

# MathCastle Putnam

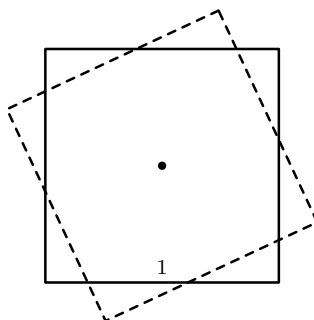
## Practice Contest 2

12 questions • 360 minutes • Write a full proof for each; the key gives the numeric target.

Two sittings of 6 problems (1–6, then 7–12), 3 hours each; the first sitting is handed in before the second is opened. One generating-function problem appears on every set.

---

1. How many ordered pairs  $(P, Q)$  of polynomials with integer coefficients satisfy  $P(x)^2 + Q(x)^2 = (x^2 + 1)^2(x^2 + 4)$  identically in  $x$ ? (Pairs are ordered: two pairs count as different whenever they differ at all, for example by swapping  $P$  and  $Q$  or by changing the sign of one of them.)
2. A unit square is rotated about its centre through an angle chosen uniformly at random from  $[0, 2\pi)$ . Find the expected area of the region common to the rotated square and the original square. Give your answer in exact form.



3. Let  $\varphi$  denote Euler's totient function. Evaluate  $\sum_{n=1}^{\infty} \frac{\varphi(n)}{2^n + 1}$ . Give your answer as a fraction in lowest terms.
4. Let  $A$  be a  $5 \times 5$  matrix with real entries such that  $A^2 = A^T$ , where  $A^T$  is the transpose of  $A$ . As  $A$  ranges over all such matrices, the trace of  $A$  takes only finitely many distinct values. Find the sum of all of these values.
5. For  $n \geq 2$ , let  $\sigma$  be a permutation of  $\{1, 2, \dots, n\}$  chosen uniformly at random. Let  $L$  be the length of the cycle of  $\sigma$  that contains 1, and let  $M$  be the length of the cycle of  $\sigma$  that contains 2 (so  $L = M$  when 1 and 2 lie in the same cycle). Find  $\lim_{n \rightarrow \infty} \frac{\mathbb{E}[LM]}{n^2}$ .
6. For each positive integer  $n$ , let  $D_n$  be the determinant of the  $n \times n$  matrix whose entry in row  $i$  and column  $j$  is  $\frac{1}{i^2 + j^2}$  (for  $1 \leq i, j \leq n$ ). It can be shown that  $D_n > 0$  for all  $n$ . Find  $\lim_{n \rightarrow \infty} \frac{\ln D_n}{n^2}$ . Give your answer in exact form.

7. Evaluate the double series  $S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+n}}{mn(m+n+1)}$ .

Give  $S$  in closed form, in terms of  $\ln 2$ .

8. Let  $\mathbb{F}_5$  be the field with 5 elements. A matrix  $A$  is chosen uniformly at random from  $GL_4(\mathbb{F}_5)$ , the set of all invertible  $4 \times 4$  matrices with entries in  $\mathbb{F}_5$ . Let  $N(A)$  be the number of vectors  $v \in \mathbb{F}_5^4$  (the zero vector included) with  $Av = v$ .

Find the expected value  $\mathbb{E}[N(A)^3]$ .

9. Consider the  $5 \times 5$  matrices  $A$  every one of whose 25 entries is 0 or 1. How many of these  $2^{25}$  matrices satisfy  $A^3 = 0$ , where the product is computed with ordinary integer arithmetic (not modulo 2)?

10. The cube  $[-1, 1]^3$  is inscribed in the ball  $B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 3\}$ , so all eight vertices of the cube lie on the boundary sphere. The planes  $x = \pm 1$  and  $y = \pm 1$  of four of the cube's faces cut  $B$  into nine pieces. Find the volume of the piece that contains the cube, namely  $\{(x, y, z) \in B : |x| \leq 1, |y| \leq 1\}$ .

Give the exact value in terms of  $\pi$  and radicals.

11. A rotation  $A \in SO(3)$  (a real  $3 \times 3$  orthogonal matrix of determinant 1) is chosen at random according to the Haar measure, the unique probability measure on  $SO(3)$  invariant under multiplication on the left and on the right by rotations. Equivalently,  $A$  is the rotation  $x \mapsto qxq^{-1}$  of the pure quaternions, where  $q$  is a uniformly random unit quaternion.

Find the expected value  $\mathbb{E}[(\text{tr } A)^6]$ .

12. Let  $R$  be the set of points  $(a, b, c, d) \in \mathbb{R}^4$  such that all four roots of  $x^4 + ax^3 + bx^2 + cx + d$  are real and lie in the interval  $[-1, 1]$  (roots counted with multiplicity). Find the four-dimensional volume of  $R$ , as a fraction in lowest terms.

## Answer Key — Putnam Practice Contest 2

1. 24
2.  $\frac{4\ln 2}{\pi}$
3.  $\frac{10}{9}$
4. 12
5.  $\frac{7}{24}$
6.  $\ln 2 - \frac{\pi}{2}$
7.  $2(1 - \ln 2)^2$
8. 64
9. 8161
10.  $\frac{16\pi - 4\sqrt{3}\pi + 8}{3}$
11. 15
12.  $\frac{64}{1575}$

## Solutions — Putnam Practice Contest 2

1. **Answer:** 24. **The idea.** Over the Gaussian integers a sum of two squares factors:  $P^2 + Q^2 = (P + iQ)(P - iQ)$ . The ring  $\mathbb{Z}[i][x]$  has unique factorisation, so  $P + iQ$  must be built from the Gaussian linear factors of the right-hand side, and conjugation pins down how.

**Work it out.** Over  $\mathbb{Z}[i]$ ,  $f(x) = (x^2 + 1)^2(x^2 + 4) = (x + i)^2(x - i)^2(x + 2i)(x - 2i)$ , a product of monic (hence primitive and irreducible) linear factors, and  $\mathbb{Z}[i]$  is a UFD, so by Gauss's lemma  $\mathbb{Z}[i][x]$  is a UFD. Put  $G = P + iQ \in \mathbb{Z}[i][x]$ . Its complex conjugate (conjugating coefficients) is  $\overline{G} = P - iQ$ , and  $G\overline{G} = f$ . Since  $f$  is monic, the content of  $G$  is a unit, so  $G = u(x + i)^a(x - i)^b(x + 2i)^c(x - 2i)^d$  with  $u \in \{\pm 1, \pm i\}$  and  $a + b \leq 2$ ,  $c + d \leq 1$  matching factors of  $f$ . Then  $\overline{G} = \bar{u}(x - i)^a(x + i)^b(x - 2i)^c(x + 2i)^d$  and  $G\overline{G} = (x + i)^{a+b}(x - i)^{a+b}(x + 2i)^{c+d}(x - 2i)^{c+d}$ . Comparing with  $f$  forces  $a + b = 2$  and  $c + d = 1$ . So  $a \in \{0, 1, 2\}$ ,  $c \in \{0, 1\}$  and  $u$  has 4 choices.

Conversely each such  $G$  has Gaussian-integer coefficients, so  $P = \operatorname{Re} G$  and  $Q = \operatorname{Im} G$  (taken coefficientwise) have integer coefficients, and  $P^2 + Q^2 = G\overline{G} = f$ . Different triples  $(u, a, c)$  give different  $G$  by unique factorisation, and  $G$  determines  $(P, Q)$ . The count is  $4 \cdot 3 \cdot 2 = 24$ .

**Check.** Two sample pairs:  $a = 1, c = 1, u = 1$  gives  $G = (x^2 + 1)(x + 2i)$ , i.e.  $(P, Q) = (x^3 + x, 2x^2 + 2)$ , and clearly  $P^2 + Q^2 = (x^2 + 1)^2(x^2 + 4)$ . With  $a = 2, c = 1, u = 1$ ,  $G = (x + i)^2(x + 2i) = x^3 - 5x + i(4x^2 - 2)$ , and  $(x^3 - 5x)^2 + (4x^2 - 2)^2 = x^6 + 6x^4 + 9x^2 + 4 = (x^2 + 1)^2(x^2 + 4)$ . Independently, a computer search confirms 24:  $\deg P, \deg Q \leq 3$  (leading terms cannot cancel), so  $P, Q$  are determined by their values at 0, 1, 2, 3, which must be lattice points on the circles of radii  $\sqrt{4}, \sqrt{20}, \sqrt{200}, \sqrt{1300}$ ; interpolating every combination and testing the identity exactly leaves exactly 24 pairs.

2. **Answer:**  $\frac{4 \ln 2}{\pi}$ . **The idea.** The overlap is the original square with four congruent right-triangular corners cut off. Along one side of the original square, the two legs and the hypotenuse of the cut-off triangles fill the whole side, which gives the hypotenuse, hence the area, in closed form; the area turns out to be the surprisingly simple  $2/(1 + \sin \theta + \cos \theta)$ .

**Work it out.** Rotating by  $\pi/2$  maps the square to itself, so the overlap area  $A(\theta)$  has period  $\pi/2$ , and the expected value is  $\frac{2}{\pi} \int_0^{\pi/2} A(\theta) d\theta$ . Also  $A(0) = A(\pi/2) = 1$ , so take  $0 < \theta < \pi/2$ . Both squares have inradius  $\frac{1}{2}$  about the common centre, and by symmetry (the configuration is invariant under the rotation by  $\pi/2$ ) each side of the rotated square cuts off one corner of the original square, and vice versa, so the overlap is an octagon: the original square minus four congruent right triangles. In one such triangle, the hypotenuse  $c$  lies along a side of the rotated square and the legs lie along two sides of the original square; its acute angles are  $\theta$  and  $\frac{\pi}{2} - \theta$ , so its legs are  $c \cos \theta$  and  $c \sin \theta$ .

Now walk along one side of the original square. From corner to corner it meets: a leg of one cut triangle, then a side of the octagon, then a leg of the next cut triangle. The octagon side lying on the original square equals the hypotenuse  $c$  of a triangle cut from the *rotated* square, and by the symmetry swapping the two squares (reflection in a line through the centre at angle  $\theta/2$  to the sides) that triangle is congruent to ours. Adjacent triangles meet this side with their two different legs, so  $c \cos \theta + c + c \sin \theta = 1$ , i.e.  $c = \frac{1}{1 + \sin \theta + \cos \theta}$ . Therefore, writing  $s = \sin \theta + \cos \theta$  and using  $\sin 2\theta = s^2 - 1$ ,  $A(\theta) = 1 - 4 \cdot \frac{1}{2} c^2 \sin \theta \cos \theta = 1 - \frac{s^2 - 1}{(1 + s)^2} = 1 - \frac{s - 1}{s + 1} = \frac{2}{1 + \sin \theta + \cos \theta}$ .

With  $t = \tan(\theta/2)$  we have  $1 + \sin \theta + \cos \theta = \frac{(1+t^2)+2t+(1-t^2)}{1+t^2} = \frac{2(1+t)}{1+t^2}$  and  $d\theta = \frac{2dt}{1+t^2}$ , so  $\int_0^{\pi/2} \frac{d\theta}{1+\sin \theta + \cos \theta} = \int_0^1 \frac{dt}{1+t} = \ln 2$ . The expected area is  $\frac{2}{\pi} \cdot 2 \ln 2 = \frac{4 \ln 2}{\pi} \approx 0.8825$ .

**Check.** At  $\theta = 0$  the formula gives  $2/2 = 1$ , correct. At  $\theta = \pi/4$  it gives  $\frac{2}{1+\sqrt{2}} = 2(\sqrt{2}-1)$ , the area of a regular octagon with inradius  $\frac{1}{2}$ , namely  $8 \cdot (\frac{1}{2})^2 \tan 22.5^\circ = 2(\sqrt{2}-1)$ . Averaging a quantity that lies between  $2(\sqrt{2}-1) \approx 0.828$  and 1 should give something in between, and 0.8825 is. Numerically clipping the rotated square against the original and integrating gives the same value to 58 digits.

3. **Answer:**  $\frac{10}{9}$ . **The idea.** Generating functions of the Lambert type  $\sum a_n \frac{x^n}{1-x^n}$  turn divisor sums into ordinary power series. For  $\varphi$ , the divisor sum  $\sum_{d|m} \varphi(d) = m$  makes the Lambert series collapse to  $\frac{x}{(1-x)^2}$ . Our summand is  $\frac{x^n}{1+x^n}$  at  $x = \frac{1}{2}$  (a plus sign, not a minus), and a one-line identity converts it into two Lambert series of the good kind.

**Work it out.** For  $0 < x < 1$ , expanding each geometric series and grouping by  $m = nk$  (all terms are nonnegative, so rearranging is allowed):  $L(x) = \sum_{n \geq 1} \varphi(n) \frac{x^n}{1-x^n} = \sum_{n \geq 1} \sum_{k \geq 1} \varphi(n) x^{nk} = \sum_{m \geq 1} x^m \sum_{d|m} \varphi(d) = \sum_{m \geq 1} m x^m = \frac{x}{(1-x)^2}$ . Here we used  $\sum_{d|m} \varphi(d) = m$ : classify the fractions  $\frac{1}{m}, \frac{2}{m}, \dots, \frac{m}{m}$  by their reduced denominator  $d$ ; exactly  $\varphi(d)$  of them have denominator  $d$ .

Next,  $\frac{y}{1+y} = \frac{y}{1-y} - \frac{2y^2}{1-y^2}$ , since the right side is  $\frac{y(1+y)-2y^2}{1-y^2} = \frac{y(1-y)}{(1-y)(1+y)}$ . With  $y = x^n$  and summing (both series converge absolutely),  $\sum_{n \geq 1} \varphi(n) \frac{x^n}{1+x^n} = L(x) - 2L(x^2) = \frac{x}{(1-x)^2} - \frac{2x^2}{(1-x^2)^2}$ .

Since  $\frac{1}{2^{n+1}} = \frac{x^n}{1+x^n}$  at  $x = \frac{1}{2}$ , the sum is  $\frac{1/2}{1/4} - \frac{2 \cdot \frac{1}{4}}{9/16} = 2 - \frac{8}{9} = \frac{10}{9}$ .

**Check.** The first terms are  $\frac{1}{3} + \frac{1}{5} + \frac{2}{9} + \frac{2}{17} + \frac{4}{33} + \frac{2}{65} + \frac{6}{129} + \dots \approx 0.333 + 0.200 + 0.222 + 0.118 + 0.121 + 0.031 + 0.047 = 1.072$ , and the tail is tiny (the terms decay like  $n/2^n$ ), consistent with  $10/9 \approx 1.111$ ; a 260-term partial sum agrees with  $10/9$  to 60 digits.

4. **Answer:** 12. **The idea.** The condition says  $A^T$  is a polynomial in  $A$ , so  $A$  commutes with its transpose:  $A$  is normal. Normal real matrices are orthogonally block-diagonal with  $1 \times 1$  and rotation-type  $2 \times 2$  blocks, and the condition cuts the possible eigenvalues down to four.

**Work it out.**  $A$  is normal:  $AA^T = A \cdot A^2 = A^3 = A^2 \cdot A = A^T A$ .

*Eigenvalues.* For a normal matrix,  $Av = \lambda v$  implies  $A^T v = \bar{\lambda} v$  (because  $\|(A^T - \bar{\lambda} I)v\| = \|(A - \lambda I)v\|$  when  $A$  is normal; here  $A^T = A^*$  as  $A$  is real). But  $A^T v = A^2 v = \lambda^2 v$ , so  $\bar{\lambda} = \lambda^2$ . Taking absolute values,  $|\lambda| = |\lambda|^2$ , so  $\lambda = 0$  or  $|\lambda| = 1$ ; in the latter case  $\lambda^2 = \bar{\lambda} = 1/\lambda$ , so  $\lambda^3 = 1$ . Hence  $\lambda \in \{0, 1, \omega, \bar{\omega}\}$  with  $\omega = e^{2\pi i/3}$ , and  $\omega, \bar{\omega}$  occur with equal multiplicity since  $A$  is real.

*Structure.* A real normal matrix is orthogonally similar to a block-diagonal matrix with blocks  $[\lambda]$  for real eigenvalues and, for each pair  $p \pm qi$ , a  $2 \times 2$  block with rows  $(p, -q)$  and  $(q, p)$ . So  $A$  has  $a$  eigenvalues 1,  $b$  pairs  $\omega, \bar{\omega}$ , and  $5 - a - 2b$  zeros, with  $\text{tr } A = a + b(\omega + \bar{\omega}) = a - b$  and  $a + 2b \leq 5$ .

*All of these occur.* The rotation by  $120^\circ$ ,  $R$ , satisfies  $R^2 = R^{-1} = R^T$ , and  $[1], [0]$  trivially satisfy the condition, so any block-diagonal matrix built from  $a$  blocks  $[1]$ ,  $b$  blocks  $R$  and zeros satisfies  $A^2 = A^T$  (squaring and transposing both act blockwise).

*Values.*  $b = 0$ :  $a - b \in \{0, \dots, 5\}$ .  $b = 1$  ( $a \leq 3$ ):  $\{-1, 0, 1, 2\}$ .  $b = 2$  ( $a \leq 1$ ):  $\{-2, -1\}$ . The union is  $\{-2, -1, 0, 1, 2, 3, 4, 5\}$ , whose sum is 12.

**Check.** The extremes make sense:  $\text{tr } A = 5$  for  $A = I$ , and  $\text{tr } A = -2$  for  $A = R \oplus R \oplus [0]$ ; a trace of  $-3$  would need three rotation blocks, which do not fit in dimension 5. Solving  $\bar{\lambda} = \lambda^2$  with  $\lambda = u + vi$  directly ( $u^2 - v^2 = u$ ,  $2uv = -v$ ) gives exactly  $0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ .

5. **Answer:**  $\frac{7}{24}$ . **The idea.** The cycle through a fixed point of a random permutation has uniformly distributed length, and once that cycle is removed, what remains is again a uniformly random permutation of the leftover points. So first reveal the cycle of 1, then ask where 2 is.

**Work it out.**  $L$  is uniform. Trace the cycle of 1:  $1 \rightarrow \sigma(1) \rightarrow \sigma^2(1) \rightarrow \dots$ . The number of permutations in which this cycle has length exactly  $k$  is  $(n-1)(n-2) \cdots (n-k+1) \cdot (n-k)! = (n-1)!$  (choose the  $k-1$  successive images, then permute the rest arbitrarily). So  $P(L = k) = \frac{1}{n}$  for  $1 \leq k \leq n$ .

*Given  $L = k$ .* By the same count, conditioned on the cycle of 1 being a particular  $k$ -cycle, the rest of  $\sigma$  is a uniform permutation of the remaining  $n-k$  points; and by symmetry the  $k-1$  other points of the cycle of 1 form a uniformly random  $(k-1)$ -subset of  $\{2, \dots, n\}$ . Hence 2 lies in the cycle of 1 with probability  $\frac{k-1}{n-1}$ , in which case  $M = k$ . Otherwise 2 is one of the  $n-k$  remaining points, and  $M$  is the cycle length of a fixed point in a uniform permutation of  $n-k$  points: uniform on  $\{1, \dots, n-k\}$ , with mean  $\frac{n-k+1}{2}$ . Therefore  $\mathbb{E}[LM] = \frac{1}{n} \sum_{k=1}^n \left[ \frac{k-1}{n-1} k^2 + \frac{n-k}{n-1} k \frac{n-k+1}{2} \right]$ .

*The limit.* Divide by  $n^2$  and put  $x = k/n$ : the bracket is  $n^2 \left[ x^3 + \frac{1}{2}x(1-x)^2 \right] + O(n)$  uniformly in  $k$ , so  $\frac{\mathbb{E}[LM]}{n^2}$  is a Riemann sum converging to  $\int_0^1 \left( x^3 + \frac{1}{2}x(1-x)^2 \right) dx = \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{12} = \frac{7}{24}$ .

**Check.** For  $n = 5$  the bracketed terms for  $k = 1, \dots, 5$  are  $\frac{5}{2}, 4, \frac{27}{4}, 13, 25$ , with sum  $\frac{205}{4}$ , so  $\mathbb{E}[LM] = \frac{41}{4}$ , which matches a brute-force average over all 120 permutations. Independently, writing  $LM = \sum_{i,j} [i \sim 1][j \sim 2]$  and using the fact that the cycles of a random permutation split any  $k$  fixed points into blocks  $B$  with probability  $\prod (|B|-1)!/k!$  gives  $\mathbb{E}[LM]$  as a rational function of  $n$  whose limit over  $n^2$  is again  $\frac{7}{24}$ . (Heuristically: with probability about  $\frac{1}{2}$  in the limit the two points share a cycle, and  $L, M$  are of order  $n$ .)

6. **Answer:**  $\ln 2 - \frac{\pi}{2}$ . **The idea.** The matrix is a Cauchy matrix  $\left[ \frac{1}{x_i + y_j} \right]$  with  $x_i = y_i = i^2$ , whose determinant is an explicit product. After taking logarithms, the  $\ln n$  pieces cancel at order  $n^2$ , and what is left is a sum of Riemann sums for three double integrals of logarithms, each with an integrable singularity.

**Work it out.** *Cauchy's formula.*  $\det \left[ \frac{1}{x_i + y_j} \right] = \frac{\prod_{i < j} (x_j - x_i)(y_j - y_i)}{\prod_{i,j} (x_i + y_j)}$ . (Proof sketch: multiplying by  $\prod_{i,j} (x_i + y_j)$  gives a polynomial of degree  $n^2 - n$  that vanishes when  $x_i = x_j$  or  $y_i = y_j$ , so it is a constant multiple of the numerator, which also has degree  $n^2 - n$ ; the constant is 1, by induction on  $n$  after multiplying by  $x_n + y_n$  and letting  $x_n = -y_n$ .) With  $x_i = y_i = i^2$ ,  $D_n = \frac{\prod_{1 \leq i < j \leq n} (j^2 - i^2)^2}{\prod_{i,j=1}^n (i^2 + j^2)} > 0$ ,  $\ln D_n = 2 \sum_{i < j} [\ln(j-i) + \ln(j+i)] - \sum_{i,j} \ln(i^2 + j^2)$ .

*Removing the scale.* Write  $\ln(j-i) = \ln n + \ln \frac{j-i}{n}$ ,  $\ln(j+i) = \ln n + \ln \frac{j+i}{n}$  and  $\ln(i^2 + j^2) = 2 \ln n + \ln \frac{i^2 + j^2}{n^2}$ . The  $\ln n$  terms total  $2 \cdot \binom{n}{2} \cdot 2 \ln n - n^2 \cdot 2 \ln n = -2n \ln n = o(n^2)$ . So we need the limits of  $S_1 = \frac{2}{n^2} \sum_{i < j} \ln \frac{j-i}{n}$ ,  $S_2 = \frac{2}{n^2} \sum_{i < j} \ln \frac{j+i}{n}$ ,  $S_3 = \frac{1}{n^2} \sum_{i,j} \ln \left( \left( \frac{i}{n} \right)^2 + \left( \frac{j}{n} \right)^2 \right)$ .

*The Riemann sums converge.* (1) Grouping by  $d = j - i$ :  $S_1 = \frac{2}{n} \sum_{d=1}^{n-1} h\left(\frac{d}{n}\right)$  with  $h(t) = (1-t) \ln t$ . Since  $h'(t) = \frac{1}{t} - 1 - \ln t > 0$  on  $(0, 1)$ ,  $h$  is increasing, so  $\int_0^{1-1/n} h \leq \frac{1}{n} \sum_{d=1}^{n-1} h\left(\frac{d}{n}\right) \leq$

$\int_{1/n}^1 h$ , and both bounds tend to  $\int_0^1 (1-t) \ln t \, dt = -\frac{3}{4}$  because  $h$  is integrable. So  $S_1 \rightarrow -\frac{3}{2}$ . (2) and (3) Let  $g(x, y)$  be  $\ln(x+y)$  or  $\ln(x^2+y^2)$ ; it is increasing in each variable. Comparing  $g(\frac{i}{n}, \frac{j}{n})$  with its average over the cells  $[\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}]$  (below it) and  $[\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}]$  (above it) gives  $\iint_{[0,1]^2} g \leq \frac{1}{n^2} \sum_{i,j=1}^n g(\frac{i}{n}, \frac{j}{n}) \leq \iint_{[1/n, 1+1/n]^2} g$ , and both bounds tend to  $\iint_{[0,1]^2} g$  since  $g$  is integrable near the origin (the only singularity). For  $S_2$ ,  $\sum_{i < j} = \frac{1}{2}(\sum_{i,j} - \sum_{i=j})$  and the diagonal contributes  $\frac{1}{n^2} \sum_i \ln \frac{2i}{n} = O(\frac{1}{n})$ . Hence  $\lim \frac{\ln D_n}{n^2} = \iint_{[0,1]^2} [\ln |x-y| + \ln(x+y) - \ln(x^2+y^2)] \, dx \, dy$ .

*The integrals.*  $\iint \ln |x-y| = \int_{-1}^1 (1-|t|) \ln |t| \, dt = -\frac{3}{2}$ . The density of  $x+y$  is  $s$  on  $[0, 1]$  and  $2-s$  on  $[1, 2]$ , so  $\iint \ln(x+y) = \int_0^1 s \ln s \, ds + \int_1^2 (2-s) \ln s \, ds = -\frac{1}{4} + (4 \ln 2 - 2) - (2 \ln 2 - \frac{3}{4}) = 2 \ln 2 - \frac{3}{2}$ . Finally  $\int_0^1 \ln(x^2+y^2) \, dy = \ln(1+x^2) - 2 + 2x \arctan \frac{1}{x}$  (integrate by parts), and  $\int_0^1 \ln(1+x^2) \, dx = \ln 2 - 2 + \frac{\pi}{2}$ ,  $\int_0^1 2x \arctan \frac{1}{x} \, dx = [x^2 \arctan \frac{1}{x}]_0^1 + \int_0^1 \frac{x^2}{1+x^2} \, dx = \frac{\pi}{4} + 1 - \frac{\pi}{4} = 1$ . So  $\iint \ln(x^2+y^2) = \ln 2 - 3 + \frac{\pi}{2}$ . The limit is  $-\frac{3}{2} + (2 \ln 2 - \frac{3}{2}) - (\ln 2 - 3 + \frac{\pi}{2}) = \ln 2 - \frac{\pi}{2} \approx -0.8776$ .

**Check.** For  $n = 6$  the determinant computed directly equals the Cauchy product exactly. The three double integrals, computed as genuine two-dimensional quadratures (the last in polar coordinates), reproduce  $\ln 2 - \frac{\pi}{2}$  to 58 digits. At  $n = 400$  the exact log-sum gives  $\frac{\ln D_n}{n^2} \approx -0.889$ , already close to  $-0.8776$ ; the error is  $O(\frac{\ln n}{n})$ , and fitting  $\ln D_n \approx an^2 + bn \ln n + cn + d \ln n + e$  to the exact values at  $n = 250, 500, \dots, 4000$  gives  $a \approx -0.8776491$ , matching  $\ln 2 - \frac{\pi}{2} = -0.87764914 \dots$  to seven digits.

7. **Answer:**  $2(1 - \ln 2)^2$ . **The idea.** The awkward factor  $\frac{1}{m+n+1}$  is an integral,  $\int_0^1 x^{m+n} \, dx$ . Once it is written that way the double sum splits into a product of two identical one-variable sums, each of which is a logarithm.

**Work it out.** First, the series converges absolutely:  $mn(m+n+1) \geq mn \cdot 2\sqrt{mn}$ , and  $\sum_{m,n} \frac{1}{2(mn)^{3/2}} = \frac{1}{2} \zeta(3/2)^2 < \infty$ . So we may rearrange freely, and by Tonelli (the terms  $\frac{x^{m+n}}{mn}$  are nonnegative and their integrals sum to a finite value) we may swap sum and integral:  $S = \int_0^1 \sum_{m,n \geq 1} \frac{(-x)^m (-x)^n}{mn} \, dx = \int_0^1 \left( \sum_{m \geq 1} \frac{(-x)^m}{m} \right)^2 \, dx$ . For  $0 \leq x < 1$ ,  $\sum_{m \geq 1} \frac{(-x)^m}{m} = -\ln(1+x)$ , so  $S = \int_0^1 \ln^2(1+x) \, dx = \int_1^2 \ln^2 u \, du$ . An antiderivative of  $\ln^2 u$  is  $u \ln^2 u - 2u \ln u + 2u$  (differentiate to check:  $\ln^2 u + 2 \ln u - 2 \ln u - 2 + 2 = \ln^2 u$ ). Hence  $S = (2 \ln^2 2 - 4 \ln 2 + 4) - (0 - 0 + 2) = 2 \ln^2 2 - 4 \ln 2 + 2 = 2(1 - \ln 2)^2 \approx 0.18832$ .

**Check.** Group the terms by  $s = m + n$ . Since  $\frac{1}{mn} = \frac{1}{s} \left( \frac{1}{m} + \frac{1}{n} \right)$ ,  $\sum_{m+n=s} \frac{1}{mn} = \frac{2H_{s-1}}{s}$ , so  $S = \sum_{s \geq 2} \frac{(-1)^s 2H_{s-1}}{s(s+1)} = \frac{2}{6} - \frac{2 \cdot \frac{3}{2}}{12} + \frac{2 \cdot \frac{11}{6}}{20} - \dots$ . Its partial sums 0.333, 0.083, 0.267, 0.128, 0.237, ... oscillate and close in on 0.18832; summed to 60 digits it agrees with  $2(1 - \ln 2)^2$ .

8. **Answer:** 64. **The idea.**  $N(A)^3$  is the number of ordered triples  $(v_1, v_2, v_3)$  of vectors that  $A$  fixes, i.e. the number of fixed points of  $A$  acting on  $X = (\mathbb{F}_5^4)^3$ . By Burnside's lemma the average number of fixed points of a group element equals the number of orbits. So we must count the orbits of  $G = GL_4(\mathbb{F}_5)$  on triples of vectors, and a triple is determined up to  $G$  by its linear relations.

**Work it out.** Burnside:  $\frac{1}{|G|} \sum_{A \in G} |\text{Fix}(A)| = \#\{\text{orbits of } G \text{ on } X\}$ , and  $|\text{Fix}(A)| = N(A)^3$ , so  $\mathbb{E}[N(A)^3]$  is the number of orbits.

For a triple  $\mathbf{v} = (v_1, v_2, v_3)$  let  $R(\mathbf{v}) = \{c \in \mathbb{F}_5^3 : c_1 v_1 + c_2 v_2 + c_3 v_3 = 0\}$ , a subspace of  $\mathbb{F}_5^3$  (the kernel of  $c \mapsto \sum c_i v_i$ ). Invertible  $A$  preserves it:  $\sum c_i A v_i = A \sum c_i v_i$  is zero exactly when  $\sum c_i v_i$  is. Conversely, suppose  $R(\mathbf{v}) = R(\mathbf{w})$ . Define  $\phi : \text{span}(\mathbf{v}) \rightarrow \text{span}(\mathbf{w})$  by  $\phi(\sum c_i v_i) =$

$\sum c_i w_i$ . It is well defined and injective because  $\sum c_i v_i = \sum c'_i v_i$  iff  $c - c' \in R(\mathbf{v}) = R(\mathbf{w})$  iff  $\sum c_i w_i = \sum c'_i w_i$ ; it is onto, so it is a linear isomorphism between spaces of the same dimension  $d$ . Extend a basis of each span to a basis of  $\mathbb{F}_5^4$  and extend  $\phi$  to an invertible  $A$ ; then  $Av_i = w_i$ . So orbits correspond injectively to relation spaces.

Every subspace  $R \subseteq \mathbb{F}_5^3$  occurs:  $\mathbb{F}_5^3/R$  has dimension at most  $3 \leq 4$ , so there is a linear map  $T: \mathbb{F}_5^3 \rightarrow \mathbb{F}_5^4$  with kernel exactly  $R$ , and  $v_i = T(e_i)$  has  $R(\mathbf{v}) = R$ . Hence the number of orbits equals the number of subspaces of  $\mathbb{F}_5^3$ : one of dimension 0,  $\frac{5^3-1}{5-1} = 31$  lines, 31 planes (dual to lines), and one of dimension 3. The expectation is  $1 + 31 + 31 + 1 = 64$ .

**Check.** Count directly instead.  $A$  fixes a triple iff it is the identity on  $U = \text{span}(\mathbf{v})$ ; if  $\dim U = d$ , writing  $A$  in a basis extending one of  $U$  shows there are  $5^{d(4-d)} |GL_{4-d}(\mathbb{F}_5)|$  such  $A$ . Summing  $\Pr(A \text{ fixes } \mathbf{v})$  over all  $5^{12}$  triples, sorted by  $d$ , gives exactly 64. The same reasoning with  $GL_3(\mathbb{F}_3)$  predicts the number of subspaces of  $\mathbb{F}_3^3$ , namely  $1 + 13 + 13 + 1 = 28$ , and a brute-force average over all 11232 matrices of  $GL_3(\mathbb{F}_3)$  gives 28.

9. **Answer:** 8161. **The idea.** Read  $A$  as a directed graph on vertices  $1, \dots, 5$  with an edge  $i \rightarrow j$  when  $a_{ij} = 1$  (loops allowed). Because all entries are nonnegative,  $(A^3)_{ij}$  is exactly the number of directed walks of length 3 from  $i$  to  $j$ , so  $A^3 = 0$  says: there is no walk with 3 edges. Such a graph is acyclic, and labelling each vertex by the length of the longest path leaving it cuts the graph into three levels with a rigid structure that can be counted directly.

**Work it out.** Suppose  $A^3 = 0$ . A cycle (a loop included) could be walked around for 3 steps, so the graph has no cycles, and every path has at most 2 edges. Let  $h(v) \in \{0, 1, 2\}$  be the number of edges in a longest path starting at  $v$ , and let  $H_k = \{v : h(v) = k\}$ , of sizes  $a, b, c$  with  $a + b + c = 5$ . Then:

- (i)  $v \in H_0$  has no out-edges.
- (ii)  $v \in H_1$  has at least one out-edge, and every out-neighbour  $w$  has  $h(w) = 0$  (if  $h(w) \geq 1$  we would get a path of length 2 from  $v$ ).
- (iii)  $v \in H_2$ : every out-neighbour  $w$  has  $h(w) \leq 1$  (otherwise a path of length 3), and at least one has  $h(w) = 1$  (the second vertex of a longest path).

Conversely, take any split of the vertices into labelled sets  $H_0, H_1, H_2$  and any edges obeying (i)–(iii). Every edge goes to a strictly smaller label, so every walk has at most 2 edges and  $A^3 = 0$ ; and the labels are the longest-path lengths (a vertex of  $H_1$  has an edge into  $H_0$  and nothing further; a vertex of  $H_2$  has an edge to some vertex of  $H_1$ , which has an edge onward, while no walk exceeds 2 edges), so the graph determines the split. Thus the matrices correspond bijectively to (split, edges obeying (i)–(iii)). For a given split each vertex of  $H_1$  picks a nonempty subset of  $H_0$ :  $2^a - 1$  ways; each vertex of  $H_2$  picks a subset of  $H_0 \cup H_1$  meeting  $H_1$ :  $2^a(2^b - 1)$  ways. Hence  $\# = \sum_{a+b+c=5} \binom{5}{a,b,c} (2^a - 1)^b (2^a(2^b - 1))^c$ .

The terms with  $c = 0$  (these are the matrices with  $A^2 = 0$ ) are  $1 + 75 + 490 + 270 + 5 = 841$  for  $(a, b) = (5, 0), (4, 1), (3, 2), (2, 3), (1, 4)$ . With  $c = 1$ :  $(3, 1)$ :  $20 \cdot 7 \cdot 8 = 1120$ ;  $(2, 2)$ :  $30 \cdot 9 \cdot 12 = 3240$ ;  $(1, 3)$ :  $20 \cdot 1 \cdot 14 = 280$ ; total 4640. With  $c = 2$ :  $(2, 1)$ :  $30 \cdot 3 \cdot 16 = 1440$ ;  $(1, 2)$ :  $30 \cdot 1 \cdot 36 = 1080$ ; total 2520. With  $c = 3$ :  $(1, 1)$ :  $20 \cdot 1 \cdot 2^3 = 160$ . (Terms with  $a = 0$  vanish, and  $b = 0 < c$  vanishes.) Altogether  $841 + 4640 + 2520 + 160 = 8161$ .

**Check.** Small sizes: for  $2 \times 2$  the formula gives 3 (the zero matrix and the two off-diagonal units), and for  $3 \times 3$  it gives  $25 = 13 + 12$ : the 13 matrices with  $A^2 = 0$  plus, for each of the 6 orderings  $x \rightarrow y \rightarrow z$ , the path with or without the shortcut edge  $x \rightarrow z$ . An exhaustive computer check of all  $2^{25}$  matrices, multiplying the 0/1 matrices directly, also finds 8161.

10. **Answer:**  $\frac{16\pi-4\sqrt{3}\pi+8}{3}$ . **The idea.** Integrate over the square  $Q = [-1, 1]^2$  in the  $xy$ -plane: above and below every point of  $Q$  the piece is a full vertical chord of the ball. Polar coordinates on one eighth of the square reduce everything to a single integral  $\int_0^1 \frac{(2-t^2)^{3/2}}{1+t^2} dt$ , and the identity  $2 - t^2 = 3 - (1 + t^2)$  splits it into a circular area and an arctangent that happens to be  $\arctan \sqrt{3} = \frac{\pi}{3}$ .

**Work it out.** Every  $(x, y) \in Q$  has  $x^2 + y^2 \leq 2 < 3$ , so the vertical line through it meets  $B$  in the segment  $|z| \leq \sqrt{3 - x^2 - y^2}$ , and a point with  $|x| > 1$  or  $|y| > 1$  is not in the piece. Hence  $V = \iint_Q 2\sqrt{3 - x^2 - y^2} dA$ . The integrand and  $Q$  are invariant under the 8 symmetries of the square, and the triangle  $0 \leq y \leq x \leq 1$  is, in polar coordinates,  $0 \leq \theta \leq \frac{\pi}{4}$ ,  $0 \leq \rho \leq \sec \theta$ . So  $V = 16 \int_0^{\pi/4} \int_0^{\sec \theta} \sqrt{3 - \rho^2} \rho d\rho d\theta = \frac{16}{3} \int_0^{\pi/4} \left( 3\sqrt{3} - (3 - \sec^2 \theta)^{3/2} \right) d\theta = 4\sqrt{3}\pi - \frac{16}{3}J$ , where  $J = \int_0^{\pi/4} (2 - \tan^2 \theta)^{3/2} d\theta$  (using  $3 - \sec^2 \theta = 2 - \tan^2 \theta \geq 1$  on this range). Put  $t = \tan \theta$ ,  $d\theta = \frac{dt}{1+t^2}$ , and write  $2 - t^2 = 3 - (1 + t^2)$ :  $J = \int_0^1 \frac{(2-t^2)^{3/2}}{1+t^2} dt = 3K - L$ ,  $K = \int_0^1 \frac{\sqrt{2-t^2}}{1+t^2} dt$ ,  $L = \int_0^1 \sqrt{2-t^2} dt$ .

$L$  is the area under the circle of radius  $\sqrt{2}$  over  $[0, 1]$ : a right triangle with legs 1, 1 (area  $\frac{1}{2}$ ) plus a sector of angle  $\frac{\pi}{4}$  (area  $\frac{1}{2} \cdot 2 \cdot \frac{\pi}{4}$ ), so  $L = \frac{1}{2} + \frac{\pi}{4}$ .

For  $K$  put  $t = \sqrt{2} \sin \varphi$ ,  $0 \leq \varphi \leq \frac{\pi}{4}$ : then  $\sqrt{2-t^2} dt = 2 \cos^2 \varphi d\varphi$  and  $2 \cos^2 \varphi = 3 - (1 + 2 \sin^2 \varphi)$ , so  $K = \int_0^{\pi/4} \frac{3 d\varphi}{1+2 \sin^2 \varphi} - \frac{\pi}{4}$ . Dividing top and bottom by  $\cos^2 \varphi$  and setting  $u = \tan \varphi$  gives  $\int_0^{\pi/4} \frac{d\varphi}{1+2 \sin^2 \varphi} = \int_0^1 \frac{du}{1+3u^2} = \frac{\arctan \sqrt{3}}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$ . Hence  $K = \frac{\pi}{\sqrt{3}} - \frac{\pi}{4}$  and  $J = \sqrt{3}\pi - \frac{3\pi}{4} - \frac{1}{2} - \frac{\pi}{4} = \sqrt{3}\pi - \pi - \frac{1}{2}$ . Therefore  $V = 4\sqrt{3}\pi - \frac{16}{3} \left( \sqrt{3}\pi - \pi - \frac{1}{2} \right) = \frac{16\pi-4\sqrt{3}\pi+8}{3} \approx 12.16663$ .

**Check.** Sanity bounds: the piece contains the cube (volume 8) and lies in  $B$  (volume  $4\sqrt{3}\pi \approx 21.77$ ). Inclusion-exclusion gives an independent route: the piece is  $B$  minus the four caps  $x > 1$ ,  $x < -1$ ,  $y > 1$ ,  $y < -1$ , plus the four places where two caps overlap (such as  $x, y > 1$ ); no point lies in three caps, because any three of them include an opposite pair such as  $x > 1$  and  $x < -1$ . Each cap has height  $h = \sqrt{3} - 1$  and volume  $\frac{\pi h^2(3\sqrt{3}-h)}{3} = \frac{\pi(6\sqrt{3}-8)}{3}$ , so the formula predicts that each overlap has volume  $\frac{2\sqrt{3}\pi-4\pi+2}{3} \approx 0.10548$ ; a direct double integral of  $2\sqrt{3-x^2-y^2}$  over  $\{x, y \geq 1, x^2+y^2 \leq 3\}$  agrees to 40 digits. Slicing horizontally instead (at height  $z$  the section is  $Q$  cut by a disk of radius  $\sqrt{3-z^2}$ ) and integrating numerically also gives 12.1666300288....

11. **Answer:** 15. **The idea.** Write  $q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} u$  with  $u$  a unit pure quaternion; then  $x \mapsto qxq^{-1}$  is the rotation by  $\theta$  about  $u$ , whose trace is  $1+2\cos \theta = 4\cos^2 \frac{\theta}{2} - 1$ . So  $\text{tr } A = 4t^2 - 1$  where  $t$  is the real coordinate of a uniform point on the sphere  $S^3$ . That coordinate has the semicircle density, and the moment becomes the constant term of a Laurent polynomial.

**Work it out.** *Distribution of  $t$ .* For a uniform point of  $S^3 \subset \mathbb{R}^4$ , the slice at first coordinate  $t$  is a 2-sphere of radius  $\sqrt{1-t^2}$ , area  $4\pi(1-t^2)$ , met at slant factor  $1/\sqrt{1-t^2}$ ; so  $t$  has density proportional to  $\sqrt{1-t^2}$  on  $[-1, 1]$ , which normalises to  $\frac{2}{\pi}\sqrt{1-t^2}$ .

Put  $t = \cos \varphi$ ,  $\varphi \in [0, \pi]$ : the density becomes  $\frac{2}{\pi} \sin^2 \varphi d\varphi$ , and  $4t^2 - 1 = 1 + 2\cos 2\varphi$ . With  $\psi = 2\varphi \in [0, 2\pi]$  and  $\sin^2 \varphi = \frac{1-\cos \psi}{2}$ :  $\mathbb{E}[(\text{tr } A)^6] = \frac{1}{2\pi} \int_0^{2\pi} (1+2\cos \psi)^6 (1-\cos \psi) d\psi$ . (Equivalently the rotation angle has density  $\frac{1-\cos \psi}{\pi}$  on  $[0, \pi]$ .) With  $z = e^{i\psi}$ ,  $1+2\cos \psi = 1+z+z^{-1}$  and  $1-\cos \psi = 1-\frac{1}{2}(z+z^{-1})$ , and  $\frac{1}{2\pi} \int_0^{2\pi}$  extracts the constant term. If  $T_k$  is the coefficient of  $z^k$  in  $(1+z+z^{-1})^6$ , the answer is  $T_0 - \frac{1}{2}(T_1 + T_{-1}) = T_0 - T_1$ .

Counting words of six letters from  $\{+1, 0, -1\}$ :  $T_0 = \sum_j \frac{6!}{j!j!(6-2j)!} = 1 + 30 + 90 + 20 = 141$  ( $j$  = number of  $+1$ s), and  $T_1 = \sum_j \frac{6!}{(j+1)!j!(5-2j)!} = 6 + 60 + 60 = 126$ . So the expectation is  $141 - 126 = 15$ .

**Check.**  $\text{tr } A$  is the character of the 3-dimensional (spin-1) representation  $V_1$ , so  $\mathbb{E}[\chi^6]$  is the number of trivial summands in  $V_1^{\otimes 6}$ . Using  $V_l \otimes V_1 = V_{l-1} \oplus V_l \oplus V_{l+1}$  ( $l \geq 1$ ) and  $V_0 \otimes V_1 = V_1$ , the trivial multiplicities of  $V_1^{\otimes k}$  for  $k = 0, \dots, 6$  are 1, 0, 1, 1, 3, 6, 15 (the Riordan numbers); for instance  $k = 2$  gives 1, matching  $\mathbb{E}[(\text{tr } A)^2] = 1$ . A direct symbolic integration of  $\frac{1}{\pi} \int_0^\pi (1 + 2 \cos \psi)^6 (1 - \cos \psi) d\psi$  also gives 15.

12. **Answer:**  $\frac{64}{1575}$ . **The idea.** Parametrise  $R$  by the roots. On the region  $-1 \leq r_1 < r_2 < r_3 < r_4 \leq 1$  the map from roots to coefficients is one-to-one, and its Jacobian determinant is, up to sign, the Vandermonde product  $\Delta = \prod_{i < j} (r_j - r_i)$ . So the volume is  $\int \Delta$  over the ordered region, and de Bruijn's identity turns that fourfold integral of a determinant into a  $4 \times 4$  Pfaffian of double integrals.

**Work it out.** *Change of variables.* The coefficient map is  $\Phi(r) = (-e_1, e_2, -e_3, e_4)$  in the elementary symmetric functions. Its Jacobian determinant  $J$  is a polynomial in the  $r_i$  of degree  $0 + 1 + 2 + 3 = 6$ . If  $r_i = r_j$  the columns  $\partial \Phi / \partial r_i$  and  $\partial \Phi / \partial r_j$  coincide (swap symmetry), so  $J$  vanishes; hence  $(r_i - r_j) \mid J$  for every pair and  $\Delta \mid J$ . Both have degree 6, so  $J = \pm \Delta \cdot \text{const}$ , and comparing the coefficient of  $r_2 r_3^2 r_4^3$  gives  $|J| = |\Delta|$ . Every point of  $R$  whose roots are distinct has exactly one preimage in the open ordered region  $O = \{-1 < r_1 < \dots < r_4 < 1\}$ , and  $\Phi$  is a diffeomorphism there ( $J \neq 0$ ). Points of  $R$  with a repeated root or a root at  $\pm 1$  form the image of a set of lower dimension under a polynomial map, hence have volume 0. So  $\text{vol}(R) = \int_O \Delta dr$ ,  $\Delta = \det [r_i^j]_{1 \leq i \leq 4, 0 \leq j \leq 3} > 0$  on  $O$ .

*De Bruijn.* For an even number of functions,  $\int_{x_1 < \dots < x_4} \det[\phi_j(x_i)] dx = \text{Pf}[a_{jk}]$  with  $a_{jk} = \iint \text{sgn}(y - x) \phi_j(x) \phi_k(y) dx dy$ . Here  $\phi_j(x) = x^j$  on  $[-1, 1]$ . The inner integral is  $\int_{-1}^1 \text{sgn}(y - x) x^j dx = \frac{2y^{j+1} - 1 - (-1)^{j+1}}{j+1}$ , and integrating against  $y^k$  gives  $a_{01} = \frac{4}{3}$ ,  $a_{03} = \frac{4}{5}$ ,  $a_{12} = -\frac{4}{15}$ ,  $a_{23} = \frac{4}{21}$ ,  $a_{02} = a_{13} = 0$  (the zero entries have  $j + k$  even, where the integrand is odd under  $(x, y) \mapsto (-y, -x)$ ). The Pfaffian of a  $4 \times 4$  antisymmetric matrix is  $a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}$ , so  $\text{vol}(R) = \frac{4}{3} \cdot \frac{4}{21} - 0 + \frac{4}{5} \cdot \left(-\frac{4}{15}\right) = \frac{16}{63} - \frac{16}{75} = \frac{16 \cdot 12}{4725} = \frac{64}{1575}$ .

(Alternatively, Selberg's integral with  $\alpha = \beta = 1$ ,  $\gamma = \frac{1}{2}$  gives  $\int_{[0,1]^4} |\Delta| = \frac{1}{1050}$ ; rescaling to  $[-1, 1]$  multiplies by  $2^{4+6}$ , and dividing by  $4!$  for the ordering gives  $\frac{1024}{25200} = \frac{64}{1575}$  again.)

**Check.** The same method in lower degree: for  $x^2 + ax + b$  it gives  $\int_{-1 < r_1 < r_2 < 1} (r_2 - r_1) = \frac{4}{3}$ , and indeed the region  $\{|a| \leq 1 + b, b \leq a^2/4, |a| \leq 2\}$  between the parabola and its two tangent lines  $b = \pm a - 1$  has area  $\frac{4}{3}$ ; for cubics it gives  $\frac{16}{45}$ . Expanding  $\Delta$  and integrating iteratively in  $r_1$ , then  $r_2, r_3, r_4$  with exact arithmetic gives  $\frac{64}{1575} \approx 0.0406$  directly.