MathCastle

AIME

Complete course booklet • 11 topics

Contents

1 Advanced Number Theory

CRT, orders, and totients

1.1 Systems of Congruences

Chinese Remainder Theorem

If the moduli are pairwise coprime, a system $x \equiv a_i \pmod{m_i}$ has a unique solution modulo $\prod m_i$.

In plain terms. Several "remainder" conditions with no shared factors always combine into a single condition — one remainder mod the product.

Example. $x \equiv 2 \pmod 3$ and $x \equiv 3 \pmod 5$ give $x \equiv 8 \pmod{15}$ (check: $8 = 2 \cdot 3 + 2$ and $8 = 5 + 3$).

1.2 Orders and Totients

Euler's theorem and totient

If $\gcd(a, n) = 1$, then $a^{\varphi(n)} \equiv 1 \pmod n$, where $\varphi(n) = n \prod_{p|n}(1 - \frac{1}{p})$.

In plain terms. Like Fermat but for any modulus: raising to the "totient power" resets to 1. The totient counts how many numbers below n share no factor with it.

Example. $\varphi(10) = 10(1 - \frac{1}{2})(1 - \frac{1}{5}) = 4$, so $3^4 = 81 \equiv 1 \pmod{10}$.

Order of an element

The order of a mod n is the least $d > 0$ with $a^d \equiv 1$, and it always divides $\varphi(n)$.

In plain terms. It is the length of the repeating cycle of powers of a — and that length must divide the totient, which narrows the possibilities.

Example. Order of 2 mod 7: $2, 4, 1, \ldots$, so the order is 3, which divides $\varphi(7) = 6$.

2 Advanced Counting

Recursion, expectation, and the binomial theorem

2.1 Building Recurrences

Condition on the last step

Count by splitting on the final move: a tiling ending in a square leaves a_{n-1} ways, one ending in a domino leaves a_{n-2}, so $a_n = a_{n-1} + a_{n-2}$.

In plain terms. Ask "what could the last piece be?" Each choice reduces to a smaller version of the same problem, giving a recurrence.

Example. Ways to tile a $1 \times n$ strip with squares and dominoes: $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5$ — the Fibonacci numbers.

Binomial theorem

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$.

In plain terms. Expanding a power of a sum spreads it into terms weighted by "choose" numbers; plugging in clever x, y evaluates tricky sums.

Example. Set $x = y = 1$: $\sum_k \binom{n}{k} = 2^n$. Set $x = 1, y = -1$: the alternating sum is 0.

2.2 Expected Value

Linearity of expectation

The expected value of a sum equals the sum of expected values, even when the parts depend on each other.

In plain terms. To find an average total, add up the average of each little piece — you never need the pieces to be independent.

Example. Expected number of heads in 10 flips: each flip contributes $\frac{1}{2}$, so the total is $10 \cdot \frac{1}{2} = 5$.

3 Polynomials

Newton and the roots-of-unity filter

3.1 Symmetric Functions of Roots

Newton's identities

Power sums $p_k = \sum r_i^k$ come from the Vieta sums e_j: $p_1 = e_1$, $p_2 = e_1 p_1 - 2e_2$, $p_3 = e_1 p_2 - e_2 p_1 + 3e_3$.

In plain terms. You can compute the sum of the kth powers of the roots step by step from the coefficients — without ever finding the roots.

Example. If roots sum to $e_1 = 3$ and pair-sum to $e_2 = 1$, then $p_2 = \sum r_i^2 = 3 \cdot 3 - 2 \cdot 1 = 7$.

3.2 The Roots of Unity Filter

Extracting every nth coefficient

To sum the coefficients of $x^0, x^n, x^{2n}, \dots$ in P, average P over the nth roots of unity: $\frac{1}{n}\sum_j P(\omega^j)$.

In plain terms. Plugging the symmetric roots of unity in and averaging cancels every term except the ones whose exponent is a multiple of n.

Example. $\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\cdots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$ via the cube roots of unity.

4 Geometry: Coordinates & Complex

Analytic and complex methods

4.1 Analytic Geometry

Shoelace formula

A polygon with vertices (x_i, y_i) in order has area $\frac{1}{2}\left|\sum_i (x_i y_{i+1} - x_{i+1} y_i)\right|$.

In plain terms. List the corner coordinates in a loop, cross-multiply neighbors, and half the total gives the area — no need to cut the shape up.

Example. Triangle $(0,0), (4,0), (0,3)$: area $= \frac{1}{2}|0\cdot 0 - 4\cdot 0 + 4\cdot 3 - 0\cdot 0 + 0 - 0| = 6$.

4.2 Complex Numbers as Points

Rotations by multiplication

Rotating a point z about the origin by θ is multiplication by $e^{i\theta}$; about a center c, use $c + (z - c)e^{i\theta}$.

In plain terms. Treat plane points as complex numbers; a rotation is then just one multiplication, which makes symmetric figures easy.

Example. Rotating $z = 1$ by $90°$ gives $1 \cdot i = i$ — the point $(0,1)$, exactly a quarter turn.

4.3 Cevians: Ceva & Menelaus

Ceva's theorem

Cevians AD, BE, CF (with D, E, F on sides BC, CA, AB) are concurrent if and only if $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB} = 1$.

In plain terms. Three lines from the corners of a triangle to the opposite sides pass through one common point exactly when the three side-splitting ratios multiply to 1. Two known ratios always lock in the third — no lengths of the triangle itself are ever needed.

Example. If $BD : DC = 2 : 3$ and $CE : EA = 1 : 2$ and the cevians are concurrent, then

$\frac{2}{3} \cdot \frac{1}{2} \cdot \frac{AF}{FB} = 1$ forces $AF : FB = 3 : 1$.

Menelaus's theorem

A transversal line crossing lines BC, CA, AB of triangle ABC at D, E, F satisfies $\dfrac{BD}{DC} \cdot \dfrac{CE}{EA} \cdot \dfrac{AF}{FB} = 1$ (unsigned; with signed ratios the product is -1).

In plain terms. Same product as Ceva, but for a straight LINE slicing across the triangle instead of three concurrent cevians — the line always cuts one side externally, on its extension. Use it whenever a problem draws a line through two marked points and asks where it crosses a third side.

Example. With $BD : DC = 2 : 3$ on BC and M the midpoint of CA ($CM : MA = 1$), the line DM meets line AB at X with $\frac{AX}{XB} \cdot \frac{2}{3} \cdot 1 = 1$, so $AX : XB = 3 : 2$ — and X lies outside segment AB.

Choosing between them — and mass points

Concurrency of three cevians $\Rightarrow$ Ceva. One straight line cutting across the triangle $\Rightarrow$ Menelaus. For the RATIO ALONG a cevian (like $AP : PD$), assign masses inverse to the side ratios: the balance point is the cevian intersection, and $\dfrac{AP}{PD} = \dfrac{\text{mass at } D}{\text{mass at } A}$.

In plain terms. The two theorems answer "where does the third cut land?"; mass points answer "how far along the cevian is the crossing?". Together they dispatch nearly every ratio-chasing problem without coordinates.

Example. With $BD : DC = 1 : 2$ and $CE : EA = 1 : 2$, masses $4, 2, 1$ at B, C, A make D carry 6, so cevians AD and BE meet at P with $AP : PD = 6 : 1$.

5 Cevians & Mass Points

Ratio-chasing: Ceva, Menelaus, masses, and areas

5.1 The Ratio-Chasing Recipes

Which theorem for which picture

Three cevians through ONE point: Ceva, $\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1$. One LINE crossing all three sides (one externally): Menelaus, same product (unsigned) equals 1.

In plain terms. Ceva is for concurrency, Menelaus is for a transversal line. In both, two known side-ratios always determine the third — draw the picture first and ask "one point or one line?"

Example. Cevians concurrent with $BD : DC = 4 : 5$ and $AF : FB = 2 : 3$: Ceva forces $CE : EA = \frac{15}{8}$.

Mass points for ratios ALONG a cevian

Assign masses inversely to the side ratios so each cevian foot balances; then $\dfrac{AP}{PD} = \dfrac{\text{mass}(D)}{\text{mass}(A)}$ at the intersection P.

In plain terms. Put weights on the corners so every marked point is a balance point; the crossing point ratio reads off as mass-over-mass. Rescale one pair when two cevians disagree about a shared vertex.

Example. $BD : DC = 2 : 3$ and $AF : FB = 3 : 4$ give masses $4, 3, 2$ at A, B, C, so D carries 5 and $AP : PD = 5 : 4$.

Area ratios: same height, same base

Triangles sharing an apex have areas in the ratio of their bases; triangles on the same base have areas in the ratio of their heights. Chain these to convert cevian ratios into area ratios.

In plain terms. A cevian splits a triangle into two pieces whose areas are exactly the base ratio — walking around a figure multiplying such ratios computes almost any area fraction with no lengths at all.

Example. If $BD : DC = 2 : 3$, cevian AD splits $\triangle ABC$ into areas $2 : 3$; a second cevian subdivides those pieces the same way.

6 Trig Formulas That Win

The identities contest problems are built from

6.1 Sum, Double, Triple

Addition and double-angle

$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$; $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$; $\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$; $\sin 2a = 2 \sin a \cos a$, $\cos 2a = 1 - 2\sin^2 a$.

In plain terms. Every other identity is a rearrangement of these. When an equation mixes angles, rewrite everything in terms of ONE angle with the addition formulas.

Example. $\tan a = \frac{1}{2}$, $\tan b = \frac{1}{3}$: $\tan(a+b) = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = 1$, so $a + b = 45°$.

Triple angle in disguise

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Cubic expressions in sin or cos are usually a triple angle wearing a costume.

In plain terms. See $8\cos^3\theta - 6\cos\theta$? That is $2\cos 3\theta$. Solving the cubic head-on is the trap; recognizing the pattern is the whole problem.

Example. $8\cos^3\theta - 6\cos\theta = 1 \Rightarrow \cos 3\theta = \frac{1}{2} \Rightarrow \theta = 20^\circ$ (acute).

6.2 Products & Sums That Collapse

The 60-degree product identity

$\sin\theta\,\sin(60^\circ - \theta)\,\sin(60^\circ + \theta) = \dfrac{\sin 3\theta}{4}$, and the same shape holds for cosine and tangent (tan version multiplies to $\tan 3\theta$).

In plain terms. Three angles in arithmetic progression around 60° multiply into one triple angle — the engine behind products like $\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{8}$.

Example. $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ = \sqrt{3}$.

Pairing tricks for long sums

Pair to a constant: $\sin^2 k^\circ + \sin^2(90-k)^\circ = 1$; $\tan k^\circ \tan(90-k)^\circ = 1$; $(1+\tan k^\circ)(1+\tan(45-k)^\circ) = 2$. Long sums and products almost always fold in half.

In plain terms. Do not evaluate 89 terms — find the partner that makes each pair trivial, count the pairs, and handle the lone middle term separately.

Example. $\sum_{k=1}^{89} \sin^2 k^\circ = 44 + \sin^2 45^\circ = \frac{89}{2}$.

7 Recurrences & Characteristic Equations

Closed forms for linear recursions

7.1 The Characteristic Equation Method

Distinct roots

For $a_{n+2} = p\,a_{n+1} + q\,a_n$, solve $x^2 = px + q$. With distinct roots r, s: $a_n = Ar^n + Bs^n$, where A, B come from the first two terms.

In plain terms. Guess $a_n = x^n$, and the recurrence becomes a quadratic in x. Every solution is a mix of the two root-powers; the starting values fix the mix.

Example. $a_{n+2} = a_{n+1} + 2a_n$ has roots $2, -1$; with $a_1 = 1, a_2 = 2$ the mix is $a_n = 2^{n-1}$ exactly.

Repeated roots

If the characteristic equation has a double root r, the general solution gains a factor of n: $a_n = (A + Bn)\, r^n$.

In plain terms. One root cannot carry two initial conditions alone — the extra n in front supplies the second degree of freedom.

Example. $a_n = 6a_{n-1} - 9a_{n-2}$ has $(x-3)^2$: with $a_1 = 3, a_2 = 18$, $a_n = n \cdot 3^n$.

Fibonacci and Binet

Fibonacci is the recurrence $x^2 = x + 1$ with roots $\varphi = \frac{1+\sqrt{5}}{2}$ and $\hat{\varphi} = \frac{1-\sqrt{5}}{2}$: $F_n = \dfrac{\varphi^n - \hat{\varphi}^n}{\sqrt{5}}$.

In plain terms. The golden ratio IS the characteristic root of Fibonacci. Since $|\hat{\varphi}| < 1$, F_n is the nearest integer to $\varphi^n/\sqrt{5}$ — closed forms tame huge indices instantly.

Example. Sums telescope too: $F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$, so the first ten sum to $144 - 1 = 143$.

8 Telescoping & Recursion

Sums that collapse and sequences that build

8.1 Making Long Computations Short

Telescoping sums

Partial fractions turn $\dfrac{1}{k(k+1)}$ into $\dfrac{1}{k} - \dfrac{1}{k+1}$, so the sum **telescopes**: $\sum_{k=1}^{n} \dfrac{1}{k(k+1)} = 1 - \dfrac{1}{n+1}$.

In plain terms. Write each term as a difference; everything in the middle cancels.

Example. $\sum_{k=1}^{9} \dfrac{1}{k(k+1)} = 1 - \dfrac{1}{10} = \dfrac{9}{10}$.

Linear recursions

When each state comes from the previous ones — like $a_n = a_{n-1} + a_{n-2}$ — compute forward from the base cases rather than hunting for a formula.

In plain terms. Build the answer step by step; ten steps of arithmetic beat an hour of algebra.

Example. Climbing 5 stairs taking 1 or 2 at a time: $1, 2, 3, 5, 8$ — there are 8 ways (a Fibonacci-type recursion).

Circle power tools together

Ptolemy's theorem ($AC \cdot BD = AB \cdot CD + AD \cdot BC$ in a cyclic quadrilateral) and the **Power of a Point** ($AP \cdot PB = CP \cdot PD$ for chords through P) often chain together in one problem.

In plain terms. In circle problems, name the theorem each length fact comes from and the algebra falls out.

Example. Cyclic $ABCD$ with diameter $AC = 25$, $AB = 7$, $BC = 24$, $CD = 20$, $DA = 15$: Ptolemy gives $BD = \dfrac{7 \cdot 20 + 15 \cdot 24}{25} = 20$.

(diagram in the online version)

9 Heavy Machinery, Proved

Stewart, Ptolemy, and roots of unity — with their proofs

9.1 Stewart's Theorem and the Bisector, Proved

Proof of Stewart's theorem

Let cevian $AD = d$ hit BC with $BD = m$, $DC = n$. Apply the **law of cosines** in $\triangle ABD$ and $\triangle ACD$ at the angles at D — which are supplementary, so their cosines are NEGATIVES of each other. Multiply the first equation by n, the second by m, and ADD: the cosine terms cancel, leaving $b^2m + c^2n = a(d^2 + mn)$. ∎

In plain terms. Two law-of-cosines equations, weighted so the unknown angle disappears. That cancellation IS the theorem.

Example. $AB = 7$, $AC = 9$, $BC = 8$, $BD = 3$: $49 \cdot 5 + 81 \cdot 3 - 8d^2 = 8 \cdot 3 \cdot 5$ gives $d = \sqrt{46}$.

The angle bisector length, derived from Stewart

A bisector from A splits BC in ratio $c : b$ (bisector ratio theorem), so $m = \frac{ac}{b+c}$, $n = \frac{ab}{b+c}$. Substituting into Stewart and simplifying collapses to $t^2 = bc - mn$ — the bisector length needs only the two sides and the two pieces. ∎

In plain terms. Stewart plus the ratio the bisector forces = a two-term formula.

Example. Sides $4, 6$ around the bisected angle, pieces $2, 3$: $t^2 = 24 - 6 = 18$, $t = 3\sqrt{2}$.

9.2 Ptolemy's Theorem, Proved

Proof of Ptolemy's theorem

In cyclic $ABCD$, mark E on diagonal BD with $\angle BCE = \angle ACD$. Equal inscribed angles give $\triangle BCE \sim \triangle ACD$ and $\triangle CDE \sim \triangle CAB$; the two similarity ratios produce $BE \cdot AC = AB \cdot CD$

and $DE \cdot AC = AD \cdot BC$. Adding, and using $BE + DE = BD$: $AC \cdot BD = AB \cdot CD + AD \cdot BC$. ■

In plain terms. One clever point splits a diagonal into two pieces, each measured by a similar triangle; adding the pieces is the theorem.

Example. Equilateral $\triangle ABC$ with P on arc BC: Ptolemy on $ABPC$ collapses to $PA = PB + PC$.

Reading equality: when Ptolemy becomes an inequality

For a NON-cyclic quadrilateral the same construction gives $AC \cdot BD < AB \cdot CD + AD \cdot BC$ — so equality is a TEST for being cyclic. Many AIME problems hide the cyclic condition exactly here.

In plain terms. If the products balance exactly, the four points share a circle.

Example. A quadrilateral with $AC \cdot BD = AB \cdot CD + AD \cdot BC$ must be cyclic — no angle chasing required.

9.3 Roots of Unity, Proved

Why $x^n - 1$ factors over the nth roots of unity

By **De Moivre** (proved by induction on the angle-addition formula), the n numbers $\omega_k = \operatorname{cis}\frac{2\pi k}{n}$ all satisfy $x^n = 1$; a degree-n polynomial has at most n roots, so these are ALL of them and $x^n - 1 = \prod_k (x - \omega_k)$. ■

In plain terms. Spinning by $\frac{1}{n}$ of a turn n times lands you back at 1 — and a polynomial cannot have more roots than its degree.

Example. $x^4 - 1 = (x-1)(x+1)(x-i)(x+i)$.

Proof: the distances from one vertex multiply to n

Divide $x^n - 1$ by $x - 1$: $\prod_{k=1}^{n-1}(x - \omega_k) = 1 + x + \cdots + x^{n-1}$. Evaluate at $x = 1$: the right side is n, and the left side is exactly the product of distances from vertex 1 to every other vertex of the regular n-gon. ■

In plain terms. A polynomial identity, evaluated at one point, computes a geometric product no ruler could.

Example. A regular pentagon on the unit circle: the four distances from one vertex multiply to 5.

10 Complete Syllabus & Gap Fillers

Everything the AIME tests, with the pieces that were missing

10.1 What the AIME Tests

The checklist

Integer answers 000–999, fifteen problems, three hours. Number theory (CRT, orders, totients, Legendre/Kummer, base representations, digit sums, LTE-lite); algebra (Vieta and Newton sums, symmetric substitutions, functional equations, floor/fractional-part equations, logs and exponentials); counting (recursion, bijections, inclusion–exclusion with structure, expected value, generating functions in easy cases, roots-of-unity filter); geometry (coordinates, complex numbers, mass points, Ceva/Menelaus, power of a point, trig in triangles, 3D solids); sequences (telescoping, periodic recurrences, characteristic equations); trigonometry (product-to-sum, roots of unity products like $\prod \sin \frac{k\pi}{n}$).

10.2 Gap Fillers

Floor functions and base representations

$\lfloor x \rfloor$ problems are block counts: $\sum_{k=1}^{2026} \lfloor \frac{k}{7} \rfloor$ groups k by quotient; "n divisible by $\lfloor \sqrt{n} \rfloor$" gives exactly three per square block. Base-b digit conditions are bijections with strings: base-3 numbers with no digit 2 are binary strings read in base 3.

Iterated functions and periodicity

Digit sums preserve residues mod 9 and shrink fast — $f(f(f(7^{2026})))$ is a single digit found by size bounds plus mod 9. Rational recurrences like $x_{n+1} = \frac{x_n+1}{1-x_n}$ are tangent-addition in disguise and have small periods; compute one cycle and reduce the index.

Generating functions, the easy cases

The coefficient of x^n in $(1+x)^a(1+x)^b$ is Vandermonde ($\sum \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$; $\sum \binom{10}{k}^2 = \binom{20}{10}$). Stars and bars is the coefficient in $(1-x)^{-k}$. Differentiating $\sum x^n$ evaluates $\sum kx^k$ — how $\sum k\binom{20}{k} = 20 \cdot 2^{19}$ and $\sum \frac{k}{2^k} = 2$ both arise.

11 Formulas & Methods

The AIME toolkit: every formula, and the method that makes it a solution

11.1 Number Theory & Algebra

Number theory

CRT; orders and primitive roots; φ, τ, σ multiplicative; Legendre and Kummer; LTE $v_p(a^n -$

$b^n) = v_p(a-b) + v_p(n)$; Wilson; quadratic residues (-1 QR iff $p \equiv 1 \bmod 4$); base-b digit sums $\equiv n \pmod{b-1}$; $\gcd(a^m-1, a^n-1) = a^{\gcd(m,n)} - 1$; SFFT $(x+a)(y+b) = c + ab$.

For "remainder when divided by 1000": CRT with 8 and 125. For "how many $n \le N$": count by residue classes or by inclusion–exclusion. For Diophantine equations: factor (SFFT), bound, reduce mod a small prime, or descend.

Algebra and polynomials

Newton's sums; roots of unity filter; $\prod_{k=1}^{n-1}(1-\omega^k) = n$ and $\prod \sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}$; Lagrange interpolation; finite differences (degree-d polynomial has constant dth differences); floor: $\sum \lfloor k/m \rfloor$ by blocks, $\lfloor \sqrt{n} \rfloor$ constant on square blocks; $\{x\}$ equations by casework on $\lfloor x \rfloor$; symmetric substitutions $u = x + \frac{1}{x}$; telescoping and periodic recurrences.

For values of a polynomial at consecutive integers, use finite differences or an auxiliary polynomial with known roots. For floor equations, set $n = \lfloor x \rfloor$ and solve $n \le x < n+1$ per case.

11.2 Counting, Probability & Expectation

Counting

Bijections (Dyck paths, compositions, stars and bars with bounds by PIE); recursion by first choice; generating functions in easy cases; Burnside for symmetric colorings $\frac{1}{|G|}\sum |\mathrm{Fix}(g)|$; Vandermonde and hockey stick; distributing with restrictions via complements.

State the recurrence, compute small cases, and identify (Fibonacci, Catalan, tribonacci) before closing the form. Bounded stars-and-bars: subtract the overshoot cases.

Probability and expectation

Linearity with indicators; expected waiting times by first-step equations; Markov-chain states for "until two in a row"; geometric probability (unit square regions, broken stick); conditional probability by restricting the sample space; symmetry arguments for "equally likely orderings."

Define the states and write $E_i = 1 + \sum p_{ij} E_j$. For continuous problems draw the region and compute areas; for "same color/pair" problems count favorable pairs directly.

11.3 Geometry & Trigonometry

Geometry

Power of a point; radical axis; Ptolemy; Brahmagupta; Stewart; mass points and Ceva/Menelaus; Heron; $rs = \frac{abc}{4R}$; Euler $OI^2 = R(R-2r)$; area ratios via shared angles $\frac{[ADE]}{[ABC]} = \frac{AD \cdot AE}{AB \cdot AC}$; coordinates with a smart origin; complex numbers on the unit circle for cyclic configurations; 3D: distances via coordinates, volumes by $\frac{1}{3}Bh$, cross-sections.

Compute the area two or three ways. Put the circumcenter at the origin when a circle dominates. For ratio-chasing, mass points first, coordinates second.

Trigonometry

Product-to-sum and sum-to-product; tan addition; $\sum_{k=1}^{n-1} \cos \frac{2k\pi}{n} = -1$; $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} = \frac{1}{8}$ (doubling trick); telescoping via $\tan(a) - \tan(b) = \frac{\sin(a-b)}{\cos a \cos b}$; $\sin x \sin(60 - x) \sin(60 + x) = \frac{1}{4} \sin 3x$; De Moivre for $\cos n\theta$ polynomials.

Turn products into sums, sums of cosines into roots of unity, and telescope whenever consecutive angles appear. Multiply by sin of the smallest angle to trigger the doubling trick.

The universal problem-solving loop

Read twice. Restate the goal in your own words. List the givens. Pick a representation. Try small cases. Look for symmetry, an invariant, or an extremal object. Compute, then CHECK against a second method or a sanity bound.

At AMC 10/12 and AIME level, the second method is what separates a 4-second guess from a 12-second certainty: compute a probability two ways (complement and direct), a length via two theorems, a count via a recurrence and a formula. Casework must be organized by a stated criterion so nothing is double-counted; algebra must be checked by substituting back.