

# MathCastle AMC 12

## Practice Contest 14

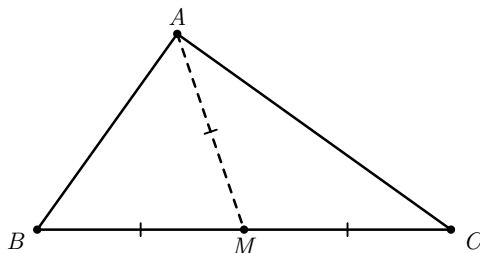
25 questions • 75 minutes • 6 points per correct answer, 1.5 per blank, 0 per wrong.

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1. Positive real numbers  $a$ ,  $b$  and  $c$  satisfy  $ab = 2$ ,  $bc = 3$  and  $ca = 6$ . What is the value of the product  $abc$ ?

(A) 6   (B) 11   (C) 12   (D) 18   (E) 36

2. In triangle  $ABC$ , let  $M$  be the midpoint of side  $\overline{BC}$ . It is given that  $AM = \frac{1}{2}BC$ . What is the measure of angle  $BAC$ , in degrees?



(A) 30   (B) 45   (C) 60   (D) 90   (E) 120

3. What is the sum of all the *distinct* real roots of the equation  $(x^2 - 3x + 2)(x^2 - 5x + 6)(x^2 - 7x + 12) = 0$ ?

(A) 6   (B) 10   (C) 12   (D) 15   (E) 20

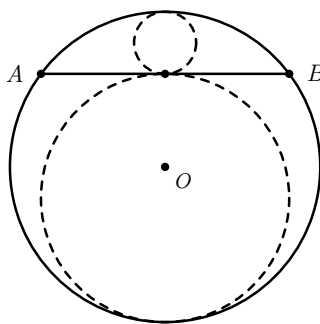
4. Let  $x = \cos 36^\circ$ . What is the exact value of  $4x^2 - 2x$ ? (You may use the double-angle and product-to-sum formulas; no table of values is needed.)

(A)  $\frac{1}{2}$    (B)  $\frac{\sqrt{5}-1}{2}$    (C) 1   (D)  $\frac{1+\sqrt{5}}{2}$    (E) 2

5. How many positive integers less than 1000 have digits that strictly decrease from left to right? (One-digit numbers count; for example 9, 70 and 531 all qualify, but 55 and 120 do not.)

(A) 165   (B) 174   (C) 175   (D) 184   (E) 219

6. A circle of radius 5 has a chord  $\overline{AB}$  of length 8. Two smaller circles are drawn, one on each side of the chord: each is tangent to  $\overline{AB}$  at its midpoint and tangent to the big circle. What is the sum of the radii of the two smaller circles?

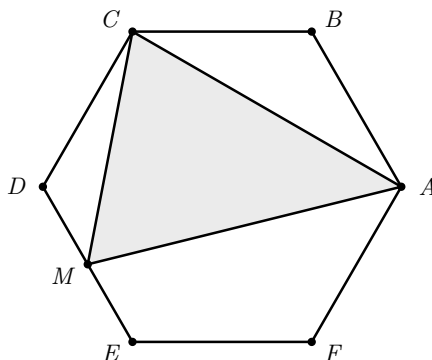


- (A) 3   (B) 4   (C) 5   (D) 6   (E) 8

7. Seven fair coins are flipped. What is the probability that the number of heads exceeds the number of tails by at least 2?

- (A)  $\frac{1}{8}$    (B)  $\frac{29}{128}$    (C)  $\frac{1}{4}$    (D)  $\frac{35}{128}$    (E)  $\frac{1}{2}$

8. Regular hexagon  $ABCDEF$  has side length 1, and  $M$  is the midpoint of side  $\overline{DE}$ . What is the area of triangle  $ACM$ ?



- (A)  $\frac{\sqrt{3}}{2}$    (B) 1   (C)  $\frac{5\sqrt{3}}{8}$    (D)  $\sqrt{3}$    (E)  $\frac{3\sqrt{3}}{2}$

9. Let  $f(x) = x^2 - 4x$  for every real number  $x$ . For how many real numbers  $x$  is  $f(f(x)) = f(x)$ , that is, applying  $f$  a second time changes nothing?

- (A) 1   (B) 2   (C) 3   (D) 4   (E) 5

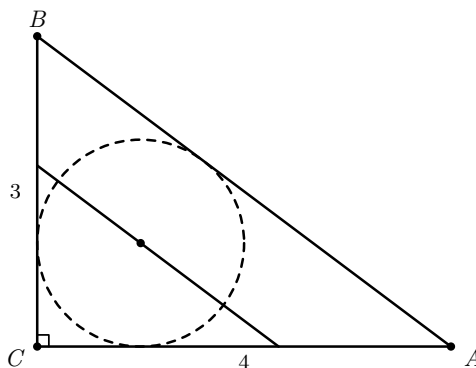
10. How many integers  $n$  with  $1 \leq n \leq 2026$  have the property that  $n$  and  $2n$  have the same number of digits when written in base ten?

- (A) 1026   (B) 1027   (C) 1471   (D) 1475   (E) 2026

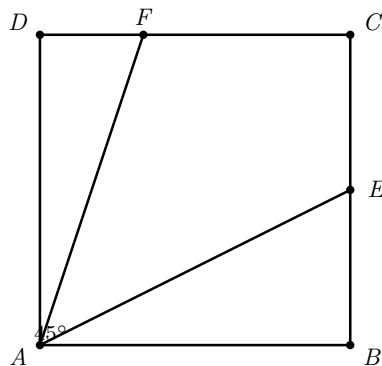
11. What is the value of  $\sin^2 0^\circ + \sin^2 15^\circ + \sin^2 30^\circ + \sin^2 45^\circ + \cdots + \sin^2 165^\circ$ , the sum of  $\sin^2(15k)^\circ$  for  $k = 0, 1, \dots, 11$ ?

- (A) 4   (B)  $\frac{9}{2}$    (C) 5   (D)  $\frac{11}{2}$    (E) 6

12. For how many integers  $k$  does the equation  $x^3 - 3x + k = 0$  have three distinct real roots?  
**(A)** 3    **(B)** 4    **(C)** 5    **(D)** 6    **(E)** 7
13. How many six-digit positive integers use each of the digits 1, 1, 2, 2, 3, 3 exactly once (two of each) and have no two adjacent digits equal?  
**(A)** 24    **(B)** 30    **(C)** 36    **(D)** 48    **(E)** 90
14. A right triangle has legs of lengths 3 and 4, and a circle is inscribed in it. The line through the centre of this circle parallel to the hypotenuse cuts off a smaller right triangle at the right-angle vertex. What is the area of that smaller triangle?

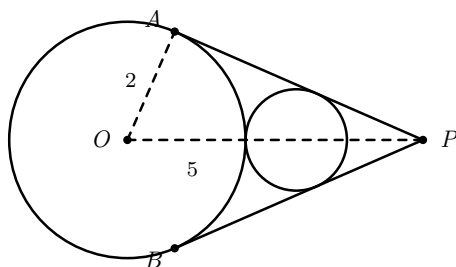


- (A)**  $\frac{49}{24}$     **(B)**  $\frac{9}{4}$     **(C)**  $\frac{5}{2}$     **(D)** 3    **(E)**  $\frac{7}{2}$
15. A regular octahedron has edge length 2. Each of its eight faces is parallel to exactly one other face. What is the distance between two such opposite (parallel) faces?  
**(A)** 1    **(B)**  $\frac{2\sqrt{3}}{3}$     **(C)**  $\sqrt{2}$     **(D)**  $\frac{2\sqrt{6}}{3}$     **(E)** 2
16. How many ordered pairs  $(x, y)$  of integers (positive, negative or zero) satisfy the equation  $x^2 + xy + y^2 = 7$ ?  
**(A)** 12    **(B)** 14    **(C)** 16    **(D)** 18    **(E)** 24
17. For how many integers  $n$  with  $1 \leq n \leq 100$  is  $n!$  divisible by  $2^{n-1}$ ? (For example  $n = 4$  qualifies, since  $4! = 24$  is divisible by  $2^3$ .)  
**(A)** 1    **(B)** 5    **(C)** 6    **(D)** 7    **(E)** 50
18. Square  $ABCD$  has side length 1. Point  $E$  is the midpoint of side  $\overline{BC}$ , and point  $F$  lies on side  $\overline{CD}$  so that  $\angle EAF = 45^\circ$ . What is the length of  $DF$ ?



- (A)  $\frac{1}{4}$    (B)  $\frac{1}{3}$    (C)  $\frac{2}{5}$    (D)  $\frac{1}{2}$    (E)  $\frac{2}{3}$

19. How many ordered triples  $(a, b, c)$  of nonnegative integers satisfy  $a + b + c = 20$  and have the property that  $a$ ,  $b$  and  $c$  are the side lengths of a triangle with positive area?  
 (A) 20   (B) 24   (C) 30   (D) 33   (E) 36
20. In how many ways can the numbers 1, 2, 3, 4, 5, 6 be arranged in a row so that, reading from left to right, the numbers first strictly increase and then strictly decrease, with the number 6 somewhere strictly between the two ends? (For example, 2 4 6 5 3 1 is such an arrangement.)  
 (A) 15   (B) 16   (C) 24   (D) 28   (E) 30
21. Consider the graph of  $y = |x^2 - 4|$  and the line  $y = x + k$ . For most values of  $k$  the two graphs meet in 0, 2 or 4 points. What is the sum of all real numbers  $k$  for which they meet in *exactly* 3 points?  
 (A) 2   (B)  $\frac{17}{4}$    (C)  $\frac{25}{4}$    (D) 8   (E) 10
22. A polynomial  $P(x)$  with integer coefficients satisfies  $P(1) = 5$  and  $P(5) = 1$ . It is also known that  $P(k) = k$  for some integer  $k$ . What is  $k$ ?  
 (A) -3   (B) 0   (C) 1   (D) 2   (E) 3
23. A circle of radius 2 has centre  $O$ , and a point  $P$  lies at distance 5 from  $O$ . The two tangent lines from  $P$  touch the circle at  $A$  and  $B$ . A second, smaller circle is tangent to segment  $\overline{PA}$ , to segment  $\overline{PB}$ , and to the minor arc  $AB$  of the first circle (it sits in the region bounded by those three). What is the radius of the smaller circle?



- (A)  $\frac{1}{2}$  (B)  $\frac{3}{5}$  (C)  $\frac{2}{3}$  (D)  $\frac{6}{7}$  (E) 1

24. Real numbers  $x$  and  $y$  satisfy  $x^2 + y^2 = 4$ . What is the largest possible value of  $x^3y - xy^3$ ?

- (A) 1 (B) 2 (C)  $2\sqrt{2}$  (D)  $2\sqrt{3}$  (E) 4

25. A *descent* in an arrangement  $p_1, p_2, \dots, p_6$  of the numbers 1 through 6 is a position  $i$  with  $p_i > p_{i+1}$ . How many arrangements of 1, 2, 3, 4, 5, 6 have exactly one descent?

- (A) 57 (B) 60 (C) 62 (D) 64 (E) 120

## Answer Key — AMC 12 Practice Contest 14

1. (A) 6
2. (D) 90
3. (B) 10
4. (C) 1
5. (B) 174
6. (C) 5
7. (B)  $\frac{29}{128}$
8. (C)  $\frac{5\sqrt{3}}{8}$
9. (D) 4
10. (C) 1471
11. (E) 6
12. (A) 3
13. (B) 30
14. (A)  $\frac{49}{24}$
15. (D)  $\frac{2\sqrt{6}}{3}$
16. (A) 12
17. (D) 7
18. (B)  $\frac{1}{3}$
19. (E) 36
20. (E) 30
21. (C)  $\frac{25}{4}$
22. (E) 3
23. (D)  $\frac{6}{7}$
24. (E) 4
25. (A) 57

## Solutions — AMC 12 Practice Contest 14

1. **Answer: (A) 6. The idea.** Do not solve for  $a, b, c$  one at a time. Multiplying the three equations together gives  $(ab)(bc)(ca) = a^2b^2c^2 = (abc)^2$ , which is exactly the square of what is asked.

**Work it out.**  $(abc)^2 = 2 \cdot 3 \cdot 6 = 36$ , so  $abc = 6$  (positive, since all three numbers are positive).

**Check.** Dividing  $abc = 6$  by each equation:  $c = 3$ ,  $a = 2$ ,  $b = 1$ , and indeed  $ab = 2$ ,  $bc = 3$ ,  $ca = 6$ . ✓ The traps: 36 forgets the square root;  $11 = 2 + 3 + 6$  adds; 12 and 18 multiply only two of the equations.

2. **Answer: (D) 90. The idea.**  $AM = \frac{1}{2}BC = BM = CM$ , so  $A, B$  and  $C$  are all at the same distance from  $M$ : they lie on a circle centred at  $M$  with  $\overline{BC}$  as a diameter. An angle inscribed in a semicircle is a right angle.

**Work it out.**  $\angle BAC = 90^\circ$ .

**Check.** In the triangles  $ABM$  and  $ACM$ , both isosceles,  $\angle BAM = \angle ABM$  and  $\angle CAM = \angle ACM$ ; adding,  $\angle BAC = \angle B + \angle C$ , and since the three angles sum to  $180^\circ$ ,  $\angle BAC = 90^\circ$ . ✓ The traps: 60 and 30 assume an equilateral or 30-60-90 picture; 45 and 120 are guesses.

3. **Answer: (B) 10. The idea.** Factor each quadratic:  $x^2 - 3x + 2 = (x - 1)(x - 2)$ ,  $x^2 - 5x + 6 = (x - 2)(x - 3)$ ,  $x^2 - 7x + 12 = (x - 3)(x - 4)$ . The roots 2 and 3 each appear twice, so Vieta on the degree-6 polynomial would overcount.

**Work it out.** The distinct roots are 1, 2, 3, 4, with sum 10.

**Check.** Vieta gives the sum *with* multiplicity:  $3 + 5 + 7 = 15 = 1 + 2 + 2 + 3 + 3 + 4$ , and removing the repeated 2 and 3 leaves 10. ✓ The traps: 15 counts multiplicity; 20 and 12 miscount; 6 stops at the first two factors.

4. **Answer: (C) 1. The idea.**  $4x^2 - 2x = 2(2x^2 - 1) - 2x + 2 = 2\cos 72^\circ - 2\cos 36^\circ + 2$  by the double-angle formula, so the question is really about  $\cos 36^\circ - \cos 72^\circ$ . That difference is a classic: multiply by  $2\sin 36^\circ$  and use product-to-sum.

**Work it out.** Let  $D = \cos 36^\circ - \cos 72^\circ$ . Then  $2\sin 36^\circ D = 2\sin 36^\circ \cos 36^\circ - 2\sin 36^\circ \cos 72^\circ = \sin 72^\circ - (\sin 108^\circ - \sin 36^\circ) = \sin 36^\circ$ , since  $\sin 108^\circ = \sin 72^\circ$ . So  $D = \frac{1}{2}$ , and  $4x^2 - 2x = 2 - 2D = 1$ .

**Check.**  $\cos 36^\circ = \frac{1 + \sqrt{5}}{4} \approx 0.809$ :  $4(0.809)^2 - 2(0.809) = 2.618 - 1.618 = 1.000$ . ✓ (Indeed  $x = \frac{1 + \sqrt{5}}{4}$  is a root of  $4x^2 - 2x - 1 = 0$ .) The traps:  $\frac{\sqrt{5}-1}{2}$  and  $\frac{1+\sqrt{5}}{2}$  are  $2\cos 72^\circ$  and  $2\cos 36^\circ$  themselves;  $\frac{1}{2}$  is  $D$ ; 2 forgets the  $-2D$ .

5. **Answer: (B) 174. The idea.** A strictly decreasing string of digits is determined by *which* digits it uses: any set of distinct digits can be arranged in decreasing order in exactly one way, and the leading digit is automatically the largest, hence nonzero (unless the set is  $\{0\}$  alone).

**Work it out.** One-digit numbers: 1 through 9, that is 9 (the digit 0 alone is not a positive integer). Two-digit: choose 2 of the 10 digits,  $\binom{10}{2} = 45$  (sets like  $\{1, 0\}$  give 10, which is fine). Three-digit:  $\binom{10}{3} = 120$ . Total  $9 + 45 + 120 = 174$ .

**Check.** A computer count of the integers below 1000 with strictly decreasing digits gives 174. The traps: 175 counts 0; 165 forgets the one-digit numbers; 184 uses  $\binom{10}{1} + \binom{10}{2} + \binom{10}{3} + 9$ ; 219 adds  $\binom{10}{2}$  twice or includes four-digit cases.

6. **Answer: (C) 5. The idea.** By symmetry each small circle is centred on the diameter perpendicular to the chord, so it touches the chord at its midpoint  $N$  and touches the big circle at the end of that diameter. Hence its diameter is exactly the distance from  $N$  to the big circle along the perpendicular, i.e. the height of the circular segment on its side.

**Work it out.** The chord is at distance  $\sqrt{5^2 - 4^2} = 3$  from the centre, so the two heights are  $5 - 3 = 2$  and  $5 + 3 = 8$ ; the radii are 1 and 4, with sum 5.

**Check.** The two diameters add up to the full diameter 10 regardless of where the chord is, so the radii always add to 5. ✓ The traps: 4 and 6 miscount one height; 3 is the distance to the chord; 8 is the larger diameter.

7. **Answer: (B)  $\frac{29}{128}$ . The idea.** With 7 flips the difference heads minus tails is odd, so "at least 2 more heads" means at least 3 more, i.e. at least 5 heads. Count those outcomes.

**Work it out.**  $\binom{7}{5} + \binom{7}{6} + \binom{7}{7} = 21 + 7 + 1 = 29$  outcomes out of  $2^7 = 128$ : probability  $\frac{29}{128}$ .

**Check.** By symmetry the probability of "at least 2 more tails" is also  $\frac{29}{128}$ , and the remaining outcomes, exactly 4-3 either way, number  $2\binom{7}{4} = 70$ ; indeed  $29 + 29 + 70 = 128$ . ✓ The traps:  $\frac{35}{128}$  counts 4 heads as well ( $\binom{7}{4} = 35$ ) and then drops a term;  $\frac{1}{4}$  and  $\frac{1}{8}$  are rough guesses;  $\frac{1}{2}$  is "more heads than tails".

8. **Answer: (C)  $\frac{5\sqrt{3}}{8}$ . The idea.** As a point  $P$  slides along segment  $\overline{DE}$ , the area of triangle  $ACP$  changes linearly (the base  $AC$  is fixed and the height varies linearly). So the area at the midpoint  $M$  is the average of the areas at  $D$  and at  $E$ :  $[ACM] = \frac{1}{2}([ACD] + [ACE])$ .

**Work it out.**  $AD$  is a diameter of the circumscribed circle (length 2), so  $\angle ACD = 90^\circ$  with  $CD = 1$  and  $AC = \sqrt{3}$ :  $[ACD] = \frac{1}{2} \cdot \sqrt{3} \cdot 1 = \frac{\sqrt{3}}{2}$ . Triangle  $ACE$  is equilateral with side  $\sqrt{3}$ :  $[ACE] = \frac{\sqrt{3}}{4} \cdot 3 = \frac{3\sqrt{3}}{4}$ . Average:  $\frac{1}{2} \left( \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{4} \right) = \frac{5\sqrt{3}}{8}$ .

**Check.** Coordinates with the centre at the origin:  $A = (1, 0)$ ,  $C = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ ,  $M = (-\frac{3}{4}, -\frac{\sqrt{3}}{4})$ ; the shoelace formula gives  $\frac{5\sqrt{3}}{8} \approx 1.083$ . ✓ The traps:  $\frac{\sqrt{3}}{2}$  and  $\frac{3\sqrt{3}}{2}$  are  $[ACD]$  and the hexagon's area;  $\sqrt{3}$  and 1 are guesses.

9. **Answer: (D) 4. The idea.** Put  $y = f(x)$ . The condition says  $f(y) = y$ , so  $y$  must be a fixed point of  $f$ :  $y^2 - 4y = y$  gives  $y(y - 5) = 0$ , so  $y = 0$  or  $y = 5$ . Now count the  $x$  with  $f(x) = 0$  and with  $f(x) = 5$ .

**Work it out.**  $f(x) = 0$ :  $x(x - 4) = 0$ , so  $x = 0$  or 4.  $f(x) = 5$ :  $x^2 - 4x - 5 = (x - 5)(x + 1) = 0$ , so  $x = 5$  or  $-1$ . Four values in all:  $-1, 0, 4, 5$ .

**Check.**  $f(-1) = 5$ ,  $f(5) = 5$ . ✓  $f(4) = 0$ ,  $f(0) = 0$ . ✓ Expanding  $f(f(x)) - f(x)$  gives a quartic  $x(x - 4)(x - 5)(x + 1)$ , with exactly these four real roots. The traps: 2 counts only the fixed points of  $f$ ; 3 misses  $x = -1$ ; 5 and 1 are miscounts.

10. **Answer: (C) 1471. The idea.** Doubling keeps the number of digits exactly when  $2n$  stays below the next power of ten: for a  $k$ -digit  $n$  (so  $10^{k-1} \leq n < 10^k$ ),  $2n < 10^k$  means  $n < 5 \cdot 10^{k-1}$ . In other words,  $n$  keeps its digit count under doubling exactly when its leading digit is 1, 2, 3 or 4.

**Work it out.** One digit: 1, 2, 3, 4 (4 numbers). Two digits: 10 to 49 (40). Three digits: 100 to 499 (400). Four digits up to 2026: 1000 to 2026 all have leading digit 1 or 2, and indeed  $2 \cdot 2026 = 4052$  still has four digits: 1027 numbers. Total  $4 + 40 + 400 + 1027 = 1471$ .

**Check.**  $n = 49$ : 98, two digits.  $\checkmark$   $n = 50$ : 100, three digits.  $\checkmark$  A computer count gives 1471. The traps: 1027 counts only the four-digit block; 1026 is an off-by-one of that; 1475 counts 444 three-digit numbers plus 4; 2026 assumes doubling never adds a digit.

11. **Answer: (E) 6. The idea.** Pair each angle with its complement:  $\sin^2 \theta + \sin^2(90^\circ - \theta) = \sin^2 \theta + \cos^2 \theta = 1$ . The twelve angles  $0^\circ, 15^\circ, \dots, 165^\circ$  split into such pairs, provided we handle the angles beyond  $90^\circ$  (use  $\sin(180^\circ - \theta) = \sin \theta$  to fold them back).

**Work it out.** Folding, the twelve values  $\sin^2(15k)^\circ$  are:  $\sin^2 0^\circ$ , then  $\sin^2 15^\circ, \sin^2 30^\circ, \dots, \sin^2 75^\circ$  each *twice* (once from  $\theta$ , once from  $180^\circ - \theta$ ), then  $\sin^2 90^\circ$ . So the sum is  $0 + 1 + 2(\sin^2 15^\circ + \sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ + \sin^2 75^\circ) + 1$ . In the bracket,  $15^\circ$  pairs with  $75^\circ$  and  $30^\circ$  with  $60^\circ$  (each pair sums to 1), and  $\sin^2 45^\circ = \frac{1}{2}$ : bracket  $= 2 + \frac{1}{2} = \frac{5}{2}$ . Total  $1 + 5 = 6$ .

**Check.** Via  $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$ : the sum is  $\frac{12}{2} - \frac{1}{2} \sum_{k=0}^{11} \cos(30k)^\circ = 6 - 0 = 6$ , because the twelve cosines of equally spaced angles around a full turn cancel.  $\checkmark$  The traps: 5 and  $\frac{11}{2}$  forget  $\sin^2 90^\circ$  or  $\sin^2 45^\circ$ ; 4 pairs only four pairs;  $\frac{9}{2}$  halves wrongly.

12. **Answer: (A) 3. The idea.** The graph of  $y = x^3 - 3x$  has a local maximum at  $x = -1$  (value 2) and a local minimum at  $x = 1$  (value  $-2$ ). The equation  $x^3 - 3x = -k$  has three distinct real roots exactly when the horizontal line  $y = -k$  cuts the graph three times, i.e. strictly between the two extreme values.

**Work it out.** Three distinct real roots  $\iff -2 < -k < 2 \iff -2 < k < 2$ . The integers are  $k = -1, 0, 1$ : three of them.

**Check.**  $k = 0$ : roots  $0, \pm\sqrt{3}$ .  $\checkmark$   $k = 2$ :  $x^3 - 3x + 2 = (x - 1)^2(x + 2)$ , a repeated root, so not distinct.  $\checkmark$   $k = 3$ : only one real root. The traps: 5 includes  $k = \pm 2$ ; 4 includes one of them; 6 and 7 count all  $|k| \leq 3$ .

13. **Answer: (B) 30. The idea.** Start from all  $\frac{6!}{2!2!2!} = 90$  arrangements and remove those with some pair of equal digits adjacent, by inclusion-exclusion over which digits form an adjacent pair. Gluing a pair into a single block shortens the word.

**Work it out.** One specified pair glued (say 11 as a block): arrangements of the block plus 2, 2, 3, 3:  $\frac{5!}{2!2!} = 30$ ; three choices of pair, 90 total. Two pairs glued:  $\frac{4!}{2!} = 12$ , three choices, 36. All three glued:  $3! = 6$ . Bad arrangements:  $90 - 36 + 6 = 60$ . Good:  $90 - 60 = 30$ .

**Check.** Direct enumeration by computer finds 30. By symmetry there are 10 good words starting with each digit; for instance the ones starting with 12 are 121323, 123123, 123132, 123213 and 123231 (five of them), and the ones starting with 13 are their mirror images with 2 and 3 swapped. The traps: 90 ignores the restriction; 36 and 24 stop inclusion-exclusion early; 48 double-counts.

14. **Answer: (A)**  $\frac{49}{24}$ . **The idea.** The small triangle is similar to the big one (parallel hypotenuses), so its area is 6 times the square of the scale factor. The scale factor is the ratio of distances from the right-angle vertex to the two parallel lines, and the incenter's line is easy to locate: the incenter of a right triangle with legs on the axes is  $(r, r)$  with  $r = \frac{3+4-5}{2} = 1$ .

**Work it out.** The hypotenuse is  $3x + 4y = 12$ ; the parallel line through  $(1, 1)$  is  $3x + 4y = 7$ . The distances from the origin to these lines are in the ratio  $7 : 12$ , so the small triangle has area  $6 \cdot \left(\frac{7}{12}\right)^2 = \frac{49}{24}$ .

**Check.** The small triangle has intercepts  $\frac{7}{3}$  and  $\frac{7}{4}$ : area  $\frac{1}{2} \cdot \frac{7}{3} \cdot \frac{7}{4} = \frac{49}{24}$ . ✓ The traps:  $\frac{9}{4}$  and  $\frac{5}{2}$  come from a wrong inradius ( $r = \frac{3}{2}$  or  $r = \frac{5}{2}$ -type slips); 3 is half the triangle;  $\frac{7}{2}$  uses the ratio  $7 : 12$  without squaring.

15. **Answer: (D)**  $\frac{2\sqrt{6}}{3}$ . **The idea.** Use volume. The octahedron is two square pyramids glued along a square of side 2; its inscribed sphere touches all eight faces at their centres, so the volume also equals  $\frac{1}{3}r \cdot (\text{surface area})$ . The distance between opposite faces is  $2r$ .

**Work it out.** Each pyramid has base area 4 and height equal to half the diagonal of the square,  $\sqrt{2}$ , so the volume is  $2 \cdot \frac{1}{3} \cdot 4 \cdot \sqrt{2} = \frac{8\sqrt{2}}{3}$ . The surface is 8 equilateral triangles of side 2:  $8 \cdot \sqrt{3} = 8\sqrt{3}$ . Then  $\frac{8\sqrt{2}}{3} = \frac{1}{3}r \cdot 8\sqrt{3}$  gives  $r = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$ , and the distance between opposite faces is  $2r = \frac{2\sqrt{6}}{3}$ .

**Check.** Coordinates: vertices  $(\pm\sqrt{2}, 0, 0)$ ,  $(0, \pm\sqrt{2}, 0)$ ,  $(0, 0, \pm\sqrt{2})$  (edge 2); the face through the three positive vertices is the plane  $x + y + z = \sqrt{2}$ , at distance  $\sqrt{2}/\sqrt{3}$  from the origin, and the opposite face is at the same distance on the other side: total  $\frac{2\sqrt{6}}{3} \approx 1.63$ . ✓ The traps:  $\sqrt{2}$  is the height of one pyramid;  $\frac{2\sqrt{3}}{3}$  is the circumradius of a face; 2 and 1 are the edge and half-edge.

16. **Answer: (A)** 12. **The idea.** Complete the square to bound one variable:  $4(x^2 + xy + y^2) = (2x + y)^2 + 3y^2$ , so  $(2x + y)^2 + 3y^2 = 28$ . Then  $3y^2 \leq 28$  forces  $|y| \leq 3$ , and for each  $y$  the value  $(2x + y)^2 = 28 - 3y^2$  must be a perfect square.

**Work it out.**  $y = 0$ : 28, not a square.  $y = \pm 1$ : 25, so  $2x + y = \pm 5$ , giving  $x = 2, -3$  (for  $y = 1$ ) and  $x = 3, -2$  (for  $y = -1$ ).  $y = \pm 2$ : 16, so  $2x + y = \pm 4$ :  $x = 1, -3$  ( $y = 2$ ) and  $x = 3, -1$  ( $y = -2$ ).  $y = \pm 3$ : 1, so  $2x + y = \pm 1$ :  $x = -1, -2$  ( $y = 3$ ) and  $x = 2, 1$  ( $y = -3$ ). That is 2 solutions for each of the six values  $y = \pm 1, \pm 2, \pm 3$ : 12 ordered pairs.

**Check.**  $(2, 1)$ :  $4 + 2 + 1 = 7$ . ✓  $(-3, 1)$ :  $9 - 3 + 1 = 7$ . ✓  $(-1, 3)$ :  $1 - 3 + 9 = 7$ . ✓ A computer search over  $|x|, |y| \leq 10$  finds exactly 12. The traps: 24 double-counts by also swapping signs that were already included; 16 and 18 include  $y = 0$  or a false square; 14 is a miscount.

17. **Answer: (D)** 7. **The idea.** Count the factors of 2 in  $n!$  by Legendre's formula,  $\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \cdots$ , which is at most  $\frac{n}{2} + \frac{n}{4} + \cdots = n$ , with a deficit that is exactly the number of 1s in the binary representation of  $n$ :  $v_2(n!) = n - s_2(n)$ . (Each floor loses the fractional part, and adding up those losses over all the binary digits gives  $s_2(n)$ .)

**Work it out.**  $v_2(n!) \geq n - 1$  means  $s_2(n) \leq 1$ , i.e.  $n$  is a power of 2: 1, 2, 4, 8, 16, 32, 64. Seven values.

**Check.**  $v_2(8!) = 4 + 2 + 1 = 7 = 8 - 1$ .  $\checkmark$   $v_2(6!) = 3 + 1 = 4 < 5$ .  $\checkmark$  A computer check over  $n \leq 100$  confirms exactly the seven powers of 2. The traps: 6 forgets  $n = 1$  ( $1! = 1$  is divisible by  $2^0$ ); 5 forgets  $n = 1, 2$ ; 50 guesses “the even numbers”; 1 assumes only  $n = 1$ .

18. **Answer: (B)**  $\frac{1}{3}$ . **The idea.** Measure angles at  $A$  with tangents. Since  $\angle BAD = 90^\circ$  and  $\angle EAF = 45^\circ$ , the two leftover angles  $\angle BAE$  and  $\angle DAF$  add to  $45^\circ$ . We know  $\tan \angle BAE = \frac{BE}{AB} = \frac{1}{2}$ , so  $\tan \angle DAF = \tan(45^\circ - \angle BAE)$  follows from the subtraction formula.

**Work it out.**  $\tan(45^\circ - \theta) = \frac{1 - \tan \theta}{1 + \tan \theta} = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}$ . Since  $\tan \angle DAF = \frac{DF}{AD} = DF$ , we get  $DF = \frac{1}{3}$ .

**Check.** With  $A$  at the origin,  $E = (1, \frac{1}{2})$  and  $F = (\frac{1}{3}, 1)$ : the dot product  $AE \cdot AF = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ , while  $|AE||AF| = \frac{\sqrt{5}}{2} \cdot \frac{\sqrt{10}}{3} = \frac{5\sqrt{2}}{6}$ , so  $\cos \angle EAF = \frac{5/6}{5\sqrt{2}/6} = \frac{1}{\sqrt{2}}$ .  $\checkmark$  (A classic alternative: rotating triangle  $ABE$  about  $A$  by  $90^\circ$  shows  $EF = BE + DF$ , and then the Pythagorean theorem in triangle  $ECF$  gives  $DF = \frac{1}{3}$ .) The traps:  $\frac{1}{2}$  assumes  $F$  mirrors  $E$ ;  $\frac{1}{4}$ ,  $\frac{2}{5}$  and  $\frac{2}{3}$  come from mis-applying the subtraction formula.

19. **Answer: (E)** 36. **The idea.** With  $a + b + c = 20$ , the triangle inequality  $a < b + c$  is the same as  $a < 20 - a$ , i.e.  $a < 10$ . So the condition is simply that all three numbers are at most 9 (and then none can be 0 either: if  $a = 0$ , then  $b + c = 20$  forces one of them to be at least 10). Count solutions of  $a + b + c = 20$  with each variable in  $\{0, 1, \dots, 9\}$  by inclusion-exclusion.

**Work it out.** All nonnegative solutions:  $\binom{22}{2} = 231$ . Solutions with  $a \geq 10$ : substitute  $a = a' + 10$ , giving  $\binom{12}{2} = 66$ ; same for  $b$  and  $c$ : 198. Two variables  $\geq 10$ :  $a' + b' + c = 0$ , one solution, three pairs: 3. Three variables  $\geq 10$ : impossible. Count:  $231 - 198 + 3 = 36$ .

**Check.** Equivalently, set  $a = 10 - a'$  etc. with  $a', b', c' \geq 1$  and  $a' + b' + c' = 10$ :  $\binom{9}{2} = 36$ .  $\checkmark$  Direct enumeration confirms. The traps:  $33 = 36 - 3$  forgets to add back the pairs; 30 and 24 come from requiring each side  $\leq 8$  or counting unordered triples; 20 is a guess.

20. **Answer: (E)** 30. **The idea.** The 6 sits at the peak. Everything to its left must be increasing and everything to its right decreasing, so an arrangement is completely determined by *which* of the numbers 1 through 5 go to the left of the 6: those are then written in increasing order, and the rest in decreasing order.

**Work it out.** There are  $2^5 = 32$  subsets of  $\{1, 2, 3, 4, 5\}$ . The empty subset puts 6 first and the full subset puts 6 last; both are excluded. That leaves  $32 - 2 = 30$  arrangements.

**Check.** For the numbers 1, 2, 3 the same reasoning gives  $2^2 - 2 = 2$  arrangements, namely 132 and 231.  $\checkmark$  Direct enumeration of the 720 permutations of 1 to 6 finds 30. The traps:  $16 = 2^4$  and 15 forget one of the five numbers or count only subsets of a certain size; 24 and 28 are miscounts of the excluded cases.

21. **Answer: (C)**  $\frac{25}{4}$ . **The idea.** The graph of  $y = |x^2 - 4|$  is the parabola  $y = x^2 - 4$  with the part below the axis ( $-2 < x < 2$ ) flipped up into the arch  $y = 4 - x^2$ , which has corners at

$(\pm 2, 0)$  and a peak at  $(0, 4)$ . As the line  $y = x + k$  slides upward, the number of intersection points changes only when the line passes through a corner or becomes tangent to the arch. An odd count, 3, happens exactly at such transitions.

**Work it out.** *Tangency to the arch:*  $4 - x^2 = x + k$  has a double root when  $x^2 + x + (k - 4)$  has discriminant 0:  $1 - 4(k - 4) = 0$ , so  $k = \frac{17}{4}$ , tangent at  $x = -\frac{1}{2}$  (inside the arch). The line also meets the two outer branches, so there are 3 points. *Through the corner  $(-2, 0)$ :*  $k = 2$ . The line  $y = x + 2$  meets the arch again at  $x = 1$  and the right branch at  $x = 3$ : 3 points. *Through the corner  $(2, 0)$ :*  $k = -2$ . The line  $y = x - 2$  meets the arch only at the corner itself and the branches nowhere else: just 1 point. In between: for  $k < -2$  the line misses the graph entirely, for  $-2 < k < 2$  there are 2 points, for  $2 < k < \frac{17}{4}$  there are 4, and for  $k > \frac{17}{4}$  there are 2. So exactly 3 points occur for  $k = 2$  and  $k = \frac{17}{4}$ , with sum  $\frac{25}{4}$ .

**Check.**  $k = 3$ : arch  $x^2 + x - 1 = 0$  gives  $x \approx 0.62, -1.62$  (both inside); branches  $x^2 - x - 7 = 0$  gives  $x \approx 3.19, -2.19$  (both outside): 4 points.  $\checkmark$   $k = 5$ : arch has no real roots; branches give 2 points.  $\checkmark$  A symbolic count of intersections at each candidate  $k$  confirms  $\{2, \frac{17}{4}\}$ . The traps:  $\frac{17}{4}$  and 2 each take only one transition; 8 and 10 include  $k = -2$  with the wrong sign or double a value.

22. **Answer: (E) 3. The idea.** For a polynomial with integer coefficients,  $a - b$  divides  $P(a) - P(b)$  whenever  $a$  and  $b$  are integers (each term  $c_j(a^j - b^j)$  is divisible by  $a - b$ ). Apply this to the pairs  $(k, 1)$  and  $(k, 5)$ .

**Work it out.**  $k - 1$  divides  $P(k) - P(1) = k - 5$ , and  $k - 5$  divides  $P(k) - P(5) = k - 1$ . Two integers that divide each other have the same absolute value:  $|k - 1| = |k - 5|$ . Since  $k - 1 \neq k - 5$ , we need  $k - 1 = -(k - 5)$ , i.e.  $k = 3$ . (Note  $k = 1$  and  $k = 5$  are impossible since  $P(1) \neq 1$  and  $P(5) \neq 5$ .)

**Check.**  $k = 3$  really can occur:  $P(x) = 6 - x$  has  $P(1) = 5$ ,  $P(5) = 1$  and  $P(3) = 3$ .  $\checkmark$  For any other candidate, say  $k = 2$ :  $k - 1 = 1$  divides  $k - 5 = -3$ , but  $-3$  does not divide 1. A computer check of the two divisibility conditions over  $|k| \leq 500$  finds only  $k = 3$ . The traps: 0, 1, 2 and  $-3$  each satisfy one of the two divisibilities but not both.

23. **Answer: (D)  $\frac{6}{7}$ . The idea.** Everything is symmetric about line  $OP$ , so the small circle's centre  $Q$  lies on  $OP$ . Two facts pin it down: (1)  $Q$  is at distance  $r$  from line  $PA$ , and since  $PA$  makes angle  $\theta$  with  $OP$  where  $\sin \theta = \frac{OA}{OP} = \frac{2}{5}$ , that distance is  $PQ \sin \theta$ ; (2) the small circle is externally tangent to the big one, so  $OQ = 2 + r$ .

**Work it out.** Let  $PQ = d$ . Then  $r = \frac{2d}{5}$  and  $OQ = 5 - d = 2 + r = 2 + \frac{2d}{5}$ . So  $3 = \frac{7d}{5}$ ,  $d = \frac{15}{7}$  and  $r = \frac{2}{5} \cdot \frac{15}{7} = \frac{6}{7}$ .

**Check.**  $OQ = 5 - \frac{15}{7} = \frac{20}{7} = 2 + \frac{6}{7}$ .  $\checkmark$  Coordinates ( $O$  at the origin,  $P = (5, 0)$ ,  $A = (\frac{4}{5}, \frac{2\sqrt{21}}{5})$ ): the distance from  $Q = (\frac{20}{7}, 0)$  to line  $PA$  is  $\frac{15}{7} \cdot \frac{2}{5} = \frac{6}{7}$ .  $\checkmark$  The traps: 1 and  $\frac{2}{3}$  use  $OQ = 2 - r$  (internal tangency) or drop the  $\sin \theta$ ;  $\frac{3}{5}$  and  $\frac{1}{2}$  use  $PQ = 3$  or halve the gap  $OP - 2 = 3$  by eye.

24. **Answer: (E) 4. The idea.** Factor:  $x^3y - xy^3 = xy(x^2 - y^2)$ . The two factors are linked by the constraint in a hidden way:  $(2xy)^2 + (x^2 - y^2)^2 = (x^2 + y^2)^2 = 16$ . So if  $u = 2xy$  and  $v = x^2 - y^2$ , the point  $(u, v)$  lies on a circle of radius 4, and we want to maximise  $\frac{uv}{2}$ .

**Work it out.** By AM-GM,  $uv \leq \frac{u^2 + v^2}{2} = 8$ , with equality when  $u = v = 2\sqrt{2}$ . So  $x^3y - xy^3 = \frac{uv}{2} \leq 4$ , and the value 4 is attained (take  $x^2 - y^2 = 2\sqrt{2}$  and  $2xy = 2\sqrt{2}$ , which does have real solutions, e.g.  $x = 2\cos 22.5^\circ$ ,  $y = 2\sin 22.5^\circ$ ).

**Check.** With  $x = 2\cos\theta$ ,  $y = 2\sin\theta$  the expression becomes  $16\sin\theta\cos\theta\cos 2\theta = 4\sin 4\theta$ , whose maximum is 4. ✓ A numerical sweep agrees. The traps: 2 and  $2\sqrt{2}$  forget the factor 2 in  $u$  or maximise only  $uv$  with the wrong radius;  $2\sqrt{3}$  and 1 are guesses.

25. **Answer: (A)** 57. **The idea.** An arrangement with exactly one descent is two increasing runs placed side by side: the first run is some subset  $A$  of  $\{1, \dots, 6\}$  written in increasing order, and the second run is the complement, also increasing. So each arrangement corresponds to a subset  $A$ , as long as the join really is a descent.

**Work it out.** Every subset  $A$  with  $A \neq \emptyset$  and  $A \neq \{1, \dots, 6\}$  gives a well-defined arrangement with *at most* one descent (at the join). The join fails to be a descent exactly when the largest element of  $A$  is smaller than the smallest element of the complement, i.e. when  $A = \{1, 2, \dots, j\}$  for some  $j$ ; those 5 subsets ( $j = 1, \dots, 5$ ) produce the identity arrangement, with no descent. Also, distinct subsets other than those give distinct arrangements (the first run's length and contents are visible). Count:  $2^6 - 2 - 5 = 57$ .

**Check.** For  $n = 3$  the same count gives  $2^3 - 2 - 2 = 4$ : the arrangements 132, 213, 231, 312. ✓ (In general this is  $2^n - n - 1$ , the Eulerian number  $\langle n \rangle_1$ .) Direct enumeration of the 720 arrangements of 1 to 6 finds 57. The traps:  $62 = 2^6 - 2$  forgets to remove the subsets that give the identity; 64 forgets both corrections; 60 and 120 are guesses from  $5!$  and  $6!/12$ .