+3)} \cdot \frac{x+3}{x-2}.$$

Cancel one $(x-2)$ and one $(x+3)$:

$$= \frac{x+2}{x+3}, \qquad x \neq -3,\ x \neq 2.$$

Tip: Cancel *before* you multiply out—it keeps the numbers small. Collect the excluded values from every original denominator.

2.5 Dividing Rational Expressions

Multiply by the Reciprocal

To divide, **flip the second fraction** (take its reciprocal) and **multiply**. Then factor and cancel as usual.

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \cdot \frac{D}{C}.$$

Example: $\dfrac{x^2-1}{x^2-4} \div \dfrac{x+1}{x-2}$

Flip the divisor and multiply:

$$\frac{x^2-1}{x^2-4} \cdot \frac{x-2}{x+1} = \frac{(x-1)(x+1)}{(x-2)(x+2)} \cdot \frac{x-2}{x+1}.$$

Cancel $(x+1)$ and $(x-2)$:

$$= \frac{x-1}{x+2}, \qquad x \neq 2,\ x \neq -2,\ x \neq -1.$$

Careful with restrictions when dividing. The factor you flip ($x+1$ here) was a denominator after flipping, so $x \neq -1$ too. Track excluded values from *both* the original bottoms and the divisor's top.

2.6 Adding & Subtracting with LIKE Denominators

Same Bottom? Combine the Tops

When the denominators are already the same, **add or subtract the numerators** and keep the common denominator. Then simplify if possible.

$$\frac{A}{D} \pm \frac{B}{D} = \frac{A \pm B}{D}.$$

Example: $\dfrac{x^2}{x-2} - \dfrac{4}{x-2}$

Subtract the numerators over the common denominator, then factor and cancel:

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2, \qquad x \neq 2.$$

Subtracting? Subtract the *whole* numerator—use parentheses. $\dfrac{A}{D} - \dfrac{B}{D} = \dfrac{A-B}{D}$, and the minus sign hits *every* term of B.

2.7 Adding & Subtracting with UNLIKE Denominators

Find the LCD

When denominators differ, build a **least common denominator (LCD)**:

1. Factor each denominator.
2. The LCD contains each different factor the greatest number of times it appears in any one denominator.
3. Rewrite each fraction over the LCD, then combine the numerators.

Example: $\frac{2}{x} + \frac{3}{x+1}$

The denominators x and $x+1$ share no factors, so the LCD is $x(x+1)$. Rewrite each fraction:

$$\begin{aligned} \frac{2}{x} \cdot \frac{x+1}{x+1} + \frac{3}{x+1} \cdot \frac{x}{x} &= \frac{2(x+1)}{x(x+1)} + \frac{3x}{x(x+1)} \\ &= \frac{2x+2+3x}{x(x+1)} = \frac{5x+2}{x(x+1)}, \qquad x \neq 0,\ x \neq -1. \end{aligned}$$

Example with factoring: $\frac{3}{x+2} + \frac{x}{x^2-4}$

Factor $x^2 - 4 = (x-2)(x+2)$, so the LCD is $(x-2)(x+2)$:

$$\begin{aligned} \frac{3(x-2)}{(x-2)(x+2)} + \frac{x}{(x-2)(x+2)} &= \frac{3x-6+x}{(x-2)(x+2)} \\ &= \frac{4x-6}{(x-2)(x+2)} = \frac{2(2x-3)}{(x-2)(x+2)}, \qquad x \neq 2,\ x \neq -2. \end{aligned}$$

Do not just multiply the denominators blindly. Factor first so the LCD is as small as possible, and only multiply each numerator by the factors its denominator is *missing*.

2.8 Complex Fractions (Intro)

A Fraction Within a Fraction

A **complex fraction** has fractions in its numerator, its denominator, or both. Two ways to simplify:

- **Multiply through** by the LCD of every little fraction, or
- Combine the top into one fraction and the bottom into one fraction, then divide (multiply

by the reciprocal).

Example: $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$

The only small denominator is x, so multiply the top and bottom by x:

$$\frac{\left(1+\frac{1}{x}\right)\cdot x}{\left(1-\frac{1}{x}\right)\cdot x}=\frac{x+1}{x-1},\qquad x\neq 0,\ x\neq 1.$$

Tip: Multiplying every part by the LCD clears all the little fractions in one clean step. Watch the excluded values: $x \neq 0$ (the inner denominator) *and* $x \neq 1$ (the new bottom).

2.9 Solving Rational Equations

Clear the Denominators

To **solve** an equation with rational expressions:

1. Note the excluded values (denominators $\neq 0$).
2. Multiply *every* term by the LCD to clear all fractions.
3. Solve the resulting equation.
4. **Check** each answer—reject any that is an excluded value (an **extraneous solution**).

Example (real solution): $\dfrac{3}{x+1}=\dfrac{2}{x-1}$

Cross-multiply (or multiply by the LCD $(x+1)(x-1)$):

$$\begin{aligned}3(x-1)&=2(x+1)\\3x-3&=2x+2\\x&=5.\end{aligned}$$

Since $x=5$ is not excluded ($x \neq -1, 1$), the solution is $x=5$.

Example (extraneous!): $\dfrac{x}{x-4}=\dfrac{4}{x-4}+2$

Excluded value: $x \neq 4$. Multiply every term by $(x-4)$:

$$\begin{aligned} x &= 4 + 2(x-4) \\ x &= 4 + 2x - 8 \\ x &= 2x - 4 \\ -x &= -4 \;\Rightarrow\; x = 4. \end{aligned}$$

But $x = 4$ is excluded, so it is **extraneous**. There is **no solution**.

Always check for extraneous solutions! An answer that makes any original denominator 0 must be thrown out—even though it came from correct algebra.

2.10 Applications: Work & Rate Problems

The Work Formula

If one worker (or pipe) finishes a job in a units of time and another in b units, each does a fraction of the job per unit of time: $\frac{1}{a}$ and $\frac{1}{b}$. Working together for t units:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{t}.$$

(Their rates add.)

Example: Two pipes filling a tank

Pipe A fills a tank in 6 hours; pipe B fills it in 3 hours. Together, how long?

$$\frac{1}{6} + \frac{1}{3} = \frac{1}{t}.$$

The LCD is 6: $\dfrac{1}{6} + \dfrac{2}{6} = \dfrac{3}{6} = \dfrac{1}{2}$. So $\dfrac{1}{t} = \dfrac{1}{2}$, giving $t = 2$ hours.

Example: Solving for an unknown rate

Maria paints a room in 3 hours. With Sam helping, they finish in 2 hours. How long would Sam take alone?

$$\frac{1}{3} + \frac{1}{s} = \frac{1}{2} \;\Rightarrow\; \frac{1}{s} = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}.$$

So $s = 6$ hours for Sam alone.

Sanity check: Working together must be *faster* than either person alone. If your "together" time is bigger than one of the individual times, recheck the setup.

2.11 Going Deeper: Advanced Rational Expressions

Continued & Nested Complex Fractions

A **continued fraction** stacks fractions inside fractions many layers deep. Simplify it **from the bottom up**: fully combine the innermost fraction, then work outward one level at a time. The same two tools from before still work—multiply by an LCD, or combine-then-flip—but here you apply them *repeatedly*.

$$2+\cfrac{1}{2+\cfrac{1}{2}} = 2+\frac{1}{\frac{5}{2}} = 2+\frac{2}{5} = \frac{12}{5}.$$

Worked Example: Simplify $\dfrac{x}{1+\cfrac{1}{1+\cfrac{1}{x}}}$

Start at the **deepest** level and climb out. First the inner denominator:

$$1+\frac{1}{x}=\frac{x+1}{x}.$$

Substitute and flip that little fraction:

$$1+\frac{1}{\frac{x+1}{x}} = 1+\frac{x}{x+1} = \frac{(x+1)+x}{x+1} = \frac{2x+1}{x+1}.$$

Now the whole expression is $\dfrac{x}{\frac{2x+1}{x+1}}$, so flip once more:

$$\frac{x}{1}\cdot\frac{x+1}{2x+1} = \frac{x(x+1)}{2x+1}, \qquad x\neq 0,\ x\neq -1,\ x\neq -\tfrac{1}{2}.$$

Each inner denominator contributes an excluded value—track them all as you unwind.

The $x+\dfrac{1}{x}$ Family

A classic trick: if you know the value of $x+\dfrac{1}{x}$, you can find $x^2+\dfrac{1}{x^2}$ and $x^3+\dfrac{1}{x^3}$ *without ever solving for* x. The key is that squaring or cubing $x+\frac{1}{x}$ produces a "+2" or "+3$(\cdots)$"

cross-term that you subtract back off.

$$\left(x+\tfrac{1}{x}\right)^2 = x^2+2+\tfrac{1}{x^2} \qquad \Rightarrow \qquad x^2+\tfrac{1}{x^2} = \left(x+\tfrac{1}{x}\right)^2-2,$$
$$\left(x+\tfrac{1}{x}\right)^3 = x^3+3\left(x+\tfrac{1}{x}\right)+\tfrac{1}{x^3} \qquad \Rightarrow \qquad x^3+\tfrac{1}{x^3} = \left(x+\tfrac{1}{x}\right)^3-3\left(x+\tfrac{1}{x}\right).$$

Worked Example: If $x+\dfrac{1}{x}=5$, find $x^2+\dfrac{1}{x^2}$ and $x^3+\dfrac{1}{x^3}$

Let $a = x+\dfrac{1}{x} = 5$. Square first:

$$x^2+\frac{1}{x^2} = a^2-2 = 5^2-2 = 25-2 = 23.$$

Now cube, using the formula above:

$$x^3+\frac{1}{x^3} = a^3-3a = 5^3-3(5) = 125-15 = 110.$$

Check the pattern another way: $x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)\left(x^2-1+\frac{1}{x^2}\right) = 5\,(23-1) = 5\cdot 22 = 110.$ ✓

Why it works: multiplying out $\left(x+\frac{1}{x}\right)^2$ or $\left(x+\frac{1}{x}\right)^3$, the "$x\cdot\frac{1}{x}=1$" cross-terms are constants. Rearranging isolates the "symmetric" power sum you want. This is the same idea as $(a+b)^2 = a^2+2ab+b^2$ with $b=\frac{1}{a}$.

Telescoping Sums via Partial Fractions

Some long sums of rational terms **collapse**. Split each term into a *difference* of simpler fractions; then consecutive pieces cancel and only the first and last survive. The workhorse identity is

$$\frac{1}{n(n+1)} = \frac{1}{n}-\frac{1}{n+1}.$$

Adding from $n=1$ to $n=N$, every $-\frac{1}{n+1}$ cancels the next term's $+\frac{1}{n+1}$:

$$\sum_{n=1}^{N}\frac{1}{n(n+1)} = \left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\cdots+\left(\tfrac{1}{N}-\tfrac{1}{N+1}\right) = 1-\frac{1}{N+1} = \frac{N}{N+1}.$$

Worked Example: Evaluate $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{9\cdot 10}$

Each term is $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Write them out and watch the middle vanish:

$$\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{9}-\tfrac{1}{10}\right).$$

Only the very first 1 and the very last $-\frac{1}{10}$ are left:

$$= 1-\frac{1}{10} = \frac{9}{10}.$$

(Matches the formula $\frac{N}{N+1}$ with $N = 9$.)

Recognizing a telescoping split: to break $\dfrac{1}{n(n+1)}$ into $\dfrac{A}{n}+\dfrac{B}{n+1}$, clear denominators: $1 = A(n+1)+Bn$. Setting $n = 0$ gives $A = 1$; setting $n = -1$ gives $B = -1$. That "cover-up" method is **partial-fraction decomposition**, and it is the engine behind every telescoping sum.

Preview: Asymptotes & Holes of Rational Functions

The *graph* of $y = \dfrac{P(x)}{Q(x)}$ is shaped by where the function blows up or flattens. After factoring and canceling:

- **Hole** at $x = c$: a factor $(x - c)$ that cancels from top *and* bottom—the graph is undefined at c but has no infinite spike (a single missing point).
- **Vertical asymptote** at $x = c$: a factor $(x-c)$ left in the *denominator* after canceling—the graph shoots to $\pm\infty$.
- **Horizontal asymptote**: compare degrees. Bottom-heavy $\Rightarrow y = 0$; equal degrees $\Rightarrow y = \frac{\text{leading coeff top}}{\text{leading coeff bottom}}$; top-heavy $\Rightarrow$ none (a slant asymptote instead).

Example: Locate the features of $y = \dfrac{x^2-1}{x^2-x-2}$

Factor top and bottom:

$$y = \frac{(x-1)(x+1)}{(x-2)(x+1)}.$$

The factor $(x+1)$ cancels, so there is a **hole** at $x = -1$. What remains is $\dfrac{x-1}{x-2}$, whose leftover denominator factor $(x-2)$ gives a **vertical asymptote** at $x = 2$. Top and bottom have equal degree (1 and 1) with leading coefficients 1 and 1, so the **horizontal asymptote** is $y = \frac{1}{1} = 1$.

Cancel first, then classify. A factor that cancels makes a *hole*; a factor that stays in the denominator makes a *vertical asymptote*. Same value of x can never be both—decide it *after* simplifying.

Average Speed Is a Harmonic Mean

For equal *distances* traveled at two speeds, the average speed is **not** the plain average of the speeds. Since time $= \frac{\text{distance}}{\text{speed}}$, cover distance d out at speed v_1 and back at v_2:

$$\overline{v} = \frac{\text{total distance}}{\text{total time}} = \frac{2d}{\frac{d}{v_1} + \frac{d}{v_2}} = \frac{2}{\frac{1}{v_1} + \frac{1}{v_2}} = \frac{2v_1v_2}{v_1 + v_2}.$$

This is the **harmonic mean**—the same "rates add" idea as the work formula.

Worked Example: Round trip at 30 and 60 mph

You drive to work at 30 mph and return along the same road at 60 mph. The average speed is *not* 45. Use the harmonic mean:

$$\overline{v} = \frac{2v_1v_2}{v_1 + v_2} = \frac{2(30)(60)}{30 + 60} = \frac{3600}{90} = 40 \text{ mph}.$$

Sanity check with a real distance: let each leg be 60 miles. Out takes $\frac{60}{30} = 2$ h, back takes $\frac{60}{60} = 1$ h, total 120 miles in 3 h $= 40$ mph. ✓ The slower leg dominates because you spend *more time* at the low speed.

Which mean? Equal *times* at two speeds $\Rightarrow$ ordinary average $\frac{v_1+v_2}{2}$. Equal *distances* $\Rightarrow$ harmonic mean $\frac{2v_1v_2}{v_1+v_2}$. The harmonic mean is always the smaller of the two—slow stretches cost extra time.

3 Data & Statistics

Center, spread, displays, scatter plots, correlation, and two-way tables

3.1 Measures of Center

The three averages

A **measure of center** is a single number that describes the "middle" of a data set.

- **Mean** (the "average"): add all the values, then divide by how many there are.

$$\text{mean} = \frac{\text{sum of values}}{\text{number of values}}$$

- **Median**: the middle value *after the data is put in order*. If there is an even number of values, average the two middle ones.
- **Mode**: the value that appears most often. A set can have no mode, one mode, or several.

Worked example: all three centers

Find the mean, median, and mode of: 7, 3, 9, 5, 6, 9, 3, 9.

Mean: sum $= 7 + 3 + 9 + 5 + 6 + 9 + 3 + 9 = 51$, and there are 8 values, so

$$\text{mean} = \frac{51}{8} = 6.375.$$

Median: order the data first: 3, 3, 5, 6, 7, 9, 9, 9. With 8 values the two middle ones are the 4th and 5th: 6 and 7.

$$\text{median} = \frac{6+7}{2} = 6.5.$$

Mode: 9 appears three times, more than any other value, so the mode is 9.

Which center is "best"?

- Use the **mean** when the data is fairly even with no extreme values.
- Use the **median** when there are **outliers** (values far from the rest), because the median is not pulled toward them.
- Use the **mode** for categories or "most popular" questions (favorite color, most common shoe size).

An **outlier** is a value much larger or smaller than the rest of the data.

Remember: Always *order the data* before finding the median. An outlier drags the **mean** toward itself but barely moves the **median**. Example: for $2, 4, 6, 8$ the mean is 5; change the 8 to 80 and the mean jumps to 23, but the median only moves from 5 to 5.

3.2 Measures of Spread

Range, quartiles, and IQR

A **measure of spread** tells how spread out the data is.

- **Range** = maximum − minimum.

- **Quartiles** split ordered data into four equal parts. $Q2$ is just the median. $Q1$ (the lower quartile) is the median of the *lower* half; $Q3$ (the upper quartile) is the median of the *upper* half.
- **Interquartile range (IQR)** $= Q3 - Q1$. It measures the spread of the middle 50% of the data and ignores outliers.

One outlier drags the mean but not the median.

(diagram in the online version)

Worked example: quartiles and IQR

Find the range, quartiles, and IQR of the ordered data

$$3,\ 5,\ 7,\ 8,\ 12,\ 13,\ 14,\ 18,\ 21.$$

There are 9 values. **Range** $= 21 - 3 = 18$.

The **median** ($Q2$) is the 5th value: 12.

Lower half (values below the median): $3, 5, 7, 8$. Its median is $Q1 = \frac{5+7}{2} = 6$.

Upper half (values above the median): $13, 14, 18, 21$. Its median is $Q3 = \frac{14+18}{2} = 16$.

$$\text{IQR} = Q3 - Q1 = 16 - 6 = 10.$$

Tip: When the number of values is *odd*, do *not* include the middle value in either half when finding $Q1$ and $Q3$. When it is *even*, split the ordered list exactly in half.

3.3 Five-Number Summary & Box Plots

The five-number summary

The **five-number summary** lists, in order:

$$\text{minimum}, \quad Q1, \quad \text{median}, \quad Q3, \quad \text{maximum}.$$

A **box-and-whisker plot** draws these five numbers on a number line: a *box* stretches from $Q1$ to $Q3$ with a line at the median, and *whiskers* reach out to the minimum and maximum. The box holds the middle 50% of the data.

Reading a box plot

Using the example above, the five-number summary is

$$\min = 3, \quad Q1 = 6, \quad \text{median} = 12, \quad Q3 = 16, \quad \max = 21.$$

So the box runs from 6 to 16, with a line at 12. The left whisker reaches 3 and the right whisker reaches 21. Because the right whisker (16 to 21) is a bit longer than the left, the data is slightly **skewed right**. Half of all values lie between 6 and 16.

Outlier rule (advanced): A value is usually called an outlier if it is more than $1.5 \times \text{IQR}$ below $Q1$ or above $Q3$.

3.4 Displaying Data & Describing Shape

Four ways to picture data

- **Dot plot:** one dot per data value stacked above a number line. Great for small sets; the mode is the tallest stack.
- **Stem-and-leaf plot:** each value is split into a *stem* (the leading digits) and a *leaf* (the last digit). Keeps every original value.
- **Frequency table:** lists each value or interval and how many times it occurs (its frequency).
- **Histogram:** a bar graph for *intervals* of data, with bars touching. The height of each bar is the frequency of that interval.

Frequency table and stem-and-leaf

Test scores: $82, 75, 91, 88, 75, 91, 63, 88, 91$.

A **frequency table** (grouped by tens):

The same data as a **stem-and-leaf plot** (stem = tens digit):

The mode is 91 (three leaves of 1 on the stem 9).

Describing the shape

- **Symmetric:** the left and right sides roughly mirror each other (a "bell" shape peaks in the middle).
- **Skewed right:** a long tail stretches toward the *high* values; most data sits on the left.
- **Skewed left:** a long tail stretches toward the *low* values; most data sits on the right.

Tip: In a right-skewed set the mean is pulled *above* the median; in a left-skewed set the mean is pulled *below* the median. In a symmetric set the mean and median are about equal.

3.5 Scatter Plots & Correlation

Plotting paired data

A **scatter plot** graphs pairs of data (x, y) as points. It shows whether two quantities are related. The **correlation** describes the pattern:

- **Positive:** as x increases, y tends to increase (points rise from left to right).
- **Negative:** as x increases, y tends to decrease (points fall from left to right).
- **No correlation:** no clear upward or downward pattern.

Correlation can also be **strong** (points hug a line tightly) or **weak** (points are loosely scattered).

Reading a scatter plot

A study records hours of sleep (x) and a reaction-time score (y) for several students. As sleep increases, the reaction-time score steadily rises, and the points cluster closely along a rising line. This is a **strong positive** correlation: more sleep is associated with better scores.

Big idea: correlation $\neq$ causation. Two things can rise together without one *causing* the other. Ice-cream sales and swimming-pool accidents both climb in summer, but ice cream does not cause accidents — hot weather (a "lurking variable") drives both.

3.6 Line of Best Fit: Slope & Intercept

Trend lines and predictions

A **line of best fit** (or trend line) is a straight line that comes as close as possible to the points of a scatter plot with a linear pattern. Written as $y = mx + b$:

- The **slope** m is the predicted change in y for each 1-unit increase in x (a rate).
- The **y-intercept** b is the predicted value of y when $x = 0$ (a starting amount).

Plug an x-value into the equation to **predict** a y-value.

Interpreting slope and intercept

A trend line for a plant's height (cm) after x weeks is

$$y = 3x + 8.$$

Predict the height at week 5: $y = 3(5) + 8 = 23$ cm.

Slope $= 3$**:** the plant grows about 3 cm each week.

Intercept $= 8$**:** the plant was about 8 cm tall at week 0 (when it was planted).

Tip: Predictions made *within* the range of the data are usually reliable. Predicting far outside the data (called extrapolation) can give silly answers — that plant will not keep growing 3 cm a week forever!

3.7 Two-Way Frequency Tables

Reading a two-way table

A **two-way frequency table** sorts data by *two* categories at once (rows and columns).

- A **joint frequency** is a count inside the table (one row *and* one column).
- A **marginal frequency** is a row total or column total (found in the margins).
- The **grand total** (bottom-right corner) is the total number of items surveyed.

A survey table

Students were asked whether they walk or ride the bus to school:

Joint: 14 sixth-graders walk. **Marginal:** 40 students are in Grade 6, and 47 students total ride the bus. **Grand total:** 70 students were surveyed.

Check: The row totals add to the grand total *and* the column totals add to the grand total. Here $40 + 30 = 70$ and $23 + 47 = 70$. If they don't match, recount!

3.8 Going Deeper: Advanced Statistics

Center and spread are only the beginning. This section pushes into the tools that show up on advanced exams and in real data analysis: variance and standard deviation, formal outlier fences, reading the correlation coefficient, making and judging predictions, weighted averages, and residuals.

One outlier drags the mean but not the median.

(diagram in the online version)

Variance & standard deviation: spread around the mean

The range and IQR describe spread using just a few values. **Standard deviation** uses *every* value, measuring the typical distance from the mean.

- A **deviation** is how far one value sits from the mean: $x - \bar{x}$ (where $\bar{x}$ is the mean).
- The **variance** σ^2 is the average of the *squared* deviations. We square so that positive and negative gaps don't cancel out.
- The **standard deviation** σ is the square root of the variance, returning to the original units.

$$\sigma^2 = \frac{\sum(x - \bar{x})^2}{n}, \qquad \sigma = \sqrt{\sigma^2}.$$

A *small* standard deviation means values cluster near the mean; a *large* one means they are widely spread.

Worked example: computing standard deviation

Find the standard deviation of $4,\ 8,\ 6,\ 5,\ 2.$

Step 1 — mean: $\bar{x} = \dfrac{4+8+6+5+2}{5} = \dfrac{25}{5} = 5.$

Step 2 — squared deviations (organized in a table):

Step 3 — variance: average the squared deviations:

$$\sigma^2 = \frac{1+9+1+0+9}{5} = \frac{20}{5} = 4.$$

Step 4 — standard deviation: $\sigma = \sqrt{4} = 2.$

So the values sit, on average, about 2 units away from the mean of 5.

Population vs. sample: Dividing by n gives the *population* standard deviation σ. When your data is only a *sample* of a larger group, statisticians divide by $n-1$ instead (the *sample* standard deviation s); this slightly larger value corrects for the sample underestimating the true spread. Check which one a problem asks for.

Outlier fences: the $1.5 \times$ IQR rule

Earlier we named outliers informally. Here is the formal test. From the five-number summary, compute $\text{IQR} = Q3 - Q1$, then build two **fences**:

$$\text{lower fence} = Q1 - 1.5\,(\text{IQR}), \qquad \text{upper fence} = Q3 + 1.5\,(\text{IQR}).$$

Any value *below* the lower fence or *above* the upper fence is an **outlier**. On a box plot, these are drawn as separate dots, and the whiskers only reach to the most extreme values still *inside* the fences.

Worked example: hunting for an outlier

A data set has five-number summary

$$\text{min} = 4, \quad Q1 = 10, \quad \text{median} = 15, \quad Q3 = 18, \quad \text{max} = 40.$$

Step 1 — IQR: $\text{IQR} = Q3 - Q1 = 18 - 10 = 8$.

Step 2 — fences:

$$\text{lower fence} = 10 - 1.5(8) = 10 - 12 = -2,$$
$$\text{upper fence} = 18 + 1.5(8) = 18 + 12 = 30.$$

Step 3 — compare: The minimum 4 is above -2, so it is fine. But the maximum 40 is above the upper fence of 30, so 40 **is an outlier**. On the box plot, the right whisker stops at the largest value at or below 30, and 40 is marked as its own dot.

The correlation coefficient r

The words "strong" and "weak" can be made precise with the **correlation coefficient** r, a number between -1 and 1 that measures how tightly points cluster around a straight line.

- The **sign** of r gives the direction: $r > 0$ is positive correlation, $r < 0$ is negative.
- The **size** of $|r|$ gives the strength: near 1 (or -1) the points hug a line; near 0 there is little linear pattern.
- $r = 1$ or $r = -1$ means the points lie *exactly* on a line; $r = 0$ means no linear relationship.

Rough guide: $|r| \geq 0.8$ is strong, $0.5 \leq |r| < 0.8$ is moderate, and $|r| < 0.5$ is weak.

Warning — r measures *linear* fit only. A tight U-shaped or curved pattern can have r near 0 even though the two variables are strongly related. And, as before, a large $|r|$ still does **not** prove causation — it only measures how well a *line* describes the cloud of points. A lurking variable can inflate r without either variable causing the other.

Worked example: predicting with a line of best fit

A shop fits the trend line $y = 2.5x + 40$ relating advertising spending x (in hundreds of dollars) to weekly sales y (in units), with $r = 0.86$.

Predict sales when the shop spends \$600 on ads, so $x = 6$:

$$y = 2.5(6) + 40 = 15 + 40 = 55 \text{ units}.$$

Interpret the slope 2.5**:** each extra \$100 of advertising (a 1-unit rise in x) is associated with about 2.5 more units sold per week.

Interpret the intercept 40**:** with no advertising ($x = 0$), the model predicts about 40 units sold — a baseline level of sales.

Judge the fit: $r = 0.86$ means $|r| \geq 0.8$, a strong positive linear relationship, so this prediction is trustworthy *within* the data range. Predicting sales for \$10,000 of ads would be risky extrapolation.

Residuals: how wrong is a prediction?

A **residual** measures the error at a single data point:

$$\text{residual} = \text{actual } y - \text{predicted } y.$$

- A **positive** residual means the actual value is *above* the line (the model under-predicted).
- A **negative** residual means the actual value is *below* the line (the model over-predicted).
- A residual of 0 means the point lands exactly on the line.

The line of best fit is precisely the line that makes the sum of the *squared* residuals as small as possible — which is why it is also called the **least-squares** line.

Residual plots: Plot each residual against its x-value. If the residuals scatter randomly above and below zero with no pattern, a line is a *good* model. If they curve or fan out, the data is not really linear and a line is the wrong tool. *Example:* for the plant line $y = 3x + 8$, if a plant was actually 25 cm at week 5 (predicted 23), its residual is $25 - 23 = +2$ cm.

Weighted averages

An ordinary mean treats every value equally. A **weighted average** lets some values count more than others by assigning each a **weight**:

$$\text{weighted average} = \frac{\sum(\text{value} \times \text{weight})}{\sum \text{weights}}.$$

Weights often represent importance, frequency, or size — like the number of credit hours behind each course grade, or the percentage each category contributes to a final grade.

Worked example: a weighted grade

A class grade is 20% homework, 30% quizzes, and 50% the final exam. A student earns 95 on homework, 80 on quizzes, and 70 on the final. Find the course grade.

Using the weights as decimals (0.20, 0.30, 0.50, which sum to 1):

$$\text{grade} = \frac{95(0.20) + 80(0.30) + 70(0.50)}{0.20 + 0.30 + 0.50}.$$

$$\begin{aligned}\text{numerator} &= 19 + 24 + 35 = 78,\\ \text{denominator} &= 1,\\ \text{grade} &= \frac{78}{1} = 78.\end{aligned}$$

The weighted grade is 78. Notice this is *below* the plain mean of the three scores ($\frac{95+80+70}{3} \approx 81.7$) because the low final exam carries the heaviest weight.

Sanity check: A weighted average always lands between the smallest and largest of the values (70 and 95 above). If your answer falls outside that range, you multiplied or divided by the wrong number. When the weights already add to 1 (or 100%), the denominator is just 1 and you can skip dividing.

4 Foundations

Everything you need before Algebra 1

4.1 Types of Real Numbers

The Number Families

Every number you will use in Algebra 1 is a **real number**. Real numbers sort into nested families:

- **Natural (counting) numbers:** $1, 2, 3, 4, \ldots$
- **Whole numbers:** the naturals plus zero: $0, 1, 2, 3, \ldots$
- **Integers:** whole numbers and their negatives: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
- **Rational numbers:** any number that can be written as a fraction $\frac{a}{b}$ where a and b are integers and $b \neq 0$. Their decimals either *end* or *repeat*.
- **Irrational numbers:** real numbers that *cannot* be written as a fraction. Their decimals go on forever with no repeating pattern.

Each family sits inside the next: every natural number is whole, every whole number is an integer, every integer is rational.

Sorting numbers into families

Classify each number in *every* family it belongs to: $7,\ -4,\ 0,\ \frac{3}{5},\ 0.\overline{6},\ \sqrt{2},\ \pi$.

- 7: natural, whole, integer, rational
- -4: integer, rational
- 0: whole, integer, rational
- $\frac{3}{5}$: rational (it is a fraction)
- $0.\overline{6} = 0.6666\cdots = \frac{2}{3}$: rational (repeating decimal)
- $\sqrt{2} = 1.41421356\ldots$: irrational (never ends or repeats)
- $\pi = 3.14159\ldots$: irrational

$\sqrt{2}$ and π are irrational, but $\sqrt{9} = 3$ is a perfectly ordinary integer. A square root is only irrational when the number underneath is *not* a perfect square.

4.2 The Number Line & Ordering Numbers

Reading the Number Line

The number line places every real number in order from small (left) to large (right). For any two numbers, the one **farther right is greater**.

- $a < b$ means a is to the left of b.
- $a > b$ means a is to the right of b.
- $a \leq b$ means a is less than *or equal to* b; likewise $a \geq b$.

Negatives flip your intuition: the more negative a number looks, the smaller it is. So $-8 < -3$, because -8 sits farther left.

Ordering from least to greatest

Order $-2,\ 3,\ -\frac{1}{2},\ 0,\ -4,\ 1.5$.

$$\text{Place each on the line:} \quad -4,\ -2,\ -\tfrac{1}{2},\ 0,\ 1.5,\ 3$$

Least to greatest: $-4 < -2 < -\frac{1}{2} < 0 < 1.5 < 3$.

When comparing fractions and decimals, rewrite them in the same form. To compare $\frac{3}{4}$ and 0.7, use $\frac{3}{4} = 0.75$, so $\frac{3}{4} > 0.7$.

4.3 Absolute Value

Distance from Zero

The **absolute value** of a number is its distance from 0 on the number line. Distance is never negative, so absolute value is never negative.

$$|5| = 5, \qquad |-5| = 5, \qquad |0| = 0.$$

A number and its opposite always have the same absolute value.

Absolute value inside expressions

Evaluate $|-7| + |3| - |-2|$.

$$\begin{aligned} |-7| + |3| - |-2| &= 7 + 3 - 2 \\ &= 8. \end{aligned}$$

Treat each absolute value like grouping: simplify inside first, then use the result.

Absolute value bars are *not* the same as parentheses. $-|6| = -6$, because you take $|6| = 6$ first and *then* apply the negative sign out front.

4.4 Operations with Integers (Sign Rules)

Adding & Subtracting Signed Numbers

Adding:

- Same signs: add the values, keep the sign. $(-4) + (-6) = -10$.
- Different signs: subtract the smaller value from the larger, keep the sign of the larger. $(-9) + 4 = -5$.

Subtracting: "add the opposite." Change subtraction to addition and flip the sign of the second number:

$$a - b = a + (-b), \qquad 5 - (-3) = 5 + 3 = 8.$$

Multiplying & Dividing Signed Numbers

The rule is the same for both operations:

- Same signs → **positive** answer.
- Different signs → **negative** answer.

$$(-6) \times (-3) = 18, \qquad (-6) \times 3 = -18, \qquad \frac{-20}{4} = -5, \qquad \frac{-20}{-5} = 4.$$

Mixed integer practice

Evaluate $-8 + 5 - (-3)$ and $(-4)(-2)(-1)$.

$$-8 + 5 - (-3) = -8 + 5 + 3 = 0.$$

$$(-4)(-2)(-1) = (8)(-1) = -8.$$

An *odd* number of negative factors gives a negative product; an *even* number gives a positive product.

The trickiest spot is subtracting a negative: $7 - (-2)$ becomes $7 + 2 = 9$. "Minus a minus is a plus."

4.5 Fractions

Simplifying Fractions

A fraction is in **lowest terms** when the numerator and denominator share no common factor except 1. Divide both by their greatest common factor (GCF).

$$\frac{12}{18} = \frac{12 \div 6}{18 \div 6} = \frac{2}{3}.$$

The Four Operations

- **Multiply:** multiply straight across, then simplify. $\dfrac{2}{3} \times \dfrac{4}{5} = \dfrac{8}{15}$.
- **Divide:** multiply by the reciprocal (flip the second fraction). $\dfrac{2}{3} \div \dfrac{4}{5} = \dfrac{2}{3} \times \dfrac{5}{4} = \dfrac{10}{12} = \dfrac{5}{6}$.
- **Add / Subtract:** rewrite with a common denominator first, then add or subtract the numerators.

$$\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{11}{12}.$$

Fractions ↔ Decimals

- **Fraction to decimal:** divide the numerator by the denominator. $\frac{3}{8} = 3 \div 8 = 0.375$.
- **Decimal to fraction:** write the decimal over its place value, then simplify. $0.35 = \frac{35}{100} = \frac{7}{20}$.

A full fraction subtraction

Compute $\dfrac{5}{6} - \dfrac{3}{4}$.

$$\begin{aligned} \frac{5}{6} - \frac{3}{4} &= \frac{10}{12} - \frac{9}{12} && \text{common denominator 12} \\ &= \frac{1}{12}. \end{aligned}$$

Only add or subtract fractions after finding a common denominator, but for multiplying and dividing you do *not* need one. Mixing these up is the most common fraction mistake.

4.6 Order of Operations (PEMDAS)

The Order

Simplify in this order:

1. **P** — Parentheses / grouping (including absolute value and fraction bars)
2. **E** — Exponents
3. **MD** — Multiplication and Division, left to right
4. **AS** — Addition and Subtraction, left to right

Multiplication and division rank *equally*; do whichever comes first reading left to right. Same for addition and subtraction.

Working through PEMDAS

Evaluate $3 + 2 \times (5-1)^2 \div 8$.

$$\begin{aligned} 3 + 2 \times (5-1)^2 \div 8 &= 3 + 2 \times (4)^2 \div 8 && \text{parentheses} \\ &= 3 + 2 \times 16 \div 8 && \text{exponent} \\ &= 3 + 32 \div 8 && \text{multiply (left first)} \\ &= 3 + 4 && \text{divide} \\ &= 7. && \text{add} \end{aligned}$$

$2 \times 16 \div 8$ is done left to right: $32 \div 8 = 4$. Do *not* do all multiplication before all division — they share a step.

4.7 Properties of Operations

The Named Properties

- **Commutative** (order): $a + b = b + a$ and $a \cdot b = b \cdot a$.
- **Associative** (grouping): $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
- **Distributive** (spread over a sum): $a(b + c) = ab + ac$.
- **Identity:** adding 0 or multiplying by 1 changes nothing: $a + 0 = a$, $a \cdot 1 = a$.

Using the distributive property

Expand $4(x + 3)$ and simplify $-2(3y - 5)$.

$$4(x+3) = 4 \cdot x + 4 \cdot 3 = 4x + 12.$$

$$-2(3y - 5) = (-2)(3y) + (-2)(-5) = -6y + 10.$$

Subtraction and division are *not* commutative: $7 - 3 \neq 3 - 7$. When you distribute a negative, every term inside changes sign.

4.8 Evaluating Expressions by Substitution

Substitute, then Simplify

To **evaluate** an algebraic expression, replace each variable with its given value (use parentheses!) and then follow PEMDAS.

Plugging in values

Evaluate $3x^2 - 2y$ when $x = -2$ and $y = 5$.

$$\begin{aligned} 3x^2 - 2y &= 3(-2)^2 - 2(5) \\ &= 3(4) - 10 \\ &= 12 - 10 = 2. \end{aligned}$$

Wrap every substituted value in parentheses. Without them, $(-2)^2 = 4$ can be mistaken for $-2^2 = -4$.

4.9 Combining Like Terms

What Makes Terms "Like"

Like terms have exactly the same variable part (same letters with the same exponents). You may add or subtract like terms by combining their coefficients; the variable part stays the same.

$$5x + 3x = 8x, \qquad 7a - 2a = 5a.$$

Constants (plain numbers) are like terms with each other.

Simplifying an expression

Simplify $6x + 4 - 2x + 9 - x$.

$$\begin{aligned} 6x + 4 - 2x + 9 - x &= (6x - 2x - x) + (4 + 9) \\ &= 3x + 13. \end{aligned}$$

x^2 and x are *not* like terms, so $x^2 + x$ cannot be combined. Keep the exponents in mind before merging.

4.10 Going Deeper: Advanced Foundations

Closure & the Field Properties of the Reals

A set of numbers is **closed** under an operation if combining any two members always lands you back *inside* that set.

- The integers are closed under $+$, $-$, and $\times$, but *not* under $\div$: $1 \div 2 = \frac{1}{2}$ escapes the integers.

- The real numbers are closed under $+$, $-$, $\times$, and $\div$ (dividing by 0 excepted).

Because the reals also obey commutativity, associativity, and distributivity, carry additive and multiplicative identities (0 and 1), and give every number an **additive inverse** $-a$ and every nonzero number a **multiplicative inverse** $\frac{1}{a}$, mathematicians call $\mathbb{R}$ a **field**. These field properties are the rules that justify every algebra step you take.

Closure is about *staying inside a set*, not about getting a "nice" answer. The naturals are not closed under subtraction because $3 - 5 = -2$ is not natural, even though -2 is a fine number.

Absolute Value as Distance: $|a - b|$

Beyond "distance from 0," absolute value measures the distance between *any* two numbers:

$$|a - b| = \text{the distance between } a \text{ and } b \text{ on the number line.}$$

Order does not matter, because $|a - b| = |b - a|$. For example, the gap between -3 and 4 is

$$|-3-4| = |-7| = 7 \qquad \text{and equally} \qquad |4 - (-3)| = |7| = 7.$$

This is why $|x - 5| = 2$ describes every point exactly 2 units from 5, namely $x = 3$ and $x = 7$.

Nested absolute value

Evaluate $\big|\,3 - |-8|\,\big| + \big|\,2 \cdot |-4| - 11\,\big|$.

$$\begin{aligned}
\big|\,3 - |-8|\,\big| + \big|\,2 \cdot |-4| - 11\,\big| &= \big|\,3 - 8\,\big| + \big|\,2 \cdot 4 - 11\,\big| && \text{innermost bars first} \\
&= |-5| + |\,8 - 11\,| && \text{simplify inside} \\
&= |-5| + |-3| \\
&= 5 + 3 = 8.
\end{aligned}$$

Work an absolute value from the *inside out*, exactly like nested parentheses.

The Real-Number Hierarchy $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

The number families nest as a chain of subsets, each written with its own symbol:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R},$$

that is, naturals $\subset$ integers $\subset$ rationals $\subset$ reals. The irrationals are the reals that are left over when you remove the rationals — they fill every gap the fractions miss, which is why the number line has *no holes*.

Why $\sqrt{2}$ is irrational (a proof sketch)

Suppose, for contradiction, that $\sqrt{2}$ *were* rational. Then we could write it in lowest terms:

$$\sqrt{2} = \frac{a}{b}, \qquad a, b \text{ integers with no common factor, } b \neq 0.$$

Squaring both sides and clearing the denominator:

$$2 = \frac{a^2}{b^2}$$
$$a^2 = 2b^2.$$

So a^2 is even, which forces a itself to be even; write $a = 2k$. Substituting back:

$$(2k)^2 = 2b^2 \implies 4k^2 = 2b^2 \implies b^2 = 2k^2,$$

so b is even too. But then a and b share the factor 2, contradicting "lowest terms." The assumption must be false, so $\sqrt{2}$ cannot be written as a fraction — it is irrational.

Efficient Order of Operations with Nested Grouping

When grouping symbols are nested — parentheses inside brackets inside braces, or a fraction bar acting as grouping above and below — always resolve the **innermost** group first and work outward. A fraction bar groups its *entire* numerator and *entire* denominator as if each were parenthesized.

$$\frac{2+3\cdot 4}{7-3} \text{ means } \frac{(2+3\cdot 4)}{(7-3)} = \frac{14}{4} = \frac{7}{2}.$$

Peeling nested brackets

Evaluate $20 - \left[3 + 2\,(6-4)^2\right]$.

$$\begin{aligned} 20 - \left[3 + 2(6-4)^2\right] &= 20 - \left[3 + 2(2)^2\right] && \text{innermost parentheses} \\ &= 20 - \left[3 + 2\cdot 4\right] && \text{exponent inside the bracket} \\ &= 20 - \left[3 + 8\right] && \text{multiply} \\ &= 20 - 11 = 9. \end{aligned}$$

A Preview of Summation (Σ) Notation

The Greek capital sigma Σ is shorthand for "add up a list." The expression

$$\sum_{i=1}^{n} a_i \;=\; a_1 + a_2 + a_3 + \cdots + a_n$$

reads: "start the counter i at 1, plug it into the rule after the Σ, and add the results up to

$i = n$." The letter i is just an **index**; it disappears once the sum is computed. For instance,

$$\sum_{k=1}^{4}(2k-1) = (2\cdot 1-1)+(2\cdot 2-1)+(2\cdot 3-1)+(2\cdot 4-1) = 1+3+5+7 = 16.$$

You will meet Σ again with sequences and series — for now, just read it as a compact "add these terms."

Evaluating a tricky multi-variable expression

Evaluate $\dfrac{a^2 - b\,c}{|\,b-a\,|}$ when $a=-3$, $b=4$, and $c=-2$.

$$\begin{aligned}
\frac{a^2-bc}{|\,b-a\,|} &= \frac{(-3)^2-(4)(-2)}{|\,4-(-3)\,|} && \text{substitute, each value in parentheses}\\
&= \frac{9-(-8)}{|\,7\,|} && \text{exponent and product on top}\\
&= \frac{9+8}{7} && \text{subtracting a negative}\\
&= \frac{17}{7}.
\end{aligned}$$

Simplify the numerator and denominator *separately* (the fraction bar groups each), then divide.

With several variables, substitute *every* occurrence at once and wrap each value in parentheses before simplifying. A single missed sign — like reading $-b\,c$ as positive — derails the whole evaluation.

5 Equations & Inequalities

Everything you need to solve, rearrange, and graph.

5.1 What Is an Equation?

The Big Idea: Balance

An **equation** says that two things are equal, joined by an equals sign, like $x+3=7$. Think of the $=$ sign as the middle of a balance scale. Whatever you do to one side you **must** do to the other, or the scale tips and the equation is no longer true.

To *solve* an equation means to find the value of the variable that makes it true. We do this by using **inverse operations** (opposite operations) to undo what has been done to the variable, one step at a time, until the variable is alone.

Inverse pairs: addition $\leftrightarrow$ subtraction, and multiplication $\leftrightarrow$ division. To undo $+5$, subtract 5. To undo $\times 4$, divide by 4.

Check by substituting

Is $x = 4$ the solution of $x + 3 = 7$? Replace x with 4: $4 + 3 = 7$ ✓. Yes! Always check your answer by plugging it back in.

5.2 One-Step Equations

One inverse operation

The variable has *one* operation attached to it. Undo that operation on both sides.

Addition and subtraction

Solve $x - 8 = 5$.

$$\begin{aligned} x - 8 &= 5 \\ x - 8 + 8 &= 5 + 8 && \text{(add 8 to both sides)} \\ x &= 13 \end{aligned}$$

Solve $x + 6 = 10$: subtract 6 from both sides to get $x = 4$.

Multiplication and division

Solve $\dfrac{x}{3} = 6$.

$$\frac{x}{3} \cdot 3 = 6 \cdot 3 \quad \Rightarrow \quad x = 18.$$

Solve $5x = 35$: divide both sides by 5 to get $x = 7$.

5.3 Two-Step Equations

Undo in reverse order

When two operations act on the variable, undo them in the reverse of the order of operations: first undo addition/subtraction, then undo multiplication/division.

A two-step solve

Solve $2x + 7 = 19$.

$$\begin{aligned} 2x + 7 &= 19 && \\ 2x &= 12 && \text{(subtract 7)} \\ x &= 6 && \text{(divide by 2)} \end{aligned}$$

Check: $2(6) + 7 = 12 + 7 = 19$ ✓

5.4 Multi-Step Equations

Simplify first

Before you isolate the variable, clean up each side: use the **distributive property** to remove parentheses, then **combine like terms**. Then solve as usual.

Distributing a negative: $-3(x - 4) = -3x + 12$. The minus sign changes the sign of *every* term inside the parentheses. This is a very common place to make a mistake, so slow down here.

Distribute, then combine

Solve $3(x + 2) + 4x = 27$.

$$\begin{aligned} 3(x + 2) + 4x &= 27 && \\ 3x + 6 + 4x &= 27 && \text{(distribute)} \\ 7x + 6 &= 27 && \text{(combine like terms)} \\ 7x &= 21 && \text{(subtract 6)} \\ x &= 3 && \text{(divide by 7)} \end{aligned}$$

5.5 Variables on Both Sides

Gather the variables

Move all variable terms to one side and all number terms to the other by adding or subtracting. A good habit: move the *smaller* variable term so the coefficient stays positive.

Both sides

Solve $5x - 3 = 2x + 9$.

$$\begin{aligned} 5x - 3 &= 2x + 9 \\ 3x - 3 &= 9 && \text{(subtract } 2x\text{)} \\ 3x &= 12 && \text{(add 3)} \\ x &= 4 && \text{(divide by 3)} \end{aligned}$$

5.6 Equations with Fractions & Decimals

Clear the denominators

Multiply *every* term on both sides by the **least common denominator (LCD)**. This turns a fraction equation into a whole-number equation. For decimals, multiply every term by 10, 100, . . . to clear the decimal points.

Clearing fractions

Solve $\frac{x}{2} + \frac{1}{3} = \frac{5}{6}$. The LCD of $2, 3, 6$ is 6.

$$\begin{aligned} 6 \cdot \frac{x}{2} + 6 \cdot \frac{1}{3} &= 6 \cdot \frac{5}{6} \\ 3x + 2 &= 5 \\ 3x &= 3 \quad \Rightarrow \quad x = 1 \end{aligned}$$

Clearing decimals

Solve $0.2x + 0.5 = 1.1$. Multiply every term by 10:

$$2x + 5 = 11 \quad \Rightarrow \quad 2x = 6 \quad \Rightarrow \quad x = 3.$$

5.7 Literal Equations (Solving for a Variable)

Rearranging formulas

A **literal equation** has several letters. To "solve for" one letter means to get that letter alone, treating all the other letters as if they were numbers. Use the same inverse operations as always.

Solve a formula for one letter

Solve $A = \ell w$ for w.

$$A = \ell w \quad \Rightarrow \quad \frac{A}{\ell} = w \quad \text{(divide both sides by } \ell\text{)}.$$

Solve $P = 2\ell + 2w$ for w:

$$P - 2\ell = 2w \quad \Rightarrow \quad w = \frac{P - 2\ell}{2}.$$

5.8 Special Cases: No Solution or Infinitely Many

When the variable disappears

Sometimes, while solving, all the variable terms cancel out. Look at what is left:

- A **false** statement (like $3 = 7$) means **no solution**. No value of x works.
- A **true** statement (like $5 = 5$) means **infinitely many solutions**. Every value of x works; the two sides are really the same expression.

Both special cases

Solve $2x + 4 = 2x - 1$: subtract $2x$ to get $4 = -1$, which is false. **No solution.**

Solve $3(x + 2) = 3x + 6$: distribute to get $3x + 6 = 3x + 6$; subtract $3x$ to get $6 = 6$, always true. **Infinitely many solutions.**

5.9 Solving Inequalities

Almost like equations

Inequalities use $<, >, \leq, \geq$ instead of $=$. Solve them the same way you solve equations, with **one special rule**.

The Flip Rule: When you multiply or divide both sides of an inequality by a *negative* number, you **must flip** the inequality sign. For example, $-2x < 6$ becomes $x > -3$ (divide by -2 and flip $<$ to $>$).

An inequality with a flip

Solve $-3x+1 \geq 10$.

$$\begin{aligned} -3x+1 &\geq 10 \\ -3x &\geq 9 && \text{(subtract 1)} \\ x &\leq -3 && \text{(divide by } -3\text{, flip } \geq \text{ to } \leq\text{)} \end{aligned}$$

5.10 Graphing on a Number Line

Circles and arrows

Graph the solution of a one-variable inequality on a number line:

- Use an **open circle** $\circ$ for $<$ or $>$ (the endpoint is *not* included).
- Use a **closed (filled) circle** $\bullet$ for $\leq$ or $\geq$ (the endpoint *is* included).
- Shade an arrow to the **right** for "greater than" and to the **left** for "less than."

An open circle means the endpoint is not included.

(diagram in the online version)

Describing a graph

The graph of $x \geq -3$ has a **closed** circle at -3 and an arrow shaded to the **right**. The graph of $x < 2$ has an **open** circle at 2 and an arrow shaded to the **left**.

5.11 Compound Inequalities (And / Or)

Two conditions at once

A **compound inequality** joins two inequalities.

- **And** (written $a < x < b$): x must satisfy *both*. The graph is the overlap between the two endpoints.
- **Or**: x satisfies *either* one. The graph shades outward in two directions.

For an "and" inequality, do the same operation to all three parts.

An "and" inequality

Solve $-4 < 2x \leq 6$.

$$\frac{-4}{2} < \frac{2x}{2} \leq \frac{6}{2} \quad \Rightarrow \quad -2 < x \leq 3.$$

This means x is between -2 (open) and 3 (closed).

5.12 Absolute Value Equations & Inequalities

Distance from zero

$|x|$ means the distance of x from 0, which is never negative. So $|x| = 5$ has *two* answers: $x = 5$ or $x = -5$. Get the absolute value alone first, then split into two cases.

Two patterns for inequalities:

- "**Less** than" ($|x| < a$) becomes an **and**: $-a < x < a$.
- "**Great**er than" ($|x| > a$) becomes an **or**: $x < -a$ *or* $x > a$.

If the absolute value equals a *negative* number, there is **no solution** (distance can't be negative).

Absolute value equation

Solve $|x - 2| = 5$. Split into two cases:

$$x - 2 = 5 \;\Rightarrow\; x = 7 \qquad \text{or} \qquad x - 2 = -5 \;\Rightarrow\; x = -3.$$

Solutions: $x = 7$ or $x = -3$.

Absolute value inequality

Solve $|x + 1| \leq 4$. This is a "less than," so use **and**:

$$-4 \leq x + 1 \leq 4 \quad \Rightarrow \quad -5 \leq x \leq 3.$$

5.13 Going Deeper: Advanced Equations & Inequalities

Parameter equations: when does a solution exist?

A **parameter** is a letter that stands for a fixed but unknown number (often a, b, k, m). When a linear equation contains a parameter, the *number* of solutions can change depending on the parameter's value. Every linear equation can be rearranged into the form

$$(\,\text{coefficient}\,) \cdot x = (\,\text{constant}\,), \qquad \text{that is} \qquad Ax = B.$$

Everything then depends on whether the coefficient A is zero:

- If $A \neq 0$: exactly **one solution**, $x = \dfrac{B}{A}$.
- If $A = 0$ and $B = 0$ (you get $0 = 0$): **infinitely many** solutions.
- If $A = 0$ and $B \neq 0$ (you get $0 = B$, a contradiction): **no solution**.

The whole strategy is: collect all x-terms on one side, factor x out, then study its coefficient.

The key move: write the equation as $Ax = B$ where A and B are expressions in the parameter. The "interesting" values of the parameter are exactly those that make the coefficient $A = 0$. Test each of them separately.

All three cases at once (worked example)

For each value of k, describe the number of solutions of

$$k^2x + 5 = 9x + k + 2.$$

Gather the x-terms on the left and the constants on the right:

$$\begin{aligned} k^2x + 5 &= 9x + k + 2 \\ k^2x - 9x &= k + 2 - 5 && \text{(subtract } 9x \text{ and } 5\text{)} \\ (k^2 - 9)\,x &= k - 3 && \text{(factor } x \text{ out)} \\ (k-3)(k+3)\,x &= k - 3 && \text{(difference of squares)} \end{aligned}$$

Now the coefficient is $A = (k-3)(k+3)$, zero only when $k = 3$ or $k = -3$.

- $k = 3$: the equation becomes $0 \cdot x = 0$, true for every x $\Rightarrow$ **infinitely many solutions**.
- $k = -3$: the equation becomes $0 \cdot x = -6$, impossible $\Rightarrow$ **no solution**.
- any other k: **one solution**, $x = \dfrac{k-3}{(k-3)(k+3)} = \dfrac{1}{k+3}$.

Nested absolute values: peel from the outside in

An expression like $\big||x-3| - 2\big|$ has one absolute value *inside* another. Treat the whole inner absolute value as a single block and undo the **outer** bars first, exactly as you would solve $|\,\text{block}\,| = c$. Each branch then leaves you with an ordinary absolute-value equation to finish. Always discard any branch that asks an absolute value to equal a *negative* number.

A nested absolute-value equation (worked example)

Solve $\big||x-3|-2\big| = 4$. Let the block be $B = |x-3|$. The outer bars give two branches:

$$|x-3|-2 = 4 \qquad \text{or} \qquad |x-3|-2 = -4.$$

That is $|x-3| = 6$ or $|x-3| = -2$. The second is impossible (an absolute value is never negative), so discard it. From $|x-3| = 6$:

$$x-3 = 6 \;\Rightarrow\; x = 9 \qquad \text{or} \qquad x-3 = -6 \;\Rightarrow\; x = -3.$$

Solutions: $x = 9$ or $x = -3$. (Check: $\big||9-3|-2\big| = |6-2| = 4$ ✓.)

Sums of absolute values as distances

Read $|x-a|$ as "the distance between x and a on the number line." Then $|x-a| + |x-b|$ is the **total distance** from x to the two fixed points a and b. If a and b are d units apart, that total is *smallest* exactly when x sits between them, where it equals d and can go no lower. So:

- $|x-a| + |x-b| = c$ has **no solution** if $c < d$, and infinitely many (a whole segment) if $c = d$.
- If $c > d$, there are exactly **two** solutions, one on each side of the pair.

To solve algebraically, split the line at the critical points $x = a$ and $x = b$ and handle each region with casework.

Summed absolute values with casework (worked example)

Solve $|x-1| + |x+2| = 5$. The critical points are $x = 1$ and $x = -2$, which are $d = 3$ apart. Since $5 > 3$, expect two solutions. Break the line into three regions.

$$\begin{aligned}
x < -2: &\quad -(x-1)-(x+2) = 5 \;\Rightarrow\; -2x-1 = 5 \;\Rightarrow\; x = -3 \checkmark \\
-2 \le x \le 1: &\quad -(x-1)+(x+2) = 5 \;\Rightarrow\; 3 = 5 \text{ (false, no solution here)} \\
x > 1: &\quad (x-1)+(x+2) = 5 \;\Rightarrow\; 2x+1 = 5 \;\Rightarrow\; x = 2 \checkmark
\end{aligned}$$

The middle region gives the constant 3 (the minimum total distance), confirming the sum can never dip below 3. Solutions: $x = -3$ or $x = 2$.

Literal equations that need factoring

When you solve a formula for a letter that appears in *more than one term*, you cannot simply divide — you must first **collect** every term containing that letter on one side, then **factor**

the letter out, and only then divide by the leftover expression. This is the same "$Ax = B$" idea used for parameters, applied to rearranging formulas.

Factoring to isolate a variable

Solve $xy + 3x = 5$ for x. Both terms on the left contain x, so factor it out:

$$x(y+3) = 5 \quad \Rightarrow \quad x = \frac{5}{y+3} \quad (y \neq -3).$$

Similarly, solve $ax + b = cx + d$ for x:

$$ax - cx = d - b \ \Rightarrow \ x(a-c) = d - b \ \Rightarrow \ x = \frac{d-b}{a-c} \quad (a \neq c).$$

Counting integer solutions of an inequality

Some problems ask not for the solution set but for **how many integers** lie in it. First solve the inequality to get a range, then count the whole numbers inside, watching the endpoints:

- A *strict* endpoint ($<$ or $>$) does **not** include that boundary value.
- A *non-strict* endpoint ($\leq$ or $\geq$) **does** include it.

For a run of consecutive integers from m to n inclusive, the count is $n - m + 1$.

How many integer solutions?

How many integers satisfy $|2x - 1| < 7$? Rewrite the "less than" absolute value as an "and":

$$-7 < 2x - 1 < 7 \ \Rightarrow \ -6 < 2x < 8 \ \Rightarrow \ -3 < x < 4.$$

Both ends are strict, so -3 and 4 are excluded. The integers are $-2, -1, 0, 1, 2, 3$ — that is **6** integers.

Modeling a harder word problem (worked example)

Plan A costs \$30 per month plus \$0.10 for each text message. Plan B costs a flat \$50 per month. For how many text messages is Plan A the cheaper choice?

Let m be the number of texts. "Plan A cheaper" means its cost is *less than* Plan B's:

$$\begin{aligned} 30 + 0.10m &< 50 \\ 0.10m &< 20 && \text{(subtract 30)} \\ m &< 200 && \text{(divide by 0.10)} \end{aligned}$$

Since m counts texts, it must be a whole number, so Plan A is cheaper for $0 \leq m \leq 199$ texts (fewer than 200). At exactly 200 texts the two plans tie at \$50, and beyond that Plan B wins.

6 Linear Equations & Functions

Everything you need: the coordinate plane, functions, slope, and the equations of lines

6.1 The Coordinate Plane

Axes, Origin, and Ordered Pairs

The coordinate plane is formed by two number lines that cross at the **origin** $(0, 0)$.

- The horizontal line is the x-**axis**; the vertical line is the y-**axis**.
- Every point is named by an **ordered pair** (x, y). The first number x tells how far *right* $(+)$ or *left* $(-)$ you go; the second number y tells how far *up* $(+)$ or *down* $(-)$.
- The axes split the plane into four **quadrants**, numbered counterclockwise:

$$\text{Q I } (+,+), \quad \text{Q II } (-,+), \quad \text{Q III } (-,-), \quad \text{Q IV } (+,-).$$

Plotting and naming points

Plot $A(3, 2)$: from the origin go right 3, then up 2. This lands in Quadrant I.

Plot $B(-4, 1)$: go left 4, then up 1. This is in Quadrant II.

Name the point that is left 2 and down 5: that is $(-2, -5)$, which is in Quadrant III.

Order matters! $(3, 2)$ and $(2, 3)$ are different points. Always read (x, y): run first (left/right), then rise (up/down). Points on an axis (like $(0, 4)$ or $(-3, 0)$) belong to *no* quadrant.

6.2 Functions and Function Notation

What Is a Function?

A **function** is a rule that assigns to each input *exactly one* output. Think of it as a machine: put in x, get out one value.

We write $f(x)$ (read "f of x") for the output when the input is x. Here x is the input and $f(x)$ is the output. The letters f, g, h are just names for the rule.

Evaluating a function

Let $f(x) = 2x + 5$. To **evaluate**, substitute the input in place of x.

$$f(3) = 2(3) + 5 = 6 + 5 = 11,$$
$$f(0) = 2(0) + 5 = 0 + 5 = 5,$$
$$f(-4) = 2(-4) + 5 = -8 + 5 = -3.$$

So the ordered pairs $(3, 11)$, $(0, 5)$, and $(-4, -3)$ all lie on the graph of f.

Vertical Line Test: a graph is a function if no vertical line crosses it more than once. This checks the rule "one input, one output."

6.3 Domain and Range

Inputs and Outputs

The **domain** is the set of all allowed inputs (the x-values). The **range** is the set of all resulting outputs (the y-values).

Domain and range from a set of points

For the function $\{(1, 4), (2, 7), (5, 7), (6, -3)\}$:

- Domain (inputs): $\{1, 2, 5, 6\}$.
- Range (outputs): $\{4, 7, -3\}$. We list 7 only once even though it appears twice.

This is a function because each input has exactly one output.

For a line like $y = 2x + 1$ with no restrictions, the domain is *all real numbers* and so is the range. A *horizontal* line $y = 3$ has range $\{3\}$ only.

6.4 Rate of Change and Slope

Slope = Rise over Run

Slope m measures steepness: how much y changes for each unit that x increases. Given two points (x_1, y_1) and (x_2, y_2),

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Slope is the **rate of change** of the line.

Slope is rise over run.

(diagram in the online version)

Slope from two points

Find the slope through $(1, 2)$ and $(4, 11)$.

$$m = \frac{11 - 2}{4 - 1} = \frac{9}{3} = 3.$$

Each time x goes up by 1, y goes up by 3.

A negative slope

Find the slope through $(-2, 5)$ and $(2, -3)$.

$$m = \frac{-3 - 5}{2 - (-2)} = \frac{-8}{4} = -2.$$

The line falls: as x increases by 1, y decreases by 2.

Reading the sign of slope:

- **Positive** slope $\rightarrow$ line goes *up* left-to-right.
- **Negative** slope $\rightarrow$ line goes *down* left-to-right.
- **Zero** slope $\rightarrow$ *horizontal* line (no rise): $m = 0$.
- **Undefined** slope $\rightarrow$ *vertical* line (run is 0, and we can't divide by 0).

Subtract the y's and the x's *in the same order*. Keep the points in the same order top and bottom!

6.5 Slope-Intercept Form: $y = mx + b$

Read the Slope and Intercept Directly

In **slope-intercept form**

$$y = mx + b,$$

the number m is the **slope** and b is the **y-intercept** (the y-value where the line crosses the y-axis, at the point $(0, b)$).

Slope is rise over run.

(diagram in the online version)

Identify and graph

For $y = \frac{3}{4}x - 2$: the slope is $m = \frac{3}{4}$ and the y-intercept is $b = -2$, so the line passes through $(0, -2)$.

To graph: start at $(0, -2)$. Use the slope $\frac{3}{4}$ (rise 3, run 4): move right 4 and up 3 to reach $(4, 1)$. Connect the points.

Finding b when you know slope and a point

A line has slope $m = 5$ and passes through $(2, 3)$. Substitute into $y = mx + b$:

$$3 = 5(2) + b \;\Rightarrow\; 3 = 10 + b \;\Rightarrow\; b = -7.$$

The equation is $y = 5x - 7$.

A whole-number slope like 4 is really $\frac{4}{1}$: rise 4, run 1. To go the other direction on a line, you may move *left* and *down* instead.

6.6 Point-Slope Form: $y - y_1 = m(x - x_1)$

Build a Line from a Slope and Any Point

If a line has slope m and passes through (x_1, y_1), its equation is

$$y - y_1 = m(x - x_1).$$

This is the fastest way to write a line when you know a point and the slope.

Slope is rise over run.

(diagram in the online version)

Point-slope, then simplify

Slope $m = 2$ through $(3, 4)$:

$$y - 4 = 2(x - 3).$$

Distribute and solve for y to get slope-intercept form:

$$y - 4 = 2x - 6 \;\Rightarrow\; y = 2x - 2.$$

Watch the signs: with a point like $(-1, 5)$, the form is $y - 5 = m(x - (-1)) = m(x + 1)$. Subtracting a negative becomes adding.

6.7 Standard Form: $Ax + By = C$

Standard Form and Intercepts

Standard form is $Ax+By = C$, where A, B, C are integers. It is especially handy for finding both intercepts:

- **x-intercept:** set $y = 0$ and solve for x. Point $(x, 0)$.
- **y-intercept:** set $x = 0$ and solve for y. Point $(0, y)$.

Slope is rise over run.

(diagram in the online version)

Finding both intercepts

For $3x + 2y = 12$:

$$\begin{aligned}&x\text{-intercept: } 3x + 2(0) = 12 \Rightarrow 3x = 12 \Rightarrow x = 4, \text{ point } (4, 0).\\ &y\text{-intercept: } 3(0) + 2y = 12 \Rightarrow 2y = 12 \Rightarrow y = 6, \text{ point } (0, 6).\end{aligned}$$

Plot $(4, 0)$ and $(0, 6)$ and draw the line through them.

To convert $y = mx + b$ to standard form, move the x-term to the left and clear fractions. Example: $y = \frac{2}{3}x + 1 \Rightarrow 3y = 2x + 3 \Rightarrow -2x + 3y = 3 \Rightarrow 2x - 3y = -3$.

6.8 Writing the Equation of a Line

Three Common Situations

1. **Slope + point:** plug into point-slope $y - y_1 = m(x - x_1)$, then simplify.
2. **Two points:** first find $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, then use point-slope with either point.
3. **From a graph:** read b (where it crosses the y-axis) and count rise/run for m; write $y = mx + b$.

Line through two points

Write the line through $(1, 3)$ and $(5, 11)$.

$$m = \frac{11-3}{5-1} = \frac{8}{4} = 2.$$

Use point-slope with $(1, 3)$: $y - 3 = 2(x-1) \Rightarrow y - 3 = 2x - 2 \Rightarrow y = 2x + 1$.

Check with the other point: $2(5) + 1 = 11$. ✓

Always **check** your equation by plugging in the point you did *not* use. If both points satisfy the equation, you can be confident it is right.

6.9 Parallel and Perpendicular Lines

Comparing Slopes

- **Parallel** lines never meet: they have **equal slopes**, $m_1 = m_2$.
- **Perpendicular** lines meet at a right angle: their slopes are **negative reciprocals**, $m_1 \cdot m_2 = -1$, so $m_2 = -\dfrac{1}{m_1}$.

Slope is rise over run.

(diagram in the online version)

Parallel and perpendicular slopes

A line has slope $m = \frac{2}{3}$.

- Any **parallel** line also has slope $\frac{2}{3}$.
- A **perpendicular** line has slope $-\frac{3}{2}$ (flip and change the sign).

Write the line through $(6, 1)$ perpendicular to $y = \frac{2}{3}x + 4$: use $m = -\frac{3}{2}$.

$$y - 1 = -\tfrac{3}{2}(x-6) \Rightarrow y - 1 = -\tfrac{3}{2}x + 9 \Rightarrow y = -\tfrac{3}{2}x + 10.$$

Negative reciprocal means do *two* things: (1) flip the fraction, (2) switch the sign. So $3 = \frac{3}{1} \to -\frac{1}{3}$, and $-\frac{4}{5} \to \frac{5}{4}$.

6.10 Horizontal and Vertical Lines

Special Lines

- A **horizontal** line has the form $y = c$. Every point has the same y-value. Slope $= 0$.
- A **vertical** line has the form $x = c$. Every point has the same x-value. Slope is **undefined**. (A vertical line is *not* a function.)

Reading and writing special lines

The horizontal line through $(2, -3)$ is $y = -3$ (the y-value stays -3).

The vertical line through $(2, -3)$ is $x = 2$ (the x-value stays 2).

The line $y = 5$ passes through $(0, 5)$, $(1, 5)$, $(-4, 5)$, all with height 5.

Mix-up alert: $y = c$ is *horizontal* (flat) and $x = c$ is *vertical* (straight up-and-down). Say it aloud: "y equals a number is a floor; x equals a number is a wall."

6.11 Going Deeper: Advanced Linear Functions

Moving Freely Among the Three Forms

Every non-vertical line can be written in three equivalent ways. Fluency means switching to whichever form a problem rewards.

- **Slope-intercept** $y = mx + b$ — best for graphing and reading slope/intercept.
- **Point-slope** $y - y_1 = m(x - x_1)$ — best when you know a point and the slope.
- **Standard** $Ax + By = C$ (integers, $A \geq 0$) — best for intercepts and for vertical lines $x = c$.

Translation rules: to leave point-slope, distribute and solve for y; to reach standard form, collect x and y on one side and clear denominators; to leave standard form, solve for y so that $m = -\frac{A}{B}$ and $b = \frac{C}{B}$.

Slope is rise over run.

(diagram in the online version)

One line, all three forms

Write the line through $(-2, 1)$ and $(4, 4)$ in every form.

$$
\begin{aligned}
m &= \frac{4-1}{4-(-2)} = \frac{3}{6} = \tfrac{1}{2}, \\
\text{point-slope:} \quad y - 1 &= \tfrac{1}{2}\bigl(x - (-2)\bigr) = \tfrac{1}{2}(x+2), \\
\text{slope-intercept:} \quad y &= \tfrac{1}{2}x + 1 + 1 = \tfrac{1}{2}x + 2, \\
\text{standard form:} \quad 2y &= x + 4 \;\Rightarrow\; -x + 2y = 4 \;\Rightarrow\; x - 2y = -4.
\end{aligned}
$$

Check: for $(4, 4)$, $4 - 2(4) = 4 - 8 = -4$. ✓ And $m = -\frac{A}{B} = -\frac{1}{-2} = \frac{1}{2}$ agrees.

Distance from a Point to a Line

The **distance** from a point (x_0, y_0) to the line $Ax+By+C = 0$ is the length of the *perpendicular* segment from the point to the line:

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Move everything to one side first so the line reads $Ax + By + C = 0$. The absolute value keeps the distance positive; the denominator $\sqrt{A^2 + B^2}$ scales for the line's slant.

A perpendicular distance

Find the distance from $(1, 1)$ to the line $3x + 4y = 12$.

Rewrite as $3x + 4y - 12 = 0$, so $A = 3$, $B = 4$, $C = -12$. Then

$$d = \frac{|3(1) + 4(1) - 12|}{\sqrt{3^2 + 4^2}} = \frac{|3 + 4 - 12|}{\sqrt{25}} = \frac{|-5|}{5} = \frac{5}{5} = 1.$$

The point sits exactly 1 unit from the line.

Collinearity and the Area of a Triangle

Three points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ form a triangle whose area is given by the **shoelace formula**

$$\text{Area} = \tfrac{1}{2}\bigl|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\bigr|.$$

Collinearity test: the three points lie on one line *exactly when* this area is 0 (a “flat” triangle). Equivalently, the slope AB equals the slope BC.

Area, then a collinearity check

(a) Area of the triangle with vertices $A(1,1)$, $B(5,2)$, $C(3,6)$:

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\big|1(2-6)+5(6-1)+3(1-2)\big| \\ &= \tfrac{1}{2}\big|-4+25-3\big| = \tfrac{1}{2}(18) = 9.\end{aligned}$$

(b) Are $(1,2)$, $(3,6)$, $(5,10)$ collinear?

$$\tfrac{1}{2}\big|1(6-10)+3(10-2)+5(2-6)\big| = \tfrac{1}{2}|-4+24-20| = 0.$$

Area 0, so yes — they lie on the single line $y=2x$ (each slope equals 2).

Why the Slope Rules Are True (Two Short Proofs)

Parallel lines have equal slopes. Take $y=m_1x+b_1$ and $y=m_2x+b_2$. They intersect where $m_1x+b_1=m_2x+b_2$, i.e. $(m_1-m_2)x=b_2-b_1$. If $m_1 \neq m_2$ this has exactly one solution, so the lines *cross*. Parallel lines never cross, forcing $m_1=m_2$.

Perpendicular lines have negative-reciprocal slopes. A line of slope m points along the direction vector $(1,m)$. Rotating a direction by 90° sends $(1,m)\mapsto(-m,1)$, whose slope is $\dfrac{1}{-m}=-\dfrac{1}{m}$. Hence a perpendicular line has slope $-\frac{1}{m}$, and $m\cdot\left(-\frac{1}{m}\right)=-1$.

Vertical/horizontal edge case. The negative-reciprocal rule assumes $m \neq 0$. A *horizontal* line ($m=0$) is perpendicular to a *vertical* line (undefined slope); their "product" rule breaks down, so handle this pair by geometry, not algebra.

Piecewise and Step Functions

A **piecewise function** uses different rules on different parts of the domain. Each rule is paired with the interval where it applies:

$$f(x)=\begin{cases} x+1, & 0\le x\le 1, \\ -x+3, & 1<x\le 3. \end{cases}$$

At the "seam" $x=1$ both pieces give 2, so the graph connects. A **step function** is a piecewise function whose pieces are constant, producing a staircase of flat segments (for example, a parking rate that jumps to the next whole-hour price).

Graphing a two-piece function

The function above, drawn on a grid: a rising segment then a falling segment, meeting at $(1,2)$.

Evaluate by choosing the rule for the input: $f(0.5)=0.5+1=1.5$ (first piece), while $f(2)=$

$-2+3=1$ (second piece).

Line of Best Fit (Linear Regression)

Real data rarely lands perfectly on one line. A **line of best fit** (or *trend line*) is the straight line $y = mx+b$ that passes as close as possible to *all* the points at once. The standard method, *least squares*, chooses m and b to make the total of the squared vertical gaps (the **residuals**) between the data and the line as small as possible.

- The **slope** m estimates the average rate of change in the data.
- You can **predict** an unseen value by plugging its x into $y = mx + b$.
- A fit is trustworthy only when the points genuinely cluster near a line; a strong straight-line pattern is called a strong (linear) *correlation*.

Correlation is not causation. A best-fit line can describe a pattern and forecast within the observed range, but it does not prove that x *causes* y, and predictions far outside the data (*extrapolation*) are unreliable.

7 Systems of Equations

Everything you need: graphing, substitution, elimination, special cases, inequalities, and word problems.

7.1 What Is a System of Equations?

The Big Idea

A **system of equations** is two (or more) equations that we want to be true *at the same time*. Each linear equation is a line on the coordinate plane.

A **solution** to the system is an ordered pair (x, y) that makes *every* equation true. Graphically, it is the point where the lines **cross**.

We usually write a system with a curly brace:

$$\begin{cases} x+y=6 \\ x-y=2 \end{cases}$$

This means: find the one pair (x, y) that fits *both* equations.

Checking a proposed solution

Is $(4, 2)$ a solution of the system above?

- First equation: $x + y = 4 + 2 = 6$. ✓
- Second equation: $x - y = 4 - 2 = 2$. ✓

Both are true, so $(4, 2)$ **is** the solution. If even one equation failed, it would not be a solution.

Remember: A solution must satisfy *all* equations. Always plug your answer back into *both* equations to check.

7.2 Method 1: Solving by Graphing

How Graphing Works

Graph both lines on the same axes. The **intersection point** (x, y) is the solution, because that point lies on both lines at once.

Steps: (1) Put each equation in slope-intercept form $y = mx + b$ if helpful. (2) Plot each line using its y-intercept and slope (or a small table of points). (3) Read off the point where they cross. (4) Check by substituting.

Graphing example

Solve $\begin{cases} y = x + 1 \\ y = -2x + 4 \end{cases}$

Line 1: $y = x + 1$ passes through $(0, 1)$ and $(1, 2)$.

Line 2: $y = -2x + 4$ passes through $(0, 4)$ and $(1, 2)$.

Both tables contain the point $(1, 2)$, so the lines cross there.

Check: $y = x + 1 = 1 + 1 = 2$ ✓ and $y = -2(1) + 4 = 2$ ✓

Solution: $(1, 2)$.

Tip: Graphing shows the answer at a glance, but it is only exact when the intersection lands on a clean grid point. For messy or fractional answers, use substitution or elimination instead.

7.3 Method 2: Solving by Substitution

How Substitution Works

Solve one equation for one variable, then **substitute** that expression into the other equation. Now you have one equation with one variable, which you can solve. Finally, back-substitute to find the other variable.

Best when: one variable is already isolated (like $y = \dots$) or is easy to isolate.

Substitution, fully stepped out

Solve $\begin{cases} y = 2x + 1 \\ 3x + y = 11 \end{cases}$

Step 1. The first equation already gives $y = 2x + 1$.

Step 2. Substitute $2x + 1$ in place of y in the second equation:

$$3x + (2x + 1) = 11$$

Step 3. Combine and solve for x:

$$5x + 1 = 11 \;\Rightarrow\; 5x = 10 \;\Rightarrow\; x = 2$$

Step 4. Back-substitute $x = 2$ into $y = 2x + 1$:

$$y = 2(2) + 1 = 5$$

Check: $3(2) + 5 = 11$ ✓

Solution: $(2, 5)$.

Substitution needing rearrangement

Solve $\begin{cases} x - 3y = -1 \\ 2x + y = 5 \end{cases}$

Step 1. Isolate x in the first equation: $x = 3y - 1$.

Step 2. Substitute into the second: $2(3y - 1) + y = 5$.

Step 3. Solve: $6y - 2 + y = 5 \Rightarrow 7y = 7 \Rightarrow y = 1$.

Step 4. Then $x = 3(1) - 1 = 2$.

Check: $2(2) + 1 = 5$ ✓ Solution: $(2, 1)$.

Watch out: Use parentheses when you substitute, especially with a minus sign in front. Distribute carefully.

7.4 Method 3: Solving by Elimination (Linear Combination)

How Elimination Works

Add or subtract the two equations so that one variable **cancels**. To make a variable cancel, its coefficients must be opposites (like $+2y$ and $-2y$). If they are not, **multiply** one or both equations by a number first.

Best when: the equations are in the form $ax + by = c$ and no variable is isolated.

Elimination when coefficients already match

Solve $\begin{cases} x + y = 10 \\ x - y = 4 \end{cases}$

The $+y$ and $-y$ are already opposites. **Add** the equations:

$$(x + y) + (x - y) = 10 + 4 \;\Rightarrow\; 2x = 14 \;\Rightarrow\; x = 7$$

Back-substitute into $x + y = 10$: $7 + y = 10 \Rightarrow y = 3$.

Check: $7 - 3 = 4$ ✓ Solution: $(7, 3)$.

Elimination needing multiplication (one equation)

Solve $\begin{cases} 2x + 3y = 7 \\ 4x - y = 7 \end{cases}$

Step 1. Choose a variable to eliminate. Multiply the first equation by 2 so the x-terms match:

$$2(2x + 3y) = 2(7) \;\Rightarrow\; 4x + 6y = 14$$

Step 2. Subtract the second equation from this new one:

$$(4x + 6y) - (4x - y) = 14 - 7 \;\Rightarrow\; 7y = 7 \;\Rightarrow\; y = 1$$

Step 3. Back-substitute into $2x + 3y = 7$: $2x + 3 = 7 \Rightarrow x = 2$.

Check: $4(2) - 1 = 7$ ✓ Solution: $(2, 1)$.

Elimination needing multiplication (both equations)

Solve $\begin{cases} 3x + 4y = 10 \\ 2x + 3y = 7 \end{cases}$

To cancel x, make both x-coefficients 6. Multiply the first by 2 and the second by 3:

$$6x + 8y = 20 \qquad 6x + 9y = 21$$

Subtract: $(6x + 8y) - (6x + 9y) = 20 - 21 \Rightarrow -y = -1 \Rightarrow y = 1$.

Back-substitute into $3x + 4y = 10$: $3x + 4 = 10 \Rightarrow x = 2$.

Check: $2(2) + 3(1) = 7$ ✓ Solution: $(2, 1)$.

Tips for elimination: Line up like variables in columns before you start. When you *subtract*, change the sign of *every* term in the equation you subtract. Double-check signs, since that is the most common mistake.

7.5 How Many Solutions? Three Possibilities

One, None, or Infinitely Many

Two lines can meet in exactly three ways:

- **One solution:** the lines cross once (different slopes). This is the usual case.
- **No solution:** the lines are **parallel** (same slope, different y-intercept). They never meet.
- **Infinitely many solutions:** the two equations are really the **same line** (same slope and same intercept). Every point on the line works.

Spotting the special cases

No solution: $\begin{cases} y = 2x + 1 \\ y = 2x - 3 \end{cases}$ Same slope 2, different intercepts $\Rightarrow$ parallel $\Rightarrow$ **no solution**.

Infinitely many: $\begin{cases} 2x + y = 4 \\ 4x + 2y = 8 \end{cases}$ The second equation is just the first times 2, so they are the same line $\Rightarrow$ **infinitely many solutions**.

Algebra clue: If your work leads to a false statement like $0 = 6$, there is **no solution**. If it leads to an always-true statement like $0 = 0$, there are **infinitely many solutions**.

7.6 Choosing the Best Method

Which Method Should I Use?

- **Graphing** — great for a quick picture or when you want to *see* the answer; best with clean integer intersections.
- **Substitution** — best when a variable is already isolated or has a coefficient of 1 (easy to solve for).
- **Elimination** — best when both equations are in $ax + by = c$ form, especially if coefficients line up or match easily.

There is no single "right" method; pick whichever needs the least work for the problem in front of you.

7.7 Systems of Linear Inequalities

Shading the Solution Region

A **system of inequalities** uses $<$, $>$, $\leq$, $\geq$ instead of $=$. Instead of a single point, the solution is a whole **region** of the plane.

Steps: (1) Graph each boundary line. Use a **solid** line for $\leq$ or $\geq$ and a **dashed** line for $<$ or $>$. (2) Shade the correct side of each line. (3) The solution is the **overlap** where the shadings cross. Any point in that overlap satisfies both inequalities.

Testing points in an inequality system

Consider $\begin{cases} y \geq x - 1 \\ y < -x + 3 \end{cases}$

Use a **test point** to decide which side to shade and to check membership.

Test $(0,0)$: $0 \geq 0 - 1$ (true) and $0 < 0 + 3$ (true). Both hold, so $(0,0)$ **is** in the solution region.

Test $(3,3)$: $3 \geq 3 - 1 = 2$ (true) but $3 < -3 + 3 = 0$ (*false*). One fails, so $(3,3)$ is **not** in the region.

The solution is the overlapping band above the line $y = x - 1$ and below the line $y = -x + 3$.

Tip: $(0,0)$ is the easiest test point to plug in, as long as no boundary line passes through the origin. A point is a solution only if it satisfies *every* inequality.

7.8 Applications: Word Problems

Turning Words into a System

Many word problems hide a 2×2 system. **Steps:** (1) Define two variables clearly. (2) Write one equation for each fact in the problem. (3) Solve with any method. (4) Answer in a full sentence and check that it makes sense.

Cost / number problem

A theater sells \$5 adult tickets and \$3 child tickets. One night they sold 10 tickets for a total of \$38. How many of each?

Define: a = adult tickets, c = child tickets.

System: $\begin{cases} a + c = 10 \\ 5a + 3c = 38 \end{cases}$

Solve by substitution: $a = 10 - c$, so $5(10 - c) + 3c = 38 \Rightarrow 50 - 2c = 38 \Rightarrow c = 6$, then $a = 4$.

Check: $5(4) + 3(6) = 20 + 18 = 38$ ✓ **Answer:** 4 adult and 6 child tickets.

Mixture problem

A chemist mixes a 20% acid solution with a 50% acid solution to make 30 mL of a 40% acid solution. How much of each?

Define: x = mL of 20% solution, y = mL of 50% solution.

System: $\begin{cases} x + y = 30 \\ 0.20x + 0.50y = 0.40(30) = 12 \end{cases}$

Solve: $x = 30 - y$, so $0.20(30 - y) + 0.50y = 12 \Rightarrow 6 + 0.30y = 12 \Rightarrow y = 20$, then $x = 10$.

Check: $0.20(10) + 0.50(20) = 2 + 10 = 12$ ✓ **Answer:** 10 mL of 20% and 20 mL of 50%.

Tip: Distance–rate–time and current problems often become systems too. If a boat's speed is b and the current is c, then downstream speed is $b + c$ and upstream speed is $b - c$. Two trips give you two equations.

7.9 Going Deeper: Advanced Systems

Systems in Three Variables

A **three-variable system** has three equations in x, y, and z. A solution is an ordered triple (x, y, z) that satisfies *all three*. Geometrically each equation is a **plane** in space, and a single solution is the point where all three planes meet.

Elimination strategy: pick one variable to remove. Combine equations in *two different pairs* to eliminate that same variable, leaving a 2×2 system in the other two variables. Solve that smaller system, then back-substitute to recover the third variable.

Three-variable elimination, fully stepped out

Solve

$$\begin{cases} x + y + z = 6 \\ x - y + z = 2 \\ 2x + y - z = 1 \end{cases}$$

Step 1. Eliminate z from two pairs. Add equation (1) and equation (3):

$$(x + y + z) + (2x + y - z) = 6 + 1 \;\Rightarrow\; 3x + 2y = 7$$

Add equation (2) and equation (3):

$$(x - y + z) + (2x + y - z) = 2 + 1 \;\Rightarrow\; 3x = 3 \;\Rightarrow\; x = 1$$

Step 2. Back-substitute $x = 1$ into $3x + 2y = 7$:

$$3(1) + 2y = 7 \;\Rightarrow\; 2y = 4 \;\Rightarrow\; y = 2$$

Step 3. Find z from equation (1): $1 + 2 + z = 6 \Rightarrow z = 3$.

Check (all three): $1 + 2 + 3 = 6$ ✓, $1 - 2 + 3 = 2$ ✓, $2(1) + 2 - 3 = 1$ ✓

Solution: $(x, y, z) = (1, 2, 3)$.

Systems With a Parameter

Sometimes a coefficient is a letter (a **parameter**) and we ask: for which values does the system have one, none, or infinitely many solutions? Compare the ratios of the coefficients. For

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

- **One solution** when $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$ (different slopes).
- **Infinitely many** when $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ (same line).
- **No solution** when $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ (parallel lines).

Choosing the parameter

For which value of k does $\begin{cases} x + 2y = 3 \\ 2x + ky = m \end{cases}$ fail to have a unique solution?

The x-ratio is $\frac{1}{2}$. The y-ratio is $\frac{2}{k}$. These match when $\frac{2}{k} = \frac{1}{2}$, i.e. $k = 4$. So:

- If $k \neq 4$: exactly **one solution** for any m.
- If $k = 4$ and $m = 6$: the second equation is just $2\times$ the first, so **infinitely many solutions**.
- If $k = 4$ and $m \neq 6$: parallel lines, so **no solution**.

Symmetric Systems: the $x + y$ and xy Trick

A system is **symmetric** if swapping x and y leaves it unchanged (for example, using $x + y$, xy, or $x^2 + y^2$). Let $s = x + y$ (the sum) and $p = xy$ (the product). A key identity converts squares into these:

$$x^2 + y^2 = (x + y)^2 - 2xy = s^2 - 2p$$

Once you know s and p, the two numbers x and y are exactly the roots of

$$t^2 - s\,t + p = 0.$$

Solving a symmetric system

Solve $\begin{cases} x + y = 5 \\ x^2 + y^2 = 13 \end{cases}$

Step 1. Let $s = x + y = 5$. Use the identity $x^2 + y^2 = s^2 - 2p$:

$$13 = 5^2 - 2p = 25 - 2p \;\Rightarrow\; 2p = 12 \;\Rightarrow\; p = xy = 6$$

Step 2. Now x and y are the roots of $t^2 - 5t + 6 = 0$, which factors as $(t - 2)(t - 3) = 0$, giving $t = 2$ or $t = 3$.

Step 3. So $\{x, y\} = \{2, 3\}$. The solutions are $(2, 3)$ and $(3, 2)$.

Check: $2 + 3 = 5$ ✓ and $2^2 + 3^2 = 4 + 9 = 13$ ✓

Linear Programming: Optimizing Over a Region

When a system of linear *inequalities* defines a **feasible region**, we often want to maximize (or minimize) a linear **objective function** such as $P = 3x + 4y$. The Corner-Point Theorem says the maximum and minimum of a linear objective always occur at a **vertex (corner)** of the feasible region.

Steps: (1) Graph the constraints and find the feasible region. (2) Find each corner point (each is the intersection of two boundary lines). (3) Evaluate the objective P at every corner. (4) The largest value is the maximum; the smallest is the minimum.

Maximizing an objective

Maximize $P = 40x + 30y$ subject to

$$\begin{cases} x + y \le 8 \\ 2x + y \le 10 \\ x \ge 0,\ y \ge 0 \end{cases}$$

Find the corners. The boundary lines are $x + y = 8$ and $2x + y = 10$, together with the axes.

- $(0, 0)$ — origin.
- $(5, 0)$ — where $2x + y = 10$ meets $y = 0$.
- $(0, 8)$ — where $x + y = 8$ meets $x = 0$.
- $(2, 6)$ — solve $x + y = 8$ and $2x + y = 10$: subtract to get $x = 2$, then $y = 6$.

Evaluate $P = 40x + 30y$ at each corner:

Maximum: $P = 260$ at the corner $(2, 6)$.

Nonlinear Preview: a Line Meets a Parabola

Not every system is two lines. When one equation is a **parabola** (like $y = x^2$) and the other is a **line**, substitution still works, but the result is a **quadratic**. A line and a parabola can cross in **two** points, **one** point (tangent), or **none**, depending on the discriminant of that quadratic.

Line intersecting a parabola

Solve $\begin{cases} y = x^2 \\ y = x + 2 \end{cases}$

Step 1. Substitute $y = x^2$ into $y = x + 2$:

$$x^2 = x + 2 \ \Rightarrow\ x^2 - x - 2 = 0$$

Step 2. Factor: $(x - 2)(x + 1) = 0$, so $x = 2$ or $x = -1$.

Step 3. Find each y from $y = x + 2$: if $x = 2$, $y = 4$; if $x = -1$, $y = 1$.

Check: $y = x^2$ gives $2^2 = 4$ ✓and $(-1)^2 = 1$ ✓

Two solutions: $(2,4)$ and $(-1,1)$.

A Peek at Determinants: Cramer's Rule

For a 2×2 system

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

the **determinant** of the coefficients is

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If $D \neq 0$ there is exactly one solution, given by **Cramer's Rule**:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{ed - bf}{D}, \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{af - ec}{D}.$$

Each numerator replaces one coefficient column with the constants $\frac{e}{f}$. If $D = 0$, the system has no unique solution (either none or infinitely many).

Cramer's Rule in action

Solve $\begin{cases} 2x + 3y = 7 \\ 4x - y = 7 \end{cases}$ using determinants.

Here $a = 2,\ b = 3,\ e = 7$ and $c = 4,\ d = -1,\ f = 7$.

Main determinant: $D = \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = (2)(-1) - (3)(4) = -2 - 12 = -14.$

Solve for x: $x = \dfrac{\begin{vmatrix} 7 & 3 \\ 7 & -1 \end{vmatrix}}{D} = \dfrac{(7)(-1) - (3)(7)}{-14} = \dfrac{-28}{-14} = 2.$

Solve for y: $y = \dfrac{\begin{vmatrix} 2 & 7 \\ 4 & 7 \end{vmatrix}}{D} = \dfrac{(2)(7) - (7)(4)}{-14} = \dfrac{-14}{-14} = 1.$

Solution: $(2,1)$, matching the elimination result earlier.

Big picture: Every method here rests on the same idea — reduce a hard system to a simpler one. Three-variable elimination shrinks to a 2×2; symmetric systems shrink to a single quadratic in t; Cramer's Rule packages the answer as a ratio of determinants. Whenever a determinant is 0 (or ratios all match), watch for the "no solution / infinitely many" special cases.

8 Exponents

Every rule, tip, and worked example in one place

8.1 What an Exponent Means

Repeated Multiplication

An **exponent** is a shortcut for repeated multiplication. In the power a^n:

- a is the **base** — the number being multiplied.
- n is the **exponent** (or **power**) — how many times the base is used as a factor.

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}, \qquad 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32.$$

We read a^n as "a to the nth power." Special names: a^2 is "a squared" and a^3 is "a cubed."

Expanding and evaluating

Write each power as repeated multiplication, then evaluate.

$$\begin{aligned} 3^4 &= 3 \cdot 3 \cdot 3 \cdot 3 = 81, \\ (-2)^3 &= (-2)(-2)(-2) = -8, \\ \left(\tfrac{1}{2}\right)^3 &= \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{8}. \end{aligned}$$

Watch the sign carefully: $(-2)^4 = 16$ (four negatives multiply to a positive), but $-2^4 = -(2^4) = -16$, because without parentheses the exponent applies only to the 2, and the minus sign waits out front.

8.2 All the Exponent Rules at a Glance

The Complete Rule Sheet

For any nonzero base(s) a, b and integers m, n:

- **Product of powers:** $a^m \cdot a^n = a^{m+n}$
- **Quotient of powers:** $\dfrac{a^m}{a^n} = a^{m-n}$
- **Power of a power:** $(a^m)^n = a^{mn}$
- **Power of a product:** $(ab)^n = a^n b^n$

- **Power of a quotient:** $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- **Zero exponent:** $a^0 = 1$
- **Negative exponent:** $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$

The whole topic comes back to one idea: *same base, combine the exponents.*

8.3 Product of Powers

Add the Exponents

When you multiply powers with the *same base*, keep the base and **add** the exponents.

$$a^m \cdot a^n = a^{m+n}.$$

Why? $a^2 \cdot a^3 = (a \cdot a)(a \cdot a \cdot a) = a^5$: you are just counting all the factors.

Multiplying like bases

$$\begin{aligned} x^4 \cdot x^3 &= x^{4+3} = x^7, \\ 2^5 \cdot 2^2 &= 2^7 = 128, \\ 5a^2 \cdot 3a^6 &= (5 \cdot 3)\, a^{2+6} = 15a^8. \end{aligned}$$

Multiply the coefficients (the plain numbers) and add the exponents of the matching variables.

The bases must *match*. You cannot combine $x^3 \cdot y^2$ into a single power — different bases stay separate.

8.4 Quotient of Powers

Subtract the Exponents

When you divide powers with the *same base*, keep the base and **subtract** the bottom exponent from the top.

$$\frac{a^m}{a^n} = a^{m-n}.$$

The shared factors cancel: $\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3$.

Dividing like bases

$$\frac{x^9}{x^4} = x^{9-4} = x^5,$$
$$\frac{3^7}{3^5} = 3^2 = 9,$$
$$\frac{12b^8}{4b^3} = \frac{12}{4} b^{8-3} = 3b^5.$$

Top minus bottom — order matters. $\frac{a^2}{a^6} = a^{2-6} = a^{-4} = \frac{1}{a^4}$. A negative result just means the base "lives" in the denominator.

8.5 Power of a Power

Multiply the Exponents

When a power is raised to another power, keep the base and **multiply** the exponents.

$$(a^m)^n = a^{mn}.$$

Because $\left(a^2\right)^3 = a^2 \cdot a^2 \cdot a^2 = a^6$.

Raising a power to a power

$$\left(x^3\right)^4 = x^{3\cdot 4} = x^{12},$$
$$\left(2^2\right)^3 = 2^6 = 64,$$
$$\left(y^5\right)^2 = y^{10}.$$

Do not confuse the rules: $\left(x^3\right)^4 = x^{12}$ (multiply), but $x^3 \cdot x^4 = x^7$ (add). Ask yourself whether the powers are being *multiplied together* or *stacked*.

8.6 Power of a Product and Power of a Quotient

Distribute the Exponent

An exponent on a product or a quotient applies to *every* factor inside.

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Distributing over products and quotients

$$(2x)^3 = 2^3 x^3 = 8x^3,$$
$$\left(3a^2 b\right)^2 = 3^2\, a^{2\cdot 2}\, b^2 = 9a^4 b^2,$$
$$\left(\frac{x}{4}\right)^2 = \frac{x^2}{4^2} = \frac{x^2}{16}.$$

Notice how power of a product combines with power of a power: each inside exponent gets multiplied by the outside one.

The coefficient gets the exponent too! A common slip is $(2x)^3 = 2x^3$. The correct result is $8x^3$, because $2^3 = 8$.

8.7 Zero and Negative Exponents

The Zero Exponent

Any nonzero base raised to the 0 power equals 1:

$$a^0 = 1 \quad (a \neq 0).$$

This follows from the quotient rule: $\dfrac{a^3}{a^3} = a^{3-3} = a^0$, and any number divided by itself is 1.

Negative Exponents

A negative exponent means take the **reciprocal** (flip it), then use the positive exponent:

$$a^{-n} = \frac{1}{a^n}, \qquad \frac{1}{a^{-n}} = a^n.$$

A factor moves between numerator and denominator by flipping the sign of its exponent.

Working with zero and negative exponents

$$
\begin{aligned}
7^0 &= 1, & (5x)^0 &= 1,\\
2^{-3} &= \frac{1}{2^3} = \frac{1}{8}, & x^{-4} &= \frac{1}{x^4},\\
\frac{1}{3^{-2}} &= 3^2 = 9, & \frac{2}{x^{-3}} &= 2x^3.
\end{aligned}
$$

A negative exponent does *not* make the answer negative. $2^{-3} = \frac{1}{8}$, a positive number, not -8. The exponent's sign controls *where* the factor goes (top or bottom), not the sign of the value. And remember $a^0 = 1$, never 0.

8.8 Simplifying Combined Expressions

A Game Plan

Real problems mix several rules. A reliable order:

1. Apply power-of-a-power / power-of-a-product to clear outer exponents.
2. Multiply (add exponents) and divide (subtract exponents) like bases.
3. Rewrite any negative exponents as positive by flipping their factors.
4. Simplify the number coefficients.

A fully simplified answer usually has **no negative exponents** and each base appearing once.

Combining several rules

Simplify $\dfrac{(2x^2y)^3}{4x^4y}$.

$$
\begin{aligned}
\frac{(2x^2y)^3}{4x^4y} &= \frac{2^3x^6y^3}{4x^4y} && \text{power of a product}\\
&= \frac{8x^6y^3}{4x^4y} && 2^3 = 8\\
&= \frac{8}{4}\,x^{6-4}\,y^{3-1} && \text{quotient rule}\\
&= 2x^2y^2.
\end{aligned}
$$

Clearing negative exponents

Simplify $\dfrac{6a^{-2}b^3}{3a^2b^{-1}}$ with only positive exponents.

$$\begin{aligned}
\frac{6a^{-2}b^3}{3a^2b^{-1}} &= \frac{6}{3}\,a^{-2-2}\,b^{3-(-1)} && \text{quotient rule}\\
&= 2a^{-4}b^4\\
&= \frac{2b^4}{a^4}. && \text{move } a^{-4} \text{ down}
\end{aligned}$$

When subtracting a negative exponent, be careful: $b^{3-(-1)} = b^{3+1} = b^4$. Two minus signs make a plus.

8.9 Scientific Notation

The Form

Scientific notation writes a number as

$$c \times 10^n, \qquad \text{where } 1 \le c < 10 \text{ and } n \text{ is an integer.}$$

The factor c (the *coefficient*) has exactly one nonzero digit before the decimal point. The power of 10 records how far the decimal point moved.

- **Large** numbers use a **positive** exponent: $4\,300 = 4.3 \times 10^3$.
- **Small** numbers (between 0 and 1) use a **negative** exponent: $0.0057 = 5.7 \times 10^{-3}$.

Converting to and from standard form

To scientific notation — move the decimal so one nonzero digit stays in front, and count the moves:

$$\begin{aligned}
52\,000 &= 5.2 \times 10^4 && \text{moved left 4 places}\\
0.00081 &= 8.1 \times 10^{-4} && \text{moved right 4 places}
\end{aligned}$$

To standard form — move the decimal the number of places the exponent says (right if positive, left if negative):

$$\begin{aligned}
3.6 \times 10^5 &= 360\,000,\\
9.2 \times 10^{-3} &= 0.0092.
\end{aligned}$$

Multiplying and Dividing in Scientific Notation

Handle the coefficients and the powers of 10 separately, using the product and quotient rules on the 10's.

$$(a \times 10^m)(b \times 10^n) = (a \cdot b) \times 10^{m+n},$$

$$\frac{a \times 10^m}{b \times 10^n} = \frac{a}{b} \times 10^{m-n}.$$

At the end, adjust so the coefficient again satisfies $1 \le c < 10$.

Multiplying in scientific notation

Compute $\left(3 \times 10^4\right)\left(2 \times 10^5\right)$.

$$\begin{aligned}\left(3 \times 10^4\right)\left(2 \times 10^5\right) &= (3 \cdot 2) \times 10^{4+5} \\ &= 6 \times 10^9.\end{aligned}$$

Dividing, then fixing the coefficient

Compute $\dfrac{8.4 \times 10^7}{4 \times 10^3}$ and $\dfrac{1.5 \times 10^3}{5 \times 10^6}$.

$$\frac{8.4 \times 10^7}{4 \times 10^3} = \frac{8.4}{4} \times 10^{7-3} = 2.1 \times 10^4.$$

$$\frac{1.5 \times 10^3}{5 \times 10^6} = 0.3 \times 10^{-3} = 3 \times 10^{-4}.$$

In the second one, 0.3 is too small, so we rewrite $0.3 = 3 \times 10^{-1}$, which drops the power of 10 by one.

If your coefficient ends up 10 or larger, shift the decimal left and *add* 1 to the exponent. If it ends up less than 1, shift right and *subtract* 1. Example: $12 \times 10^5 = 1.2 \times 10^6$.

8.10 Exponential Growth vs. Decay (Intro)

The Pattern $y = a \cdot b^x$

Many real situations follow

$$y = a \cdot b^x,$$

where a is the **starting amount** (the value when $x = 0$) and b is the **growth factor**. The value of b decides the behavior:

- **Growth:** $b > 1$ — the quantity gets larger as x increases (doubling, tripling, interest).
- **Decay:** $0 < b < 1$ — the quantity gets smaller as x increases (halving, cooling, fading).

Because $b^0 = 1$, the starting value is always $y = a \cdot b^0 = a$.

Evaluating a growth model

A colony starts at 5 bacteria and doubles each hour: $y = 5 \cdot 2^x$, with x in hours. Here $b = 2 > 1$, so it is **growth**.

$$\begin{aligned} x = 0: &\quad y = 5 \cdot 2^0 = 5 \cdot 1 = 5, \\ x = 1: &\quad y = 5 \cdot 2^1 = 10, \\ x = 3: &\quad y = 5 \cdot 2^3 = 5 \cdot 8 = 40. \end{aligned}$$

Evaluating a decay model

A ball bounces to $\frac{1}{2}$ its previous height, starting at 80 cm: $y = 80 \cdot \left(\frac{1}{2}\right)^x$. Here $b = \frac{1}{2}$, and $0 < b < 1$, so it is **decay**.

$$\begin{aligned} x = 0: &\quad y = 80 \cdot \left(\tfrac{1}{2}\right)^0 = 80, \\ x = 1: &\quad y = 80 \cdot \tfrac{1}{2} = 40, \\ x = 2: &\quad y = 80 \cdot \tfrac{1}{4} = 20. \end{aligned}$$

Read b to tell the story: $b > 1$ means the amount *multiplies up* (growth); $0 < b < 1$ means it *shrinks* toward zero (decay). Either way, plug in x and use your exponent skills to evaluate.

8.11 Going Deeper: Advanced Exponents

Solving Exponential Equations by Matching Bases

If two powers with the *same base* are equal, then their exponents must be equal:

$$b^m = b^n \quad \Longrightarrow \quad m = n \qquad (b > 0,\ b \neq 1).$$

To solve an equation like $b^{\square} = N$, rewrite *both* sides as powers of one common base, then set the exponents equal and solve that (usually linear) equation.

- Ask: "What base makes both sides simple?" Often it is the smaller of the two numbers.
- Use the power-of-a-power rule to fold the outside exponent in: $\left(b^k\right)^x = b^{kx}$.

Matching bases to solve

Solve each equation for x.

$$\begin{aligned} 2^x &= 32 &&\Rightarrow & 2^x &= 2^5 &&\Rightarrow & x &= 5. \\ 9^x &= 27 &&\Rightarrow & \left(3^2\right)^x &= 3^3 &&\Rightarrow & 3^{2x} &= 3^3,\quad 2x = 3,\quad x = \tfrac{3}{2}. \\ 5^{x+1} &= 125 &&\Rightarrow & 5^{x+1} &= 5^3 &&\Rightarrow & x+1 &= 3,\quad x = 2. \end{aligned}$$

Each time: write both sides with the same base, then just compare exponents.

Rational Exponents: the Bridge to Radicals

A fractional exponent is a compact way to write a root. For $a \geq 0$ and positive integers m, n:

$$a^{1/n} = \sqrt[n]{a}, \qquad a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Read the fraction as "**root on the bottom, power on the top**": the denominator n is the index of the radical, and the numerator m is the ordinary power. You may take the root first or the power first — the answer is the same, but taking the root first keeps the numbers small.

Rational exponents both ways

Evaluate, taking the root first.

$$\begin{aligned} 8^{2/3} &= \left(\sqrt[3]{8}\right)^2 = 2^2 = 4, \\ 16^{3/4} &= \left(\sqrt[4]{16}\right)^3 = 2^3 = 8, \\ 27^{-1/3} &= \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}. \end{aligned}$$

The last one combines two ideas: the negative sign flips it, and the $1/3$ takes a cube root.

All the ordinary exponent rules still work with fractional exponents. For instance $\sqrt{a} \cdot \sqrt[3]{a} = a^{1/2} \cdot a^{1/3} = a^{1/2+1/3} = a^{5/6}$. Rewriting radicals as powers is often the fastest way to combine them.

Comparing Powers of Different Bases

You cannot just eyeball which of 2^{40} or 3^{30} is larger. The trick is to make either the *exponents* match or the *bases* match, then compare.

- **Match the exponents:** factor the exponents to pull out a common one, then compare the resulting bases.

- **Match the bases:** rewrite both as powers of a shared base when possible.

Once one part agrees, the larger base (or larger exponent) wins.

Which power is bigger?

Compare 2^{40} and 3^{30} by giving them a common exponent of 10.

$$2^{40} = \left(2^4\right)^{10} = 16^{10},$$
$$3^{30} = \left(3^3\right)^{10} = 27^{10}.$$

Both are raised to the 10th power, and $27 > 16$, so $27^{10} > 16^{10}$. Therefore $3^{30} > 2^{40}$.

Units-Digit Cycles

The **units digit** (last digit) of a power repeats in a short cycle as the exponent grows, because only the previous units digit affects the next one. For base 7:

$$7^1 = 7, \quad 7^2 = 49, \quad 7^3 = 343, \quad 7^4 = 2401, \quad 7^5 = 16807, \ \ldots$$

so the units digits run 7, 9, 3, 1, 7, 9, 3, 1, ... — a cycle of length 4. To find the units digit of 7^k, divide k by the cycle length and use the remainder to pick the spot in the cycle (a remainder of 0 lands on the *last* entry).

Finding a units digit

Find the units digit of 7^{100}.

$$\text{Cycle of units digits: } 7, 9, 3, 1 \quad \text{(length 4)}.$$
$$100 \div 4 = 25 \text{ remainder } 0 \ \Rightarrow \ \text{use the 4th (last) entry.}$$
$$\text{Units digit of } 7^{100} = 1.$$

Check the idea on a small case: $100 \equiv 0 \pmod 4$ matches $7^4 = 2401$, which indeed ends in 1.

Growth and Decay by a Percent Rate

The model $y = a \cdot b^x$ often comes from a percent change per step. Writing the growth factor as $b = 1 \pm r$ (with r a decimal rate) separates the two cases:

$$\text{Growth: } y = a\,(1+r)^x, \qquad \text{Decay: } y = a\,(1-r)^x.$$

Here a is the starting amount and r is the rate per period. Growth uses $1 + r > 1$; decay uses

$1 - r$ between 0 and 1.

A percent-growth model

A \$2000 investment grows 5% per year, so $r = 0.05$ and $b = 1.05$:

$$y = 2000\,(1.05)^x.$$

$$\begin{aligned} x = 0: &\quad y = 2000\,(1.05)^0 = 2000,\\ x = 1: &\quad y = 2000\,(1.05)^1 = 2100,\\ x = 2: &\quad y = 2000\,(1.05)^2 = 2000\,(1.1025) = 2205. \end{aligned}$$

A 12% decay would instead use $b = 1 - 0.12 = 0.88$.

The Logarithm: Undoing an Exponent (Preview)

Exponentiation asks "*what do I get* when I raise b to the power x?" The **logarithm** asks the reverse: "*what power* of b gives me y?" They are inverse operations:

$$b^x = y \quad \Longleftrightarrow \quad \log_b y = x \qquad (b > 0,\ b \neq 1).$$

So $\log_b y$ is simply the exponent you would put on b to land on y. For example, $\log_2 8 = 3$ because $2^3 = 8$, and $\log_{10} 1000 = 3$ because $10^3 = 1000$. This is the tool that solves $2^x = 10$, where the bases cannot be matched by hand.

Adding and Subtracting in Scientific Notation

For multiplying and dividing you handle the coefficients and the powers of 10 separately. But to **add or subtract**, the powers of 10 must first *match*. Rewrite the smaller-exponent number so both share the larger exponent, then add or subtract the coefficients:

$$a \times 10^n \pm b \times 10^n = (a \pm b) \times 10^n.$$

Finally, adjust the coefficient back into the range $1 \leq c < 10$ if needed.

Adding with a common power of ten

Compute $\left(3 \times 10^4\right) + \left(2 \times 10^3\right)$.

$$\begin{aligned} 3 \times 10^4 + 2 \times 10^3 &= 3 \times 10^4 + 0.2 \times 10^4 && \text{match the exponents}\\ &= (3 + 0.2) \times 10^4 && \text{add the coefficients}\\ &= 3.2 \times 10^4. \end{aligned}$$

Rewriting 2×10^3 as 0.2×10^4 lets both terms share 10^4 before adding.

Never add the powers of 10 when you are *adding* numbers — that is the rule for *multiplying*. For a sum, the exponents must already be equal; only the coefficients get added.

9 Polynomials

Vocabulary, degree, standard form, adding, subtracting, multiplying, and special products

9.1 What Is a Polynomial?

The Building Blocks

A **term** is a number, a variable, or a number multiplied by variables raised to whole-number powers. Examples of terms: $7,\ 3x,\ -5x^2,\ x^3$.

A **polynomial** is a sum of terms. In each term:

- the **coefficient** is the number multiplied by the variable (in $-5x^2$ the coefficient is -5);
- a **constant** is a term with no variable (like 2 in $3x^2 - 5x + 2$).

Naming Polynomials by the Number of Terms

- **Monomial** — one term: $4x^3$
- **Binomial** — two terms: $x + 3$
- **Trinomial** — three terms: $3x^2 - 5x + 2$
- **Polynomial** — the general word for one or more terms (four or more terms usually just get called “a polynomial”).

Example: Identify the parts of $3x^2 - 5x + 2$

This is a **trinomial** (three terms).

- Terms: $3x^2,\ -5x,\ 2$
- Coefficients: 3 (of x^2) and -5 (of x)
- Constant: 2

Watch the sign! The term $-5x$ has a *negative* coefficient. Always carry the sign in front of a term along with the term.

9.2 Degree, Leading Coefficient, and Standard Form

Degree

The **degree of a term** is the exponent on its variable ($-5x^3$ has degree 3; a constant like $7 = 7x^0$ has degree 0).

The **degree of a polynomial** is the *largest* degree among its terms.

Standard Form and Leading Coefficient

A polynomial is in **standard form** when its terms are written in *descending order* of degree (highest power first).

The **leading coefficient** is the coefficient of the first term once the polynomial is in standard form (the coefficient of the highest-degree term).

Example: Put $4 - 7x + 2x^3$ in standard form

Order the terms from highest power to lowest:

$$2x^3 - 7x + 4.$$

- Degree of the polynomial: 3
- Leading coefficient: 2
- Constant: 4

Example: Degree and leading coefficient of $6x^2 - x^5 + 4$

Standard form: $-x^5 + 6x^2 + 4$.

- Degree: 5
- Leading coefficient: -1 (the coefficient of $-x^5$)

Tip: A missing power is fine—$-x^5 + 6x^2 + 4$ simply has no x^4, x^3, or x^1 terms. Standard form only cares about ordering the terms that *are* there.

9.3 Evaluating a Polynomial

Substitute and Simplify

To **evaluate** a polynomial for a given value, replace every variable with that value (use parentheses!), then follow the order of operations.

Example: Evaluate $2x^2 - 5x + 1$ at $x = 3$

$$\begin{aligned} 2(3)^2 - 5(3) + 1 &= 2(9) - 15 + 1 \\ &= 18 - 15 + 1 \\ &= 4. \end{aligned}$$

Example: Evaluate $-x^2 + 4x$ at $x = -2$

$$\begin{aligned} -(-2)^2 + 4(-2) &= -(4) - 8 \\ &= -4 - 8 = -12. \end{aligned}$$

Careful with negatives: $-(-2)^2$ means "take the opposite of $(-2)^2$." Square first: $(-2)^2 = 4$; then apply the minus: -4. Always wrap the substituted value in parentheses.

9.4 Adding Polynomials

Combine Like Terms

Like terms have the exact same variable part (same variable, same exponent), such as $3x^2$ and $5x^2$. To **add** polynomials, drop the parentheses and combine like terms.

Example: $(4x^2 - 3x + 1) + (x^2 + 3x - 5)$

Group like terms, then add:

$$\begin{aligned} (4x^2 + x^2) + (-3x + 3x) + (1 - 5) &= 5x^2 + 0x - 4 \\ &= 5x^2 - 4. \end{aligned}$$

Only like terms combine. You cannot add $5x^2$ and -4 into one term—different powers stay separate. The answer $5x^2 - 4$ is fully simplified.

9.5 Subtracting Polynomials

Distribute the Negative

To **subtract**, distribute the minus sign to *every* term of the second polynomial (this flips each sign), and then combine like terms.

Example: $(6x^2 + 2x - 1) - (3x^2 - x + 4)$

Distribute the negative across the second group:

$$\begin{aligned} 6x^2 + 2x - 1 \ - \ 3x^2 + x - 4 &= (6x^2 - 3x^2) + (2x + x) + (-1 - 4) \\ &= 3x^2 + 3x - 5. \end{aligned}$$

The most common mistake! Subtracting $(3x^2 - x + 4)$ changes *all three* signs: $-3x^2$, $+x$, -4. It is easy to flip only the first term—flip them *all*.

9.6 Multiplying a Monomial by a Polynomial

The Distributive Property

Multiply the monomial by *each* term inside the parentheses. Multiply the coefficients, and add the exponents on matching variables ($x^a \cdot x^b = x^{a+b}$).

Example: $-4x(x^2 - 2x + 1)$

Distribute $-4x$ to each term:

$$-4x \cdot x^2 \ - \ 4x \cdot (-2x) \ - \ 4x \cdot 1 = -4x^3 + 8x^2 - 4x.$$

Exponent reminder: $x \cdot x^2 = x^{1+2} = x^3$. Add exponents when multiplying powers of the same base—do not multiply them.

9.7 Multiplying Two Binomials (FOIL)

FOIL

FOIL names the four products you make when multiplying two binomials:

$$\underbrace{\text{F}}_{\text{First}}\ \underbrace{\text{O}}_{\text{Outer}}\ \underbrace{\text{I}}_{\text{Inner}}\ \underbrace{\text{L}}_{\text{Last}}$$

Then combine the two middle (Outer + Inner) like terms.

Example: $(x+4)(x-1)$

$$\begin{aligned} \text{First: } & x \cdot x = x^2 \\ \text{Outer: } & x \cdot (-1) = -x \\ \text{Inner: } & 4 \cdot x = 4x \\ \text{Last: } & 4 \cdot (-1) = -4 \\ \text{Add: } & x^2 - x + 4x - 4 = x^2 + 3x - 4. \end{aligned}$$

Example with coefficients: $(3x-4)(2x+5)$

$$\begin{aligned} \text{F: } & 3x \cdot 2x = 6x^2 \\ \text{O: } & 3x \cdot 5 = 15x \\ \text{I: } & -4 \cdot 2x = -8x \\ \text{L: } & -4 \cdot 5 = -20 \\ & 6x^2 + 15x - 8x - 20 = 6x^2 + 7x - 20. \end{aligned}$$

FOIL is just the distributive property twice. Every term in the first binomial multiplies every term in the second. That idea is what lets us multiply bigger polynomials next.

9.8 Multiplying a Binomial by a Trinomial

Distribute Each Term

Multiply *each* term of the binomial by *each* term of the trinomial (that is $2 \times 3 = 6$ small products), then combine like terms and write in standard form.

Example: $(2x-1)(x^2+3x-5)$

Distribute $2x$, then distribute -1:

$$2x(x^2+3x-5) = 2x^3+6x^2-10x$$
$$-1(x^2+3x-5) = -x^2-3x+5$$
$$\text{Add: } 2x^3+(6x^2-x^2)+(-10x-3x)+5$$
$$= 2x^3+5x^2-13x+5.$$

Stay organized. Line up like terms in columns as you go, and count your products—for a binomial times a trinomial you should form exactly six before combining.

9.9 Special Products

Two Patterns Worth Memorizing

$$\begin{aligned} \textbf{Difference of squares:}\quad & (a+b)(a-b) = a^2-b^2 \\ \textbf{Perfect-square trinomial:}\quad & (a+b)^2 = a^2+2ab+b^2 \\ & (a-b)^2 = a^2-2ab+b^2 \end{aligned}$$

These come straight from FOIL, but recognizing the pattern saves time.

Difference of squares: $(2x+5)(2x-5)$

Here $a=2x$ and $b=5$, so a^2-b^2:

$$(2x)^2-(5)^2 = 4x^2-25.$$

Check with FOIL: $4x^2-10x+10x-25 = 4x^2-25$. The middle terms cancel.

Perfect square: $(3x-4)^2$

Here $a=3x$ and $b=4$, so $a^2-2ab+b^2$:

$$(3x)^2-2(3x)(4)+(4)^2 = 9x^2-24x+16.$$

Do NOT drop the middle term! $(a+b)^2 \neq a^2+b^2$.

The correct expansion is $(a+b)^2 = a^2+2ab+b^2$. For example, $(x+3)^2 = x^2+6x+9$, *not*

$x^2 + 9$. The $2ab$ term is the one students forget most often.

9.10 Going Deeper: Advanced Polynomials

The Binomial Theorem

FOIL and repeated distribution get slow for high powers. The **Binomial Theorem** expands $(a+b)^n$ all at once:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the **binomial coefficient** $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ counts the ways to choose k of the n factors that contribute a b.

Reading the pattern for a fixed n: the power of a starts at n and *drops* by one each term, the power of b starts at 0 and *rises* by one, and the two exponents in every term add up to n.

Pascal's Triangle

The coefficients $\binom{n}{k}$ are exactly the rows of **Pascal's triangle**. Each entry is the sum of the two entries above it, and every row begins and ends with 1.

$$\begin{array}{rccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

So $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$: the row $1, 4, 6, 4, 1$ becomes the coefficients directly.

Example: Expand $(2x+3)^3$ with the Binomial Theorem

Use row $n = 3$ of Pascal's triangle: $1, 3, 3, 1$. Take $a = 2x$, $b = 3$:

$$\begin{aligned}
(2x+3)^3 &= 1\,(2x)^3 + 3\,(2x)^2(3) + 3\,(2x)(3)^2 + 1\,(3)^3 \\
&= 8x^3 + 3(4x^2)(3) + 3(2x)(9) + 27 \\
&= 8x^3 + 36x^2 + 54x + 27.
\end{aligned}$$

The coefficient is not just the Pascal number. In $(2x+3)^3$ each term also carries the powers of the constants 2 and 3. The Pascal entry 3 became 36 once $(2x)^2$ and 3^1 were multiplied in. Always raise *both* pieces of the binomial to their powers.

Extracting One Coefficient Without Full Expansion

Sometimes you only need a single coefficient of a product. Instead of multiplying everything out, collect *only* the pairs of terms whose degrees add up to the power you want.

To find the coefficient of x^k in a product, list every way to pick one term from each factor so their exponents sum to k, multiply those coefficients, and add the results.

Example: Coefficient of x^2 in $(x^2 + 3x - 4)(2x^2 - x + 5)$

We need every pair of terms whose exponents add to 2:

$$x^2 \cdot 5 = 5x^2$$
$$3x \cdot (-x) = -3x^2$$
$$-4 \cdot 2x^2 = -8x^2$$
$$\text{Sum of coefficients: } 5 + (-3) + (-8) = -6.$$

So the coefficient of x^2 is -6, found without computing any other term.

Recovering $a^2 + b^2$ and $a^3 + b^3$ from $a + b$ and ab

The **sum** $a + b$ and **product** ab contain enough information to rebuild higher symmetric expressions. Starting from $(a + b)^2 = a^2 + 2ab + b^2$ and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, rearrange:

$$a^2 + b^2 = (a + b)^2 - 2ab,$$
$$a^3 + b^3 = (a + b)^3 - 3ab(a + b).$$

The second identity also factors as $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$; both give the same value.

Example: If $a + b = 5$ and $ab = 6$, find $a^2 + b^2$ and $a^3 + b^3$

Substitute directly into the identities:

$$a^2 + b^2 = (a + b)^2 - 2ab = 5^2 - 2(6) = 25 - 12 = 13,$$
$$a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 5^3 - 3(6)(5) = 125 - 90 = 35.$$

(Check: $a = 2, b = 3$ give $a + b = 5$, $ab = 6$, and indeed $4 + 9 = 13$, $8 + 27 = 35$.)

Extending the Special Products: $(a + b + c)^2$

The perfect-square pattern generalizes to three terms. Every term squares, and *every* distinct pair produces a doubled cross term:

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

There are three squares and three cross terms because there are three ways to pair up a, b, and c.

Count the cross terms. A common error is writing only $a^2 + b^2 + c^2$. With n terms there are n squares plus one doubled cross term for *each pair*. For example, $(x + 2 + 3y)^2 = x^2 + 4 + 9y^2 + 4x + 6xy + 12y$.

End Behavior from the Leading Term

For large $|x|$, a polynomial behaves like its **leading term** alone—the highest-degree term dominates every other. So the *degree* and the *sign of the leading coefficient* decide the end behavior:

- **Even degree:** both ends point the same way—up/up if the leading coefficient is positive, down/down if negative.
- **Odd degree:** the ends point opposite ways—down/up if positive, up/down if negative.

Example: End behavior of $-2x^3 + 40x^2 - 7$

Ignore all but the leading term $-2x^3$. The degree is odd (3) and the leading coefficient is negative (-2), so the ends point opposite ways: as $x \to +\infty$ the value $\to -\infty$, and as $x \to -\infty$ the value $\to +\infty$ (up on the left, down on the right). The large $+40x^2$ term changes the middle shape but never the ends.

Polynomial Long Division (Preview)

When a polynomial does not divide evenly, **long division** splits it into a quotient plus a remainder, mirroring long division of numbers. At each step: divide the leading term of the current dividend by the leading term of the divisor, multiply back, and subtract.

Example: Divide $x^2 + 5x + 7$ by $x + 2$

$$\begin{aligned} x^2 \div x = x, &\quad \text{and} \quad x(x+2) = x^2 + 2x, \\ (x^2+5x+7) - (x^2+2x) = 3x+7, \\ 3x \div x = 3, &\quad \text{and} \quad 3(x+2) = 3x+6, \\ (3x+7) - (3x+6) = 1 \quad \text{(remainder)}. \end{aligned}$$

So $x^2 + 5x + 7 = (x+2)(x+3) + 1$, i.e. the quotient is $x + 3$ with remainder 1.

Keep everything in standard form and hold placeholders. Before dividing, order both polynomials by descending degree and insert 0 coefficients for any missing powers (e.g. write $x^3 + 1$ as $x^3 + 0x^2 + 0x + 1$) so like terms line up during each subtraction.

10 Factoring

Turning sums into products — the reverse of multiplying

10.1 What Factoring Is (and Why It Helps)

The Big Idea

Factoring means writing an expression as a *product* (things multiplied together). It is the **reverse of distributing/multiplying**.

$$\underbrace{3(x+2)}_{\text{factored form}} \xrightarrow{\text{multiply}} \underbrace{3x+6}_{\text{expanded form}} \xrightarrow{\text{factor}} \underbrace{3(x+2)}_{\text{factored form}}$$

When we distribute we go *forward*; when we factor we go *backward*.

Why it is useful: a factored form shows the hidden building blocks of an expression. It lets us simplify fractions, and — most importantly — it lets us **solve equations** that equal zero (see the last section). Factoring is the key that unlocks quadratics.

Always check your factoring by multiplying back out. If the product equals what you started with, you are correct. This one habit catches almost every mistake.

10.2 Greatest Common Factor (GCF)

Method: Factor Out the GCF

The **GCF** of several terms is the largest factor they *all* share.

1. **Numbers:** take the largest number that divides every coefficient.
2. **Variables:** take each shared variable to its *smallest* exponent that appears.
3. Write GCF $\times$ (what is left in each term).

This uses the distributive property in reverse: $ab + ac = a(b + c)$.

Example: Factor $12x^3 + 18x^2$

Coefficients 12 and 18 share GCF 6. Both terms have x; the smaller power is x^2. So the GCF is $6x^2$.

$$\begin{aligned} 12x^3 + 18x^2 &= 6x^2(2x) + 6x^2(3) \\ &= 6x^2(2x + 3). \end{aligned}$$

Check: $6x^2(2x + 3) = 12x^3 + 18x^2$. ✓

Example: Factor $15a^2b - 20ab^2 + 5ab$

Numbers $15, 20, 5$ share GCF 5. Every term has at least one a and one b, so pull $5ab$.

$$15a^2b - 20ab^2 + 5ab = 5ab(3a - 4b + 1).$$

Check: $5ab(3a - 4b + 1) = 15a^2b - 20ab^2 + 5ab$. ✓(Do not forget the +1!)

Tip: When one whole term *is* the GCF, a 1 is left behind in its place. Missing that 1 is the most common GCF error.

10.3 Factoring by Grouping (Four Terms)

Method

With **four terms**, group them in pairs, factor the GCF from each pair, and then factor out the matching binomial.

$$ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b).$$

Example: Factor $x^3 + 3x^2 + 2x + 6$

$$\begin{aligned} x^3+3x^2+2x+6 &= (x^3+3x^2)+(2x+6)\\ &= x^2(x+3)+2(x+3)\\ &= (x+3)(x^2+2). \end{aligned}$$

Check: $(x+3)(x^2+2) = x^3+2x+3x^2+6$. ✓

Tip: The two parentheses *must match* after step two. If they do not, try grouping the terms in a different order, or factor out a negative so the signs line up.

10.4 Trinomials x^2+bx+c (Leading Coefficient 1)

Method: Find Two Numbers

To factor x^2+bx+c, find two numbers that

multiply to c and **add to** b.

If those numbers are p and q, then $x^2+bx+c = (x+p)(x+q)$.

Example: Factor $x^2+7x+12$

Need two numbers with product 12 and sum 7: those are 3 and 4.

$$x^2+7x+12 = (x+3)(x+4).$$

Check: $(x+3)(x+4) = x^2+4x+3x+12 = x^2+7x+12$. ✓

Example: Factor x^2-x-12

Product -12, sum -1: those are -4 and 3.

$$x^2-x-12 = (x-4)(x+3).$$

Check: $(x-4)(x+3) = x^2+3x-4x-12 = x^2-x-12$. ✓

Sign patterns: If $c>0$, the two numbers share the sign of b. If $c<0$, they have *opposite* signs, and the bigger one carries the sign of b.

10.5 Trinomials ax^2+bx+c (Leading Coefficient $\neq 1$)

The "ac" / Grouping Method

For $ax^2 + bx + c$:

1. Multiply $a \cdot c$.
2. Find two numbers that multiply to ac and add to b.
3. **Split** the middle term bx using those two numbers.
4. **Group** and factor.

Example: Factor $2x^2 + 7x + 3$

Here $ac = 2 \cdot 3 = 6$. Two numbers multiplying to 6, adding to 7: 6 and 1.

$$\begin{aligned} 2x^2 + 7x + 3 &= 2x^2 + 6x + 1x + 3 \\ &= 2x(x+3) + 1(x+3) \\ &= (x+3)(2x+1). \end{aligned}$$

Check: $(x+3)(2x+1) = 2x^2 + x + 6x + 3 = 2x^2 + 7x + 3$. ✓

Example: Factor $6x^2 - x - 2$

$ac = 6 \cdot (-2) = -12$. Numbers multiplying to -12, adding to -1: -4 and 3.

$$\begin{aligned} 6x^2 - x - 2 &= 6x^2 - 4x + 3x - 2 \\ &= 2x(3x-2) + 1(3x-2) \\ &= (3x-2)(2x+1). \end{aligned}$$

Check: $(3x-2)(2x+1) = 6x^2 + 3x - 4x - 2 = 6x^2 - x - 2$. ✓

10.6 Difference of Squares

The Pattern

$$a^2 - b^2 = (a+b)(a-b).$$

A *difference* of two perfect squares always factors this way.

Example: Factor $x^2 - 25$

Here $x^2 = (x)^2$ and $25 = (5)^2$, so $a = x$, $b = 5$:

$$x^2 - 25 = (x+5)(x-5).$$

Check: $(x+5)(x-5) = x^2 - 5x + 5x - 25 = x^2 - 25$. ✓

Example: Factor $9y^2 - 16$

$9y^2 = (3y)^2$ and $16 = (4)^2$:

$$9y^2 - 16 = (3y+4)(3y-4).$$

Check: $(3y+4)(3y-4) = 9y^2 - 16$. ✓

Important: A *sum* of squares $a^2 + b^2$ does **not** factor over the real numbers. Only the *difference* does.

10.7 Perfect-Square Trinomials

The Patterns

$$a^2 + 2ab + b^2 = (a+b)^2,$$
$$a^2 - 2ab + b^2 = (a-b)^2.$$

Spot these when the first and last terms are perfect squares and the middle term is *twice* the product of their roots.

Example: Factor $x^2 + 6x + 9$

$x^2 = (x)^2$, $9 = (3)^2$, and the middle term $6x = 2 \cdot x \cdot 3$. It fits!

$$x^2 + 6x + 9 = (x+3)^2.$$

Check: $(x+3)^2 = x^2 + 6x + 9$. ✓

Example: Factor $4x^2 - 20x + 25$

$4x^2 = (2x)^2$, $25 = (5)^2$, middle $-20x = -2 \cdot 2x \cdot 5$.

$$4x^2 - 20x + 25 = (2x-5)^2.$$

Check: $(2x-5)^2 = 4x^2 - 20x + 25$. ✓

10.8 Factoring Completely

Method

Always pull out the GCF first, then keep factoring each piece until nothing else factors. An expression is factored *completely* when every factor is prime.

Example: Factor completely $3x^2 - 27$

First the GCF 3, then a difference of squares:

$$\begin{aligned} 3x^2 - 27 &= 3(x^2 - 9) \\ &= 3(x+3)(x-3). \end{aligned}$$

Check: $3(x+3)(x-3) = 3(x^2-9) = 3x^2 - 27$. ✓

Example: Factor completely $2x^3 + 10x^2 + 12x$

GCF is $2x$, then factor the trinomial:

$$\begin{aligned} 2x^3 + 10x^2 + 12x &= 2x(x^2 + 5x + 6) \\ &= 2x(x+2)(x+3). \end{aligned}$$

Check: $2x(x+2)(x+3) = 2x(x^2+5x+6) = 2x^3 + 10x^2 + 12x$. ✓

Tip: After you finish, glance at every factor once more. Students often stop one step early — a leftover $x^2 - 9$ still factors!

10.9 Zero Product Property (Solving by Factoring)

The Property

If a product equals zero, then at least one factor must be zero:

$$\text{if } A \cdot B = 0, \quad \text{then } A = 0 \text{ or } B = 0.$$

So to solve an equation: get one side $= 0$, **factor**, then set each factor equal to zero.

Example: Solve $x^2 + 2x - 15 = 0$

$$\begin{aligned} x^2 + 2x - 15 &= 0 \\ (x+5)(x-3) &= 0 \\ x+5 = 0 \quad &\text{or} \quad x-3=0 \\ x = -5 \quad &\text{or} \quad x = 3. \end{aligned}$$

Check: $(-5)^2 + 2(-5) - 15 = 25 - 10 - 15 = 0.$ ✓

Tip: The equation *must* equal zero before you factor. $(x+5)(x-3) = 7$ does **not** mean $x + 5 = 7$; move everything to one side first.

10.10 Going Deeper: Advanced Factoring

Sum and Difference of Cubes

Two more patterns join the difference of squares. Notice the sign rule: the binomial matches the sign in the middle, and the trinomial has the *opposite* middle sign and no doubling.

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 - ab + b^2), \\ a^3 - b^3 &= (a-b)(a^2 + ab + b^2). \end{aligned}$$

A memory aid is **SOAP**: **S**ame sign, **O**pposite sign, **A**lways **P**ositive (for the last term).

Example: Factor $8x^3 + 27$

Recognize the cubes: $8x^3 = (2x)^3$ and $27 = (3)^3$, so $a = 2x$, $b = 3$. This is a *sum* of cubes.

$$\begin{aligned} 8x^3 + 27 &= (2x)^3 + (3)^3 \\ &= (2x+3)\big((2x)^2 - (2x)(3) + 3^2\big) \\ &= (2x+3)(4x^2 - 6x + 9). \end{aligned}$$

Check: $(2x+3)(4x^2 - 6x + 9) = 8x^3 - 12x^2 + 18x + 12x^2 - 18x + 27 = 8x^3 + 27.$ ✓

Example: Factor $x^3 - 64$

Here $x^3 = (x)^3$ and $64 = (4)^3$, a *difference* of cubes with $a = x$, $b = 4$.

$$x^3 - 64 = (x-4)(x^2 + 4x + 16).$$

Check: $(x-4)(x^2 + 4x + 16) = x^3 + 4x^2 + 16x - 4x^2 - 16x - 64 = x^3 - 64.$ ✓

Quartics in Quadratic Form

An expression like $x^4 + Bx^2 + C$ is secretly a quadratic in disguise. **Substitute** $u = x^2$ (so $u^2 = x^4$), factor the ordinary quadratic in u, then swap x^2 back in and keep factoring.

Example: Factor completely $x^4 - 13x^2 + 36$

Let $u = x^2$. Then we need two numbers multiplying to 36 and adding to -13: those are -4 and -9.

$$\begin{aligned} x^4 - 13x^2 + 36 &= u^2 - 13u + 36 \\ &= (u-4)(u-9) \\ &= (x^2-4)(x^2-9) \\ &= (x-2)(x+2)(x-3)(x+3). \end{aligned}$$

Each $x^2 - 4$ and $x^2 - 9$ is itself a difference of squares — do not stop early!
Check: $(x^2-4)(x^2-9) = x^4 - 9x^2 - 4x^2 + 36 = x^4 - 13x^2 + 36$. ✓

The Add-and-Subtract-a-Square Trick (Sophie Germain)

A *sum* of squares usually will not factor — but sometimes we can **create** a difference of squares by adding and subtracting a middle term. For $x^4 + 4$, add and subtract $4x^2$ to complete a perfect square:

$$x^4 + 4 = \underbrace{x^4 + 4x^2 + 4}_{(x^2+2)^2} - 4x^2 = (x^2+2)^2 - (2x)^2.$$

Now it is a difference of squares! This is the **Sophie Germain identity**.

Example: Factor $x^4 + 4$

Using the difference of squares $A^2 - B^2 = (A+B)(A-B)$ with $A = x^2 + 2$ and $B = 2x$:

$$\begin{aligned} x^4 + 4 &= (x^2+2)^2 - (2x)^2 \\ &= (x^2+2+2x)(x^2+2-2x) \\ &= (x^2+2x+2)(x^2-2x+2). \end{aligned}$$

Check: $(x^2+2x+2)(x^2-2x+2) = x^4 - 2x^3 + 2x^2 + 2x^3 - 4x^2 + 4x + 2x^2 - 4x + 4 = x^4 + 4$. ✓

Factoring as a Calculator Shortcut

The difference of squares works on *numbers*, not just variables. It can turn a scary subtraction into an easy multiplication:

$$a^2 - b^2 = (a+b)(a-b).$$

Example: Evaluate $2024^2 - 2023^2$ without a calculator

Let $a = 2024$ and $b = 2023$:

$$2024^2 - 2023^2 = (2024+2023)(2024-2023) = (4047)(1) = 4047.$$

No large squaring needed — the difference of consecutive squares is just their sum. ✓

Reading Off the Sum and Product of the Solutions

If a quadratic factors as $x^2 + bx + c = (x-r)(x-s)$, then multiplying out shows

$$x^2 + bx + c = x^2 - (r+s)x + rs.$$

Matching coefficients gives **Vieta's relations**:

$$r + s = -b \qquad \text{and} \qquad rs = c.$$

So you can find the *sum* and *product* of the roots straight from the equation, **without solving it**.

Watch the signs. In $(x-r)(x-s)$ the roots are r and s, but the coefficient is $-(r+s)$. For $x^2-7x+12$, the roots sum to $+7$ (not -7) and multiply to 12; indeed $x^2-7x+12 = (x-3)(x-4)$ with $3+4=7$ and $3\cdot 4 = 12$.

Preview: The Factor Theorem

How do you factor something that is not a nice pattern, like x^3-7x+6? The **Factor Theorem** (studied fully in Algebra 2) says:

$$(x-r) \text{ is a factor of a polynomial } p(x) \iff p(r) = 0.$$

In words: if plugging in $x = r$ makes the whole polynomial equal zero, then $(x-r)$ divides it evenly. So you can *test* candidate roots to discover a factor.

Example: Use the Factor Theorem to start factoring $x^3 - 7x + 6$

Test small integers. Try $x = 1$:

$$p(1) = 1^3 - 7(1) + 6 = 1 - 7 + 6 = 0.$$

Since $p(1) = 0$, the theorem guarantees $(x-1)$ is a factor. Dividing out $(x-1)$ leaves x^2+x-6, which factors normally:

$$x^3 - 7x + 6 = (x-1)(x^2+x-6) = (x-1)(x+3)(x-2).$$

Check: $(x-1)(x+3)(x-2) = (x-1)(x^2+x-6) = x^3+x^2-6x-x^2-x+6 = x^3-7x+6$. ✓

Where to look for roots: for a polynomial with integer coefficients, any integer root must *divide the constant term*. For $x^3 - 7x + 6$ the constant is 6, so only $\pm1, \pm2, \pm3, \pm6$ are worth testing. This is a first taste of the *Rational Root Theorem*.

11 Quadratics

Parabolas, factoring, square roots, completing the square, the quadratic formula, the discriminant, vertex form, and applications

11.1 What Is a Quadratic?

Definition and Standard Form

A **quadratic function** is one whose highest power of the variable is 2. Its **standard form** is

$$y = ax^2 + bx + c, \qquad a \neq 0.$$

A **quadratic equation** sets a quadratic equal to zero: $ax^2 + bx + c = 0$. The numbers a, b, and c are the **coefficients**:

- a is the coefficient of x^2 (it can never be 0, or the x^2 would vanish);
- b is the coefficient of x;
- c is the constant term.

Example: Identify a, b, c in $3x^2 - 5x + 2 = 0$

Match term by term:

$$a = 3, \qquad b = -5, \qquad c = 2.$$

Carry the sign! The middle coefficient is -5, not 5.

Example: A missing middle term, $2x^2 - 7 = 0$

Here there is no x term, so $b = 0$:

$$a = 2, \qquad b = 0, \qquad c = -7.$$

How to recognize one: A relationship is quadratic exactly when, after simplifying, the highest power of the variable is 2. $y = 4x - 1$ is *linear* (highest power 1); $y = x^2 - 9$ is *quadratic*. If you can multiply it out to reach an x^2 term with $a \neq 0$, it is quadratic.

11.2 The Parabola: The Graph of a Quadratic

Shape, Direction, and Key Points

The graph of $y = ax^2 + bx + c$ is a smooth U-shaped curve called a **parabola**.

- **Direction:** if $a > 0$ the parabola *opens up* (holds water, has a lowest point); if $a < 0$ it *opens down* (has a highest point).
- **Vertex:** the turning point. Its x-coordinate is $x = -\dfrac{b}{2a}$; find the y-coordinate by substituting that x back in.
- **Axis of symmetry:** the vertical line $x = -\dfrac{b}{2a}$ through the vertex; the two halves mirror each other.
- **Minimum vs. maximum:** an up-parabola's vertex is a *minimum*; a down-parabola's vertex is a *maximum*.
- **y-intercept:** where the curve crosses the y-axis; set $x = 0$ to get the point $(0, c)$.
- **x-intercepts (roots / zeros):** where the curve crosses the x-axis; set $y = 0$ and solve $ax^2 + bx + c = 0$. A parabola can have two, one, or no x-intercepts.

The vertex is the turning point.

(diagram in the online version)

Example: Describe the parabola $y = x^2 - 6x + 5$

Here $a = 1$, $b = -6$, $c = 5$.

$$\begin{aligned}
\text{Direction: } & a = 1 > 0 \text{ so it opens } \textbf{up} \text{ (vertex is a minimum).} \\
\text{Axis of symmetry: } & x = -\frac{b}{2a} = -\frac{-6}{2(1)} = 3. \\
\text{Vertex } y\text{: } & y = (3)^2 - 6(3) + 5 = 9 - 18 + 5 = -4. \\
\text{Vertex: } & (3, -4). \\
y\text{-intercept: } & (0,\ 5) \text{ since } c = 5.
\end{aligned}$$

The x-intercepts come from $x^2 - 6x + 5 = (x-1)(x-5) = 0$, giving $(1, 0)$ and $(5, 0)$.

Two ways to describe the same curve: the axis of symmetry and the vertex share the same x-value, $x = -\frac{b}{2a}$. Find that number *once* and you have both.

11.3 Solving by Factoring (Zero Product Property)

The Zero Product Property

If a product of two factors equals zero, then at least one factor must be zero:

$$\text{if } A \cdot B = 0, \text{ then } A = 0 \text{ or } B = 0.$$

To use it, first make one side 0, factor the quadratic, set each factor equal to 0, and solve.

Example: Solve $x^2 + 7x + 12 = 0$

Factor into two binomials (two numbers that multiply to 12 and add to 7: that is 3 and 4):

$$\begin{aligned}
(x+3)(x+4) &= 0 \\
x + 3 = 0 \quad &\text{or} \quad x + 4 = 0 \\
x = -3 \quad &\text{or} \quad x = -4.
\end{aligned}$$

Check $x = -3$: $(-3)^2 + 7(-3) + 12 = 9 - 21 + 12 = 0.$ ✓

Example: A common factor first, $x^2 + 2x = 0$

Factor out x:

$$x(x+2) = 0 \ \Rightarrow\ x = 0 \ \text{ or } \ x = -2.$$

Do *not* divide both sides by x—that would lose the solution $x = 0$.

Set it equal to zero first! The Zero Product Property only works against 0. Something like $(x+3)(x+4)=2$ tells you *nothing*; you must expand, move the 2 over, and refactor.

11.4 Solving by the Square-Root Method

Take the Square Root of Both Sides

When a quadratic has no plain x term, isolate the squared quantity and take the square root of both sides—remembering *both* signs:

$$x^2 = k \;\Rightarrow\; x = \pm\sqrt{k}, \qquad (x-h)^2 = k \;\Rightarrow\; x-h = \pm\sqrt{k}.$$

This works whenever the variable appears only inside a square.

Example: Solve $x^2 = 49$

$$x = \pm\sqrt{49} = \pm 7, \qquad \text{so } x = 7 \text{ or } x = -7.$$

Example: A shifted square, $(x-2)^2 = 16$

Take the square root of both sides:

$$\begin{aligned} x - 2 &= \pm\sqrt{16} = \pm 4 \\ x &= 2 \pm 4 \\ x &= 6 \quad \text{or} \quad x = -2. \end{aligned}$$

Example: When the root is irrational, $x^2 = 12$

$$x = \pm\sqrt{12} = \pm\sqrt{4\cdot 3} = \pm 2\sqrt{3}.$$

Always simplify the radical by pulling out perfect-square factors.

Never forget the $\pm$. Every positive number has two square roots, one positive and one negative. Dropping the $\pm$ throws away a whole solution.

11.5 Solving by Completing the Square

Building a Perfect Square

Completing the square rewrites $x^2 + bx$ as a perfect square by adding $\left(\dfrac{b}{2}\right)^2$. The steps for $x^2 + bx + c = 0$:

1. Move the constant to the right side.
2. Add $\left(\frac{b}{2}\right)^2$ to *both* sides.
3. Write the left side as $\left(x + \frac{b}{2}\right)^2$.
4. Take the square root of both sides $(\pm!)$ and solve for x.

If $a \neq 1$, divide every term by a first.

Example: Solve $x^2 + 6x - 7 = 0$ by completing the square

$$\begin{aligned} x^2 + 6x &= 7 && \text{move the constant over} \\ \left(\tfrac{6}{2}\right)^2 &= 3^2 = 9 && \text{the number to add} \\ x^2 + 6x + 9 &= 7 + 9 && \text{add 9 to both sides} \\ (x+3)^2 &= 16 && \text{left side is a perfect square} \\ x + 3 &= \pm 4 && \text{square root of both sides} \\ x &= -3 \pm 4 \\ x &= 1 \quad \text{or} \quad x = -7. \end{aligned}$$

Check $x = 1$: $1 + 6 - 7 = 0$. ✓

Example: An odd middle term, $x^2 - 3x - 5 = 0$

$$\begin{aligned} x^2 - 3x &= 5 \\ \left(\tfrac{-3}{2}\right)^2 &= \tfrac{9}{4} \\ x^2 - 3x + \tfrac{9}{4} &= 5 + \tfrac{9}{4} = \tfrac{29}{4} \\ \left(x - \tfrac{3}{2}\right)^2 &= \tfrac{29}{4} \\ x - \tfrac{3}{2} &= \pm\frac{\sqrt{29}}{2} \\ x &= \frac{3 \pm \sqrt{29}}{2}. \end{aligned}$$

An odd b just means the number you add, $\left(\frac{b}{2}\right)^2$, is a fraction. That is fine.

Whatever you do to one side, do to the other. You must add $\left(\frac{b}{2}\right)^2$ to *both* sides to keep the equation balanced. And if $a \neq 1$, divide through by a before completing the square.

11.6 The Quadratic Formula

The Formula That Always Works

For any quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

It comes from completing the square on the general equation, so it solves *every* quadratic—even ones that will not factor nicely.

Example: Solve $2x^2 + 3x - 2 = 0$ with the formula

Identify $a = 2$, $b = 3$, $c = -2$, then substitute carefully:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-3 \pm \sqrt{(3)^2 - 4(2)(-2)}}{2(2)} \\ &= \frac{-3 \pm \sqrt{9 + 16}}{4} \\ &= \frac{-3 \pm \sqrt{25}}{4} = \frac{-3 \pm 5}{4}. \end{aligned}$$

Split the $\pm$: $\quad x = \dfrac{-3 + 5}{4} = \dfrac{1}{2}$ or $x = \dfrac{-3 - 5}{4} = -2.$

Example: An irrational answer, $x^2 - 4x + 1 = 0$

Here $a = 1$, $b = -4$, $c = 1$:

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2} \\ &= \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}. \end{aligned}$$

Simplify $\sqrt{12} = 2\sqrt{3}$, then divide every term of the top by the bottom.

Use parentheses when you substitute, especially for a negative b: $-(-4) = +4$. Compute the part under the root, $b^2 - 4ac$, first—and remember b^2 is always positive, even when b is negative.

11.7 The Discriminant

The Number Under the Radical

The **discriminant** is the expression inside the square root of the quadratic formula:

$$D = b^2 - 4ac.$$

Its sign tells you how many *real* solutions the equation has, before you ever finish solving:

- $D > 0$: **two** different real solutions (parabola crosses the x-axis twice);
- $D = 0$: **one** real solution, a repeated root (parabola just touches the x-axis);
- $D < 0$: **no** real solutions (parabola never reaches the x-axis).

Example: How many real solutions does $x^2 - 3x - 4 = 0$ have?

$$D = (-3)^2 - 4(1)(-4) = 9 + 16 = 25 > 0.$$

Since $D > 0$, there are **two** real solutions. (And because 25 is a perfect square, they are even rational.)

Example: A negative discriminant, $x^2 + x + 5 = 0$

$$D = (1)^2 - 4(1)(5) = 1 - 20 = -19 < 0.$$

Since $D < 0$, there are **no real solutions**—the parabola sits entirely above the x-axis.

The discriminant is a preview. It tells you the *number and type* of solutions without solving. A perfect-square D also signals that the quadratic factors over the rationals.

11.8 Vertex Form

Reading the Vertex Directly

The **vertex form** of a quadratic is

$$y = a(x - h)^2 + k,$$

where the vertex is (h, k) and $x = h$ is the axis of symmetry. As always, $a > 0$ opens up (minimum at the vertex) and $a < 0$ opens down (maximum). Watch the sign: the form uses $(x - h)$, so $y = (x + 1)^2 - 5$ has $h = -1$.

The vertex is the turning point.

(diagram in the online version)

Example: Read the vertex of $y = 2(x - 3)^2 + 4$

Compare to $y = a(x - h)^2 + k$: $a = 2$, $h = 3$, $k = 4$.

$$\text{Vertex } (3, 4), \quad \text{opens up, so } (3, 4) \text{ is a } \textbf{minimum}.$$

Example: Convert $y = x^2 - 6x + 5$ to vertex form

Complete the square on the right side:

$$\begin{aligned} y &= (x^2 - 6x) + 5 \\ &= (x^2 - 6x + 9) + 5 - 9 && \text{add and subtract } \left(\tfrac{6}{2}\right)^2 = 9 \\ &= (x - 3)^2 - 4. \end{aligned}$$

So the vertex is $(3, -4)$—matching what $x = -\frac{b}{2a}$ gave us earlier.

Mind the signs. In $y = a(x - h)^2 + k$, the vertex is (h, k) with the sign of h *flipped* from what you see: $y = (x + 2)^2 - 3$ has vertex $(-2, -3)$.

11.9 Applications

Turning Words into a Quadratic

Many real situations are modeled by quadratics:

- **Projectile height:** an object thrown or launched has height $h = -16t^2 + v_0 t + h_0$ (feet, seconds), where v_0 is the starting speed and h_0 the starting height. Set $h = 0$ to find when it lands; use $t = -\frac{b}{2a}$ for the time of greatest height.
- **Area:** length $\times$ width often produces an x^2 term.

Solve, then *keep only answers that make sense* (a length or a time cannot be negative).

Example: A ball's height is $h = -16t^2 + 64t$

When does it hit the ground? Set $h = 0$ and factor:

$$\begin{aligned} -16t^2 + 64t &= 0 \\ -16t(t-4) &= 0 \\ t = 0 \quad \text{or} \quad t &= 4. \end{aligned}$$

$t = 0$ is the launch; the ball returns to the ground at $t = 4$ seconds.

When is it highest? At $t = -\frac{b}{2a} = -\frac{64}{2(-16)} = 2$ seconds, where $h = -16(2)^2 + 64(2) = 64$ feet.

Example: A rectangle whose length is 3 more than its width has area 54

Let the width be w. Then $w(w + 3) = 54$:

$$\begin{aligned} w^2 + 3w - 54 &= 0 \\ (w+9)(w-6) &= 0 \\ w = -9 \quad \text{or} \quad w &= 6. \end{aligned}$$

A width cannot be negative, so $w = 6$: the rectangle is 6 by 9.

Check that your answer fits reality. A quadratic often gives two solutions, but a negative time or length is not physically possible—discard it and state only the meaningful answer with its units.

11.10 Going Deeper: Advanced Quadratics

Where the Quadratic Formula Comes From

The quadratic formula is not magic—it is completing the square done *once and for all* on the general equation $ax^2 + bx + c = 0$. Because the derivation never uses specific numbers, the result works for *every* quadratic. The key move is the same as before: turn $x^2 + \frac{b}{a}x$ into a perfect square by adding $\left(\frac{b}{2a}\right)^2$ to both sides. Master this derivation and you never have to memorize the formula blindly again.

The vertex is the turning point.

(diagram in the online version)

Worked Example: Derive $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Start from the general equation and complete the square, tracking every step:

$$
\begin{aligned}
ax^2 + bx + c &= 0 && \text{start} \\
x^2 + \tfrac{b}{a}x + \tfrac{c}{a} &= 0 && \text{divide every term by } a \\
x^2 + \tfrac{b}{a}x &= -\tfrac{c}{a} && \text{move the constant over} \\
x^2 + \tfrac{b}{a}x + \left(\tfrac{b}{2a}\right)^2 &= \left(\tfrac{b}{2a}\right)^2 - \tfrac{c}{a} && \text{add } \left(\tfrac{b}{2a}\right)^2 \text{ to both sides} \\
\left(x + \tfrac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} && \text{perfect square; common denominator} \\
x + \tfrac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} && \text{square root of both sides } (\pm!) \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{subtract } \tfrac{b}{2a}\text{, combine over } 2a.
\end{aligned}
$$

The discriminant $b^2 - 4ac$ appears naturally as the numerator under the root—it was there in the algebra all along.

The Discriminant and the Nature of the Roots

For a quadratic with *rational* coefficients, the discriminant $D = b^2 - 4ac$ tells you not just how many roots, but what *kind*:

- $D > 0$ and a **perfect square**: two *rational* roots (the quadratic factors over the rationals);
- $D > 0$ but **not** a perfect square: two *irrational* roots, a conjugate pair $\frac{-b}{2a} \pm \frac{\sqrt{D}}{2a}$;
- $D = 0$: one *repeated* rational root (a "double root"); the parabola is tangent to the x-axis;
- $D < 0$: no real roots—two *complex conjugate* roots (see below).

Sum and Product of the Roots (Vieta's Formulas)

If r_1 and r_2 are the two roots of $ax^2 + bx + c = 0$, then

$$r_1 + r_2 = -\frac{b}{a}, \qquad r_1 r_2 = \frac{c}{a}.$$

You can see why by expanding $a(x - r_1)(x - r_2) = ax^2 - a(r_1 + r_2)x + a\,r_1 r_2$ and matching coefficients. For a *monic* quadratic ($a = 1$), this reads especially cleanly:

$$x^2 - (\text{sum})\,x + (\text{product}) = 0.$$

These give a fast check on any answer, and let you *build* a quadratic straight from conditions on its roots.

Worked Example: Build a quadratic from conditions

(a) Find a quadratic whose roots are 3 and -5. Use the sum and product:

$$\text{sum} = 3 + (-5) = -2, \qquad \text{product} = (3)(-5) = -15.$$

So $x^2 - (\text{sum})x + (\text{product}) = x^2 - (-2)x + (-15) = x^2 + 2x - 15 = 0$.

(b) Find a quadratic whose roots sum to 6 and multiply to 10. Directly,

$$x^2 - 6x + 10 = 0.$$

Check with the discriminant: $D = (-6)^2 - 4(1)(10) = 36 - 40 = -4 < 0$, so those two roots are a complex conjugate pair—consistent, since no two *real* numbers add to 6 and multiply to 10.

Vertex Form for Maximum/Minimum Optimization

Rewriting $y = ax^2 + bx + c$ as $y = a(x-h)^2 + k$ turns optimization into *reading off the vertex*: the extreme value of y is exactly k, reached at $x = h$.

- If $a > 0$, the vertex is the **lowest** point, so k is the *minimum* value.
- If $a < 0$, the vertex is the **highest** point, so k is the *maximum* value.

This is the standard way to answer "what is the largest area / smallest cost / greatest height?"

Worked Example: Maximize a rectangular area

A farmer has 40 m of fencing for a rectangular pen. What dimensions give the largest area? Let the width be w; then the length is $20 - w$ (since $2w + 2\ell = 40$). The area is

$$\begin{aligned} A(w) &= w(20-w) = 20w - w^2 \\ &= -\left(w^2 - 20w\right) \\ &= -\left(w^2 - 20w + 100\right) + 100 && \text{complete the square, add/subtract 100} \\ &= -(w-10)^2 + 100. \end{aligned}$$

Since $a = -1 < 0$, the vertex $(10, 100)$ is a **maximum**: the area is greatest at $w = 10$ m, giving a 10×10 square with area **100** m^2. (A square always maximizes area for a fixed perimeter.)

A First Look at Complex Roots

When $D = b^2 - 4ac < 0$, the quadratic formula asks for the square root of a negative number, which no real number provides. We introduce the **imaginary unit**

$$i = \sqrt{-1}, \qquad \text{so} \qquad i^2 = -1.$$

Then $\sqrt{-16} = \sqrt{16}\sqrt{-1} = 4i$, and the formula still works. The two roots come out as a **complex conjugate pair** $p \pm qi$: same real part $p = -\frac{b}{2a}$, opposite imaginary parts. Graphically, the parabola simply never crosses the x-axis.

Worked Example: Complex roots of $x^2 + 2x + 5 = 0$

Here $a = 1$, $b = 2$, $c = 5$, and $D = (2)^2 - 4(1)(5) = 4 - 20 = -16 < 0$:

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{-16}}{2} \\ &= \frac{-2 \pm \sqrt{16}\sqrt{-1}}{2} = \frac{-2 \pm 4i}{2} \\ &= -1 \pm 2i. \end{aligned}$$

The roots are $-1 + 2i$ and $-1 - 2i$. **Vieta check:** their sum is $-2 = -\frac{b}{a}$ ✓ and their product is $(-1)^2 - (2i)^2 = 1 - 4i^2 = 1 + 4 = 5 = \frac{c}{a}$ ✓.

The Line-Tangent-to-Parabola Condition

To find where a line $y = mx + d$ meets a parabola $y = ax^2 + bx + c$, set them equal and collect everything on one side to get a single quadratic in x. The **discriminant of that quadratic** decides the geometry:

- $D > 0$: the line *cuts* the parabola at two points (a secant);
- $D = 0$: the line *just touches* it at one point—it is **tangent**;
- $D < 0$: the line *misses* the parabola entirely.

So "tangent" is precisely the condition $D = 0$: one repeated intersection.

Worked Example: For which k is $y = x + k$ tangent to $y = x^2$?

Set the two expressions equal and move everything to one side:

$$x^2 = x + k \quad \Longrightarrow \quad x^2 - x - k = 0.$$

This is a quadratic in x with $a = 1$, $b = -1$, $c = -k$. Tangency means one repeated root, i.e. $D = 0$:

$$D = (-1)^2 - 4(1)(-k) = 1 + 4k = 0 \quad \Longrightarrow \quad k = -\tfrac{1}{4}.$$

At $k = -\frac{1}{4}$ the equation becomes $x^2 - x + \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 = 0$, so the single point of contact is $x = \frac{1}{2}$, $y = \frac{1}{4}$: the point $\left(\frac{1}{2}, \frac{1}{4}\right)$.

The discriminant ties it all together. The same expression $b^2 - 4ac$ counts real roots, classifies them as rational / irrational / complex, and—when applied to the "set them equal" equation—tells whether a line cuts, touches, or misses a parabola. And Vieta's formulas $r_1 + r_2 = -\frac{b}{a}$, $r_1 r_2 = \frac{c}{a}$ give a fast check on *any* pair of roots, real or complex.

12 Radicals

Square roots, simplifying, operations, and radical equations

12.1 Square Roots & Perfect Squares

The Big Idea

A **square root** of a number n is a value that, when multiplied by itself, gives n. Because $5 \times 5 = 25$, we say 5 is a square root of 25.

The symbol $\sqrt{}$ is called the **radical sign**. The number under it is the **radicand**. The small number tucked into the notch is the **index** (it is a hidden 2 for square roots).

$$\sqrt[\text{index}]{\text{radicand}} \qquad\qquad \sqrt{25} = \sqrt[2]{25} = 5$$

A **perfect square** is a number whose square root is a whole number (like $1, 4, 9, 16, \ldots$).

Principal Root

Every positive number really has two square roots: $5 \times 5 = 25$ *and* $(-5) \times (-5) = 25$. The symbol $\sqrt{\ }$ always means the **principal (positive) root**.

$$\sqrt{25} = 5, \qquad -\sqrt{25} = -5, \qquad \pm\sqrt{25} = \pm 5$$

Also, $\sqrt{0} = 0$, and you cannot take $\sqrt{\text{a negative number}}$ in Algebra 1 (there is no real number that squares to a negative).

Reading a radical

Evaluate $\sqrt{81}$.
Ask: what positive number times itself is 81? Since $9 \times 9 = 81$, we get $\sqrt{81} = 9$.

12.2 Perfect Squares to Know by Heart

Remember: $\sqrt{n^2} = n$ for a positive number n. Squaring and taking a square root are opposite (inverse) operations.

12.3 Estimating Irrational Square Roots

Trapped Between Two Integers

If a number is *not* a perfect square, its square root is **irrational** (a never-ending, non-repeating decimal). We estimate it by finding the two nearest perfect squares.

Estimating $\sqrt{50}$

Find perfect squares just below and above 50:

$$49 < 50 < 64 \quad\Longrightarrow\quad \sqrt{49} < \sqrt{50} < \sqrt{64} \quad\Longrightarrow\quad 7 < \sqrt{50} < 8.$$

Since 50 is very close to 49, the value $\sqrt{50} \approx 7.1$ is a good estimate.

Tip: The bigger the radicand, the bigger the root, but the root grows slowly. Whenever the radicand sits closer to the lower perfect square, the root sits closer to the lower integer.

12.4 Simplifying Square Roots (Product Property)

Product Property of Radicals

For non-negative numbers a and b:

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}.$$

To **simplify**, factor the radicand so that one factor is the *largest perfect square* you can find, then pull its root out front.

Simplifying $\sqrt{12}$

$$\begin{aligned} \sqrt{12} &= \sqrt{4 \cdot 3} && \text{(4 is the largest perfect-square factor)} \\ &= \sqrt{4} \cdot \sqrt{3} && \text{product property} \\ &= 2\sqrt{3}. && \sqrt{4} = 2 \end{aligned}$$

Simplifying $\sqrt{72}$

$$\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36} \cdot \sqrt{2} = 6\sqrt{2}.$$

If you had used a smaller factor, say $\sqrt{72} = \sqrt{9 \cdot 8} = 3\sqrt{8}$, you are not done: $\sqrt{8} = 2\sqrt{2}$, so $3\sqrt{8} = 3 \cdot 2\sqrt{2} = 6\sqrt{2}$. Same answer, more steps.

Fully simplified means the radicand has *no* perfect-square factors left (other than 1).

12.5 Radicals with Variables

Even Powers Come Out Clean

Since $\sqrt{x^2} = x$ (for $x \geq 0$), any *even* power under a square root leaves half its exponent outside:

$$\sqrt{x^6} = x^3, \qquad \sqrt{x^{10}} = x^5.$$

For an *odd* power, split off one factor: $x^5 = x^4 \cdot x$, so $\sqrt{x^5} = x^2\sqrt{x}$.

Simplifying $\sqrt{50x^3}$

$$\begin{aligned}\sqrt{50x^3} &= \sqrt{25 \cdot 2 \cdot x^2 \cdot x} \\ &= \sqrt{25} \cdot \sqrt{x^2} \cdot \sqrt{2x} \\ &= 5x\sqrt{2x}.\end{aligned}$$

Pull out the perfect-square number ($25 \to 5$) and the even power ($x^2 \to x$); leave the rest inside.

12.6 Adding & Subtracting Radicals

Combine LIKE Radicals

Radicals are **like** when they have the same index *and* the same radicand. Add or subtract only the numbers in front (the coefficients); the radical stays the same, just like combining like terms.

$$2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}.$$

Often you must simplify first before you can tell that radicals are alike.

Simplify first, then combine

$$\begin{aligned}\sqrt{12}+\sqrt{27} &= 2\sqrt{3}+3\sqrt{3} \qquad\qquad (\sqrt{12}=2\sqrt{3},\ \sqrt{27}=3\sqrt{3})\\ &= 5\sqrt{3}.\end{aligned}$$

Warning: $\sqrt{a}+\sqrt{b}\neq\sqrt{a+b}$. For example $\sqrt{9}+\sqrt{16}=3+4=7$, but $\sqrt{9+16}=\sqrt{25}=5$. You can only combine radicals that are already *alike*; unlike radicals such as $2\sqrt{2}+3\sqrt{5}$ stay separate.

12.7 Multiplying Radicals

Multiply Outsides, Multiply Insides

$$\left(a\sqrt{m}\right)\left(b\sqrt{n}\right)=ab\sqrt{mn}.$$

Multiply the coefficients together and the radicands together, then simplify the result.

A basic product

$$\left(2\sqrt{6}\right)\left(3\sqrt{2}\right)=6\sqrt{12}=6\cdot 2\sqrt{3}=12\sqrt{3}.$$

Distributing and the difference of squares

Distribute: $\sqrt{3}\left(\sqrt{3}+\sqrt{5}\right)=\sqrt{9}+\sqrt{15}=3+\sqrt{15}$.

Difference of squares $(a-b)(a+b)=a^2-b^2$:

$$\left(\sqrt{7}-\sqrt{2}\right)\left(\sqrt{7}+\sqrt{2}\right)=\left(\sqrt{7}\right)^2-\left(\sqrt{2}\right)^2=7-2=5.$$

The radicals vanish! This trick powers rationalizing below.

Key fact: $\left(\sqrt{a}\right)^2=a$. Squaring a square root undoes it.

12.8 Dividing & Rationalizing

Quotient Property & Rationalizing

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (b > 0).$$

A denominator should never keep a radical. To **rationalize**, multiply the top and bottom by the radical in the denominator so it becomes a perfect square.

Rationalizing $\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}.$$

A quotient that simplifies

$$\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{\frac{6}{3}} = \sqrt{2}. \qquad \frac{\sqrt{20}}{\sqrt{5}} = \sqrt{4} = 2.$$

12.9 The Pythagorean Theorem

$a^2 + b^2 = c^2$

In a *right* triangle, the two shorter sides (legs) a and b and the longest side (hypotenuse) c satisfy

$$a^2 + b^2 = c^2.$$

Solve for the missing side, then leave the answer in **simplified radical form** unless a decimal is asked for.

Finding a hypotenuse

Legs $a = 2$, $b = 3$:

$$\begin{aligned} c^2 &= 2^2 + 3^2 = 4 + 9 = 13 \\ c &= \sqrt{13}. && \text{(13 has no perfect-square factor)} \end{aligned}$$

Finding a leg

Hypotenuse $c = 8$, leg $a = 4$:

$$\begin{aligned} 4^2 + b^2 &= 8^2 \\ 16 + b^2 &= 64 \\ b^2 &= 48 \\ b &= \sqrt{48} = 4\sqrt{3}. \end{aligned}$$

12.10 Cube Roots (A Brief Look)

Index of 3

A **cube root** $\sqrt[3]{n}$ asks: what number cubed gives n? Because $2^3 = 8$, we get $\sqrt[3]{8} = 2$. Unlike square roots, cube roots of negatives *are* allowed: $\sqrt[3]{-27} = -3$ since $(-3)^3 = -27$.

Perfect cubes: 1, 8, 27, 64, 125. Simplify by pulling out perfect-*cube* factors.

Simplifying $\sqrt[3]{54}$

$$\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}.$$

12.11 Solving Square-Root Equations

Isolate, Square, Check

To solve an equation with a variable under a radical:

1. **Isolate** the radical on one side.
2. **Square** both sides to remove the radical.
3. **Solve** the resulting equation.
4. **Check** every answer in the *original* equation.

Solving $\sqrt{x} + 3 = 7$

$$\begin{aligned} \sqrt{x} + 3 &= 7 \\ \sqrt{x} &= 4 && \text{isolate} \\ x &= 16 && \text{square both sides} \end{aligned}$$

Check: $\sqrt{16} + 3 = 4 + 3 = 7$. ✓

Why check? Squaring can create **extraneous solutions**, answers that appear valid but fail the original equation. For instance, $\sqrt{x} = -2$ has *no* solution (a principal root is never negative), even though squaring gives $x = 4$. Always test your answers.

12.12 Going Deeper: Advanced Radicals

nth Roots and Rational Exponents

The index does not have to be 2 or 3. In general, $\sqrt[n]{a}$ asks for the number whose nth power is a. There is a powerful bridge between roots and exponents:

$$a^{1/n} = \sqrt[n]{a}, \qquad a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}.$$

The **denominator** of the exponent is the *index* (the root); the **numerator** is the ordinary *power*. Once a radical is written this way, every exponent rule you already know applies.

$$8^{2/3} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4, \qquad 16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8.$$

Rational exponents at work

Simplify $\dfrac{x^{1/2} \cdot x^{1/3}}{x^{1/6}}$ and write the answer as a radical.

$$\begin{aligned}
\frac{x^{1/2} \cdot x^{1/3}}{x^{1/6}} &= x^{\,1/2+1/3-1/6} && \text{add/subtract exponents}\\
&= x^{\,3/6+2/6-1/6} && \text{common denominator 6}\\
&= x^{4/6} = x^{2/3} && \text{reduce}\\
&= \sqrt[3]{x^2}. && \text{denominator is the index}
\end{aligned}$$

Even vs. odd index. An *odd* index (like $\sqrt[3]{\ }$ or $\sqrt[5]{\ }$) accepts negative radicands; an *even* index (like $\sqrt{\ }$ or $\sqrt[4]{\ }$) does not allow a negative radicand in the real numbers.

Rationalizing with a Conjugate

When a denominator is a *sum or difference* of terms involving radicals, one factor is not enough. Multiply top and bottom by the **conjugate**, the same two terms with the middle sign flipped. The difference of squares $(a - b)(a + b) = a^2 - b^2$ then erases the radical:

$$\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b.$$

The conjugate of $3+\sqrt{5}$ is $3-\sqrt{5}$; the conjugate of $\sqrt{7}-\sqrt{2}$ is $\sqrt{7}+\sqrt{2}$.

Rationalizing a two-term denominator

Rationalize $\dfrac{4}{3-\sqrt{5}}$.

$$\begin{aligned}
\frac{4}{3-\sqrt{5}} &= \frac{4}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} && \text{multiply by the conjugate} \\
&= \frac{4\left(3+\sqrt{5}\right)}{3^2-\left(\sqrt{5}\right)^2} && \text{difference of squares} \\
&= \frac{4\left(3+\sqrt{5}\right)}{9-5} = \frac{4\left(3+\sqrt{5}\right)}{4} \\
&= 3+\sqrt{5}.
\end{aligned}$$

No radical remains in the denominator, so the expression is fully rationalized.

Denesting a Nested Radical

A radical inside a radical such as $\sqrt{7+4\sqrt{3}}$ can sometimes be **denested** into $\sqrt{a}+\sqrt{b}$. The idea: assume $\sqrt{7+4\sqrt{3}}=\sqrt{a}+\sqrt{b}$ and square both sides:

$$7+4\sqrt{3}=a+b+2\sqrt{ab}.$$

Match the parts *without* a radical to each other and the parts *with* a radical to each other:

$$a+b=7, \qquad 2\sqrt{ab}=4\sqrt{3} \;\Rightarrow\; ab=12.$$

Find two numbers that add to 7 and multiply to 12: they are 4 and 3. Hence

$$\sqrt{7+4\sqrt{3}}=\sqrt{4}+\sqrt{3}=2+\sqrt{3}.$$

Check by squaring. Denesting is only valid if it survives a check: $\left(2+\sqrt{3}\right)^2=4+4\sqrt{3}+3=7+4\sqrt{3}$. ✓ If no nice pair of numbers exists, the radical simply cannot be denested.

Solving a radical equation with an extraneous root

Solve $\sqrt{x+7} = x+1$.

$$\begin{aligned} \sqrt{x+7} &= x+1 \\ x+7 &= (x+1)^2 && \text{square both sides} \\ x+7 &= x^2+2x+1 \\ 0 &= x^2+x-6 && \text{collect on one side} \\ 0 &= (x+3)(x-2) && \text{factor} \end{aligned}$$

So $x=-3$ or $x=2$. **Check both** in the original equation:

$$x=2: \ \sqrt{2+7} = \sqrt{9} = 3, \quad 2+1 = 3. \checkmark$$

$$x=-3: \ \sqrt{-3+7} = \sqrt{4} = 2, \quad -3+1 = -2. \text{ But } 2 \neq -2. \ \textbf{(rejected)}$$

Only $x=2$ works; $x=-3$ is **extraneous** and is thrown out.

The Distance Formula Is the Pythagorean Theorem

Place two points (x_1, y_1) and (x_2, y_2) on the plane. The horizontal gap is a leg of length $|x_2 - x_1|$ and the vertical gap is a leg of length $|y_2 - y_1|$; the straight-line distance d between the points is the hypotenuse. Feeding the legs into $a^2+b^2=c^2$ gives

$$d^2 = (x_2-x_1)^2 + (y_2-y_1)^2 \implies d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}.$$

The distance formula is nothing new to memorize, it is just the Pythagorean theorem wearing coordinates.

Distance in simplest radical form

Find the distance between $(1,2)$ and $(4,8)$.

$$\begin{aligned} d &= \sqrt{(4-1)^2 + (8-2)^2} \\ &= \sqrt{3^2+6^2} = \sqrt{9+36} \\ &= \sqrt{45} = \sqrt{9 \cdot 5} \\ &= 3\sqrt{5}. \end{aligned}$$

Combining $\sqrt{a} \pm \sqrt{b}$ Cleverly

Sums and differences of unlike radicals will not merge, but you can still simplify *products and squares* of them. Treat $\sqrt{a}$ and $\sqrt{b}$ like binomial terms and use $\left(\sqrt{a}\right)^2 = a$:

$$\left(\sqrt{a}+\sqrt{b}\right)^2 = a + 2\sqrt{ab} + b, \qquad \left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a - b.$$

The first stays irrational (a nested-looking form); the second collapses to a plain integer whenever a and b are.

Expanding a squared sum of radicals

Simplify $\left(\sqrt{6}+\sqrt{2}\right)^2$.

$$\begin{aligned}\left(\sqrt{6}+\sqrt{2}\right)^2 &= \left(\sqrt{6}\right)^2 + 2\left(\sqrt{6}\right)\left(\sqrt{2}\right) + \left(\sqrt{2}\right)^2 \\ &= 6 + 2\sqrt{12} + 2 \\ &= 8 + 2\cdot 2\sqrt{3} && \sqrt{12} = 2\sqrt{3} \\ &= 8 + 4\sqrt{3}.\end{aligned}$$

13 Course Review

Every definition, rule, and formula for the whole course — all 11 topics in one place

13.1 Foundations

Real Number Families

Every number in Algebra 1 is a **real number**, sorted into nested sets:

- **Natural:** $1, 2, 3, \ldots$ **Whole:** $0, 1, 2, 3, \ldots$
- **Integers:** $\ldots, -2, -1, 0, 1, 2, \ldots$
- **Rational:** any $\frac{a}{b}$ with integers a, b and $b \neq 0$; decimals *end or repeat*.
- **Irrational:** cannot be written as a fraction; decimals never end *and* never repeat (e.g. $\sqrt{2},\ \pi$).

Each set sits inside the next: naturals $\subset$ wholes $\subset$ integers $\subset$ rationals.

Order of Operations (PEMDAS)

1. **P** — Parentheses / grouping (also $|\cdot|$ and fraction bars)
2. **E** — Exponents
3. **MD** — Multiply and Divide, left to right
4. **AS** — Add and Subtract, left to right

Multiplication and division are *equal in rank*; do whichever comes first left to right. Same for addition and subtraction.

Properties & Like Terms

- **Commutative:** $a + b = b + a,\ \ ab = ba$
- **Associative:** $(a + b) + c = a + (b + c)$
- **Distributive:** $a(b + c) = ab + ac$
- **Identity:** $a + 0 = a,\ \ a \cdot 1 = a$

Like terms share the same variable part (same letters, same exponents). Combine by adding coefficients: $5x + 3x = 8x$. Note x^2 and x are *not* like terms.

Evaluate by substitution

Evaluate $3x^2 - 2y$ when $x = -2$, $y = 5$.

$$\begin{aligned} 3x^2 - 2y &= 3(-2)^2 - 2(5) \\ &= 3(4) - 10 = 12 - 10 = 2. \end{aligned}$$

Wrap every substituted value in parentheses: $(-2)^2 = 4$, but $-2^2 = -4$. The exponent applies only to what it touches.

13.2 Solving Equations & Inequalities

Multi-Step Equations

Goal: isolate the variable. Work *backwards* through PEMDAS.

1. Distribute and clear parentheses; clear fractions by multiplying by the LCD.
2. Combine like terms on each side.
3. Move variable terms to one side, constants to the other (do the same to both sides).
4. Divide by the coefficient.

Variables on both sides: collect all variable terms on the side that keeps the coefficient positive.

Literal Equations & Solution Counts

Literal equation: solve for one letter in terms of the others using the same isolation steps. Example: $A = \frac{1}{2}bh \Rightarrow h = \dfrac{2A}{b}$.

How many solutions?

- Variable remains, e.g. $x = 4 \Rightarrow$ **one solution**.
- True statement, e.g. $6 = 6 \Rightarrow$ **all real numbers** (identity).
- False statement, e.g. $3 = 7 \Rightarrow$ **no solution**.

Inequalities, Compound & Absolute Value

Solve inequalities like equations, with one rule: **multiplying or dividing by a negative flips the inequality sign.**

Compound: "AND" means both true (overlap, $a < x < b$); "OR" means either true (union).

Absolute value (for $k > 0$):

$$
\begin{aligned}
|x| < k \;&\Rightarrow\; -k < x < k && \text{(AND)}\\
|x| > k \;&\Rightarrow\; x < -k \;\text{ or }\; x > k && \text{(OR)}
\end{aligned}
$$

Solve a multi-step equation

Solve $3(x-4) = 5x + 2$.

$$
\begin{aligned}
3x - 12 &= 5x + 2 && \text{distribute}\\
-12 - 2 &= 5x - 3x && \text{collect terms}\\
-14 &= 2x \;\Rightarrow\; x = -7.
\end{aligned}
$$

Only flip the inequality sign when you multiply or divide *by* a negative — not when a negative simply appears. $-2x < 6 \Rightarrow x > -3$.

13.3 Linear Equations & Functions

Slope

Slope m measures steepness (rise over run):

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}.$$

Positive m rises left-to-right; negative m falls. Horizontal lines have $m = 0$; vertical lines have *undefined* slope.

Forms of a Linear Equation

- **Slope-intercept:** $y = mx + b$ (m slope, b is y-intercept)

- **Point-slope:** $y - y_1 = m(x - x_1)$
- **Standard:** $Ax + By = C \quad (A, B, C$ integers, $A \geq 0)$

Intercepts: set $y = 0$ for the x-intercept, set $x = 0$ for the y-intercept.

Parallel, Perpendicular & Function Notation

- **Parallel** lines have *equal* slopes: $m_1 = m_2$.
- **Perpendicular** lines have *opposite reciprocal* slopes: $m_1 \cdot m_2 = -1$, so $m_2 = -\dfrac{1}{m_1}$.

Function notation: $f(x)$ names the output for input x. A relation is a **function** if each input has exactly one output (passes the vertical line test). $f(3)$ means substitute 3 for x.

Equation through two points

Line through $(1, 2)$ and $(3, 8)$.

$$m = \frac{8-2}{3-1} = \frac{6}{2} = 3,$$
$$y - 2 = 3(x - 1) \;\Rightarrow\; y = 3x - 1.$$

A line perpendicular to $y = 2x + 5$ has slope $-\frac{1}{2}$: flip the fraction *and* change the sign.

13.4 Systems of Equations & Inequalities

Three Solution Methods

A solution (x, y) satisfies *both* equations — the point where the graphs cross.

- **Graphing:** plot both lines; the intersection is the solution.
- **Substitution:** solve one equation for a variable, plug into the other. Best when a variable is already isolated.
- **Elimination:** add or subtract equations (scale first) to cancel one variable. Best when lined up in standard form.

Number of Solutions

- **One solution:** lines cross once (different slopes).
- **No solution:** parallel lines, same slope, different intercept (variables cancel, false statement).

- **Infinitely many:** same line (variables cancel, true statement).

Systems of inequalities: graph each, shade; the solution is the *overlap* of shaded regions.

Solve by elimination

Solve $2x + 3y = 12$ and $2x - y = 4$.

$$\begin{aligned} (2x+3y) - (2x-y) &= 12 - 4 \\ 4y = 8 &\Rightarrow\ y = 2, \\ 2x - 2 = 4 &\Rightarrow\ x = 3. \end{aligned}$$

Solution: $(3, 2)$.

Always check your answer in *both* original equations — a point that fits only one equation is not a solution to the system.

13.5 Exponents & Exponent Rules

The Exponent Rules

For any nonzero base:

- **Product:** $x^a \cdot x^b = x^{a+b}$
- **Quotient:** $\dfrac{x^a}{x^b} = x^{a-b}$
- **Power of power:** $(x^a)^b = x^{ab}$
- **Power of product:** $(xy)^a = x^a y^a$
- **Power of quotient:** $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$
- **Zero:** $x^0 = 1$

Negative exponent: $x^{-a} = \dfrac{1}{x^a}$ — move the factor across the fraction bar to make the exponent positive.

Scientific Notation

A number written as $a \times 10^n$ where $1 \le a < 10$ and n is an integer.

- Large numbers: positive n. $5{,}200{,}000 = 5.2 \times 10^6$.
- Small numbers: negative n. $0.00047 = 4.7 \times 10^{-4}$.

Multiply/divide by combining the a values and adding/subtracting the exponents.

Simplify with exponent rules

Simplify $\dfrac{6x^5y^2}{2x^2y^5}$.

$$\frac{6x^5y^2}{2x^2y^5} = 3x^{5-2}y^{2-5} = 3x^3y^{-3} = \frac{3x^3}{y^3}.$$

$x^0 = 1$, not 0. And a negative exponent never makes a number negative — it makes a reciprocal: $2^{-3} = \frac{1}{8}$.

13.6 Polynomials

Vocabulary & Add/Subtract

A **polynomial** is a sum of terms ax^n (whole-number exponents). **Degree** is the highest exponent; name by terms: monomial (1), binomial (2), trinomial (3).

Add/Subtract: combine like terms. To subtract, distribute the minus sign to *every* term in the second polynomial first.

Multiplying & Special Products

FOIL for two binomials — First, Outer, Inner, Last:

$$(a+b)(c+d) = ac + ad + bc + bd.$$

Special products (memorize):

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a-b)^2 &= a^2 - 2ab + b^2\\ (a+b)(a-b) &= a^2 - b^2\end{aligned}$$

Multiply two binomials

Expand $(2x+3)(x-5)$.

$$\begin{aligned}(2x+3)(x-5) &= 2x^2 - 10x + 3x - 15\\ &= 2x^2 - 7x - 15.\end{aligned}$$

$(a+b)^2 \neq a^2+b^2$. The middle term $2ab$ is real: $(x+4)^2 = x^2+8x+16$.

13.7 Factoring

Factoring Strategy (in order)

1. **GCF first, always:** pull out the greatest common factor. $6x^2+9x = 3x(2x+3)$.
2. **Difference of squares:** $a^2-b^2 = (a+b)(a-b)$.
3. **Perfect-square trinomial:** $a^2 \pm 2ab + b^2 = (a \pm b)^2$.
4. **Trinomial** x^2+bx+c: find two numbers that *multiply* to c and *add* to b.
5. Leading coefficient $\neq 1$: factor by grouping (AC method).

Zero Product Property

If a product equals zero, at least one factor is zero:

$$\text{if } (x-r)(x-s) = 0, \text{ then } x = r \text{ or } x = s.$$

This turns a factored equation into solutions (the roots).

Factor a trinomial

Factor $x^2-2x-15$.

$$\begin{aligned}&\text{Two numbers: multiply to } -15, \text{ add to } -2 \Rightarrow -5, +3.\\ &x^2-2x-15 = (x-5)(x+3).\end{aligned}$$

a^2+b^2 (a *sum* of squares) does *not* factor over the real numbers. Only the *difference* of squares factors.

13.8 Quadratic Functions

Graph of $y = ax^2+bx+c$

The graph is a **parabola**. It opens *up* if $a > 0$, *down* if $a < 0$.

- **Axis of symmetry / vertex x:** $x = -\dfrac{b}{2a}$.

- **Vertex:** $\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$ — the minimum or maximum point.
- **Vertex form:** $y = a(x-h)^2 + k$ with vertex (h, k).

Ways to Solve a Quadratic

- **Factoring:** write as a product $= 0$, use zero product property.
- **Square roots:** for $x^2 = k$, then $x = \pm\sqrt{k}$.
- **Completing the square:** add $\left(\frac{b}{2}\right)^2$ to form a perfect square.
- **Quadratic formula** (always works):

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The Discriminant

The discriminant is $D = b^2 - 4ac$ (under the radical). It counts real solutions:

- $D > 0$: two real solutions.
- $D = 0$: one (repeated) real solution.
- $D < 0$: no real solutions.

Quadratic formula

Solve $x^2 + 3x - 10 = 0 \quad (a = 1,\ b = 3,\ c = -10)$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{3^2 - 4(1)(-10)}}{2(1)} = \frac{-3 \pm \sqrt{49}}{2} \\ &= \frac{-3 \pm 7}{2} \quad \Rightarrow \quad x = 2 \quad \text{or} \quad x = -5. \end{aligned}$$

Set the equation equal to 0 *before* reading off a, b, c. Keep the sign attached to each: in $x^2 - 4x = 5$, rewrite as $x^2 - 4x - 5 = 0$ so $c = -5$.

13.9 Radicals

Simplifying & Operating

A radical is simplified when no perfect-square factor remains under the root.

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}, \qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Example: $\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

Add/Subtract: only combine *like radicals* (same radicand): $2\sqrt{3} + 5\sqrt{3} = 7\sqrt{3}$.

Multiply: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

Rationalizing & Pythagorean Theorem

Rationalize a denominator by removing the root from the bottom:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}.$$

Pythagorean theorem (right triangles, legs a, b, hypotenuse c):

$$a^2 + b^2 = c^2.$$

Find the hypotenuse

Legs 6 and 8.

$$\begin{aligned} c^2 &= 6^2 + 8^2 = 36 + 64 = 100 \\ c &= \sqrt{100} = 10. \end{aligned}$$

$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. For instance $\sqrt{9+16} = \sqrt{25} = 5$, but $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

13.10 Rational Expressions

Simplify & Excluded Values

A **rational expression** is a fraction of polynomials. Factor top and bottom, then cancel *common factors*.

$$\frac{x^2 - 9}{x^2 + 4x + 3} = \frac{(x-3)(x+3)}{(x+1)(x+3)} = \frac{x-3}{x+1}.$$

Excluded values: any x making an *original* denominator 0 is not allowed (here $x \neq -1$ and $x \neq -3$).

Operations & Solving

- **Multiply:** factor, cancel, multiply across.
- **Divide:** multiply by the reciprocal of the second fraction.
- **Add/Subtract:** rewrite over a common denominator (LCD), combine numerators.
- **Solve** rational equations: multiply every term by the LCD to clear fractions, solve, then *discard* any answer that is an excluded value.

Cancel only common factors

Simplify $\dfrac{x^2+5x+6}{x+2}$.

$$\frac{x^2+5x+6}{x+2} = \frac{(x+2)(x+3)}{x+2} = x+3, \quad x \neq -2.$$

Cancel *factors*, never terms. You may not cancel the x in $\dfrac{x+3}{x}$, because $x+3$ is a sum, not a product.

13.11 Data & Statistics

Center: Mean, Median, Mode

- **Mean** (average): $\dfrac{\text{sum of values}}{\text{number of values}}$.
- **Median:** middle value of the ordered list (average the two middle values if the count is even).
- **Mode:** the value(s) that appear most often.

The median resists **outliers** (extreme values); the mean gets pulled toward them.

Spread: Range & IQR

- **Range** $=$ maximum $-$ minimum.
- **Quartiles:** Q_1 (median of lower half), Q_2 (median), Q_3 (median of upper half).
- **Interquartile range:** $\text{IQR} = Q_3 - Q_1$ — the spread of the middle 50%.

Scatter Plots & Line of Best Fit

A **scatter plot** shows paired data to reveal a relationship (**correlation**):

- **Positive:** both rise together. **Negative:** one rises as the other falls.
- **No correlation:** no clear pattern.

The **line of best fit** models the trend; use its equation $y = mx+b$ to *predict* values. Correlation does not prove causation.

Mean and median

Data: $4, 7, 7, 9, 13$.

$$\text{Mean} = \frac{4+7+7+9+13}{5} = \frac{40}{5} = 8,$$
$$\text{Median} = 7 \text{ (middle)}, \qquad \text{Mode} = 7.$$

Always *order the data* before finding the median or quartiles. The mode is the most frequent value, not the largest.