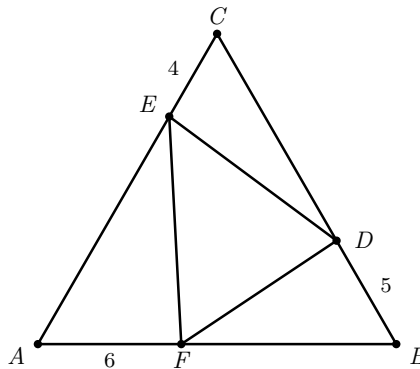


MathCastle AIME

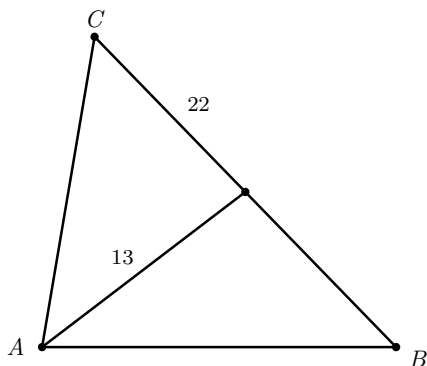
Practice Contest 9

15 questions • 180 minutes • 1 point per correct answer.

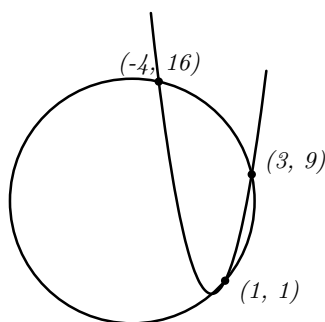
1. A sequence is defined by $a_1 = 2$ and $a_{n+1} = a_n^2 + a_n + 1$ for every $n \geq 1$ (so $a_2 = 7$, $a_3 = 57$, and the terms grow very quickly). Find the remainder when a_6 is divided by 1000.
2. In triangle ABC , $\tan A = \frac{3}{4}$ and $\tan B = \frac{8}{15}$. The cosine of angle C can be written as $-\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
3. Positive real numbers a and b satisfy $a + \frac{1}{b} = \frac{5}{4}$ and $b + \frac{1}{a} = 5$. The product ab takes exactly two possible values; their sum is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
4. Equilateral triangle ABC has side length 15. Point D lies on side $\overline{BC}$ with $BD = 5$, point E lies on side $\overline{CA}$ with $CE = 4$, and point F lies on side $\overline{AB}$ with $AF = 6$. The area of triangle DEF is $\frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers and b is not divisible by the square of any prime. Find $a + b + c$.



5. A row of 9 tiles contains four tiles labeled X , three labeled Y , and two labeled Z (tiles with the same label are not distinguishable from one another). In how many distinguishable arrangements of all nine tiles in a row is it true that no two X 's are adjacent and no two Y 's are adjacent? (Two Z 's are allowed to be adjacent.)
6. In triangle ABC , $\cos A = \frac{1}{6}$, $BC = 22$, and the median from A to side $\overline{BC}$ has length 13. Find $AB \cdot AC$.

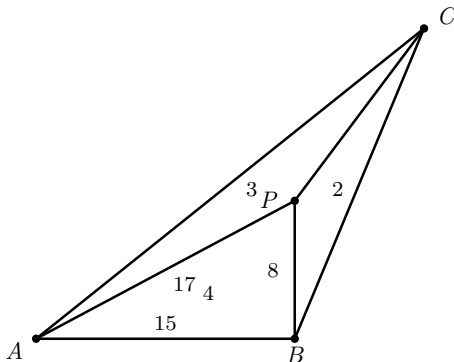


7. A circle passes through the three points $(-4, 16)$, $(1, 1)$, and $(3, 9)$, which all lie on the parabola $y = x^2$. Find R^2 , the square of the circle's radius.

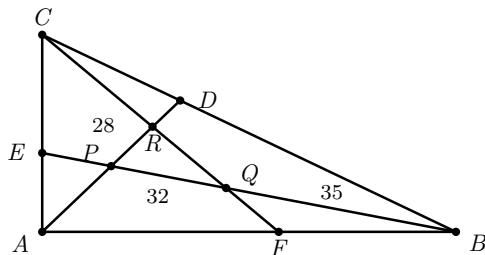


8. Find the number of ordered pairs of positive integers (x, y) such that $x^2y + xy^2 + x + y = 2304$.
9. Find the number of ordered pairs of positive integers (a, b) with $1 \leq a \leq 50$ and $1 \leq b \leq 50$ such that $a^2 \equiv b \pmod{7}$ and $b^2 \equiv a \pmod{7}$.
10. The 9 vertices of a regular 9-gon are each colored red, green, or blue so that no two vertices joined by a side of the 9-gon receive the same color. Two colorings are considered the same arrangement if one can be rotated (about the center of the 9-gon) to match the other; reflections are not identified with each other. Find the number of distinguishable colorings.
11. Real numbers w, x, y, z satisfy $w + x + y + z = 10$ and $w + 2x + 3y + 4z = 24$. Let S be the smallest possible value of $w^2 + x^2 + y^2 + z^2$ over all quadruples (w, x, y, z) satisfying both equations. If $S = m/n$ for relatively prime positive integers m and n , find $m + n$.
12. Find the number of ways to partition the set $\{1, 2, \dots, 9\}$ into three nonempty subsets, each containing an odd number of elements. (A partition is an unordered collection of subsets covering every element exactly once; the order in which the three subsets are listed does not matter.)

13. Point P lies inside triangle ABC so that $[PBC] : [PCA] : [PAB] = 2 : 3 : 4$, where $[XYZ]$ denotes the area of triangle XYZ . Given that $AB = 15$, $PA = 17$, and $PB = 8$, suppose $PC = \frac{m}{n}$ for relatively prime positive integers m and n . Find $m + n$.



14. Each of the 16 unit cells of a 4×4 grid is independently colored black or white, each color with probability $1/2$. A 2×2 sub-square is any of the nine 2×2 blocks of cells aligned with the grid lines (sub-squares are allowed to overlap one another); such a sub-square is called monochromatic if all 4 of its cells share the same color. Let X be the number of monochromatic 2×2 sub-squares. If $E[X^2] = m/n$ for relatively prime positive integers m, n , find $m + n$.
15. Let D, E, F be points lying strictly between the endpoints of sides BC, CA, AB of triangle ABC , respectively. The lines AD, BE, CF are not all three concurrent, so they meet in pairs at three distinct points: P , the intersection of lines AD and BE ; Q , the intersection of lines BE and CF ; and R , the intersection of lines CF and AD . Suppose triangle ABC has area 105, triangle AFR has area 32, triangle BDP has area 35, and triangle CEQ has area 28. Find the area of triangle PQR .



Answer Key — AIME Practice Contest 9

1. 807
2. 121
3. 21
4. 42
5. 93
6. 288
7. 85
8. 4
9. 211
10. 58
11. 131
12. 820
13. 27
14. 29
15. 7

Solutions — AIME Practice Contest 9

1. **Answer:** 807. **The idea.** We only need a_6 modulo 1000, and the rule $a_{n+1} = a_n^2 + a_n + 1$ only uses a_n through squaring and adding, both of which respect reduction mod 1000: if $a_n \equiv r \pmod{1000}$ then $a_{n+1} \equiv r^2 + r + 1 \pmod{1000}$. So we can replace every term by its last three digits as soon as we compute it, keeping the numbers small.

Work it out. $a_1 = 2$. $a_2 = 4 + 2 + 1 = 7$. $a_3 = 49 + 7 + 1 = 57$.

$$a_4 = 57^2 + 57 + 1 = 3249 + 58 = 3307 \equiv 307 \pmod{1000}.$$

$$a_5 \equiv 307^2 + 307 + 1 = 94249 + 308 = 94557 \equiv 557 \pmod{1000}.$$

$$a_6 \equiv 557^2 + 557 + 1 = 310249 + 558 = 310807 \equiv \boxed{807} \pmod{1000}.$$

Check. Computing without any reduction, the exact values are $a_1, \dots, a_6 = 2, 7, 57, 3307, 10,939,557, 119,671,117$. The last three digits of a_6 are indeed 807, confirming the shortcut of reducing at each step is valid.

2. **Answer:** 121. **The idea.** Since $A + B + C = 180^\circ$, angle C is the supplement of $A + B$, so $\cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B$. Because $\tan A$ and $\tan B$ are both positive, A and B are both acute, so their sines and cosines are both positive and come straight from the two Pythagorean triples hiding in $\frac{3}{4}$ and $\frac{8}{15}$.

Work it out. From $\tan A = \frac{3}{4}$ (a 3-4-5 triangle), $\sin A = \frac{3}{5}$ and $\cos A = \frac{4}{5}$. From $\tan B = \frac{8}{15}$ (an 8-15-17 triangle), $\sin B = \frac{8}{17}$ and $\cos B = \frac{15}{17}$.

$$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{4}{5} \cdot \frac{15}{17} - \frac{3}{5} \cdot \frac{8}{17} = \frac{60}{85} - \frac{24}{85} = \frac{36}{85}.$$

So $\cos C = -\cos(A + B) = -\frac{36}{85}$. Since $85 = 5 \cdot 17$ and $36 = 2^2 \cdot 3^2$ share no common factor, $m + n = 36 + 85 = 121$.

Check. $\cos C < 0$ makes C obtuse, which is consistent: $A = \arctan \frac{3}{4} \approx 36.87^\circ$ and $B = \arctan \frac{8}{15} \approx 28.07^\circ$ sum to about 64.94° , well under 180° , leaving $C \approx 115.06^\circ$ for a valid triangle. A fully independent numerical computation of $\cos(180^\circ - \arctan \frac{3}{4} - \arctan \frac{8}{15})$ gives $-0.42352941\dots$, matching $-\frac{36}{85} = -0.42352941\dots$ to high precision.

3. **Answer:** 21. **The idea.** Multiplying the two given equations produces ab , $\frac{1}{ab}$, and two constant 1's, so it turns the system into a single equation in $t = ab$ alone.

Work it out. Multiply: $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = ab + \frac{1}{ab} + 2$. This equals

$$\frac{5}{4} \cdot 5 = \frac{25}{4}, \text{ so with } t = ab,$$

$$t + \frac{1}{t} = \frac{25}{4} - 2 = \frac{17}{4} \implies 4t^2 - 17t + 4 = 0 \implies (4t - 1)(t - 4) = 0,$$

giving $t = 4$ or $t = \frac{1}{4}$.

Both values actually occur: from $a + \frac{1}{b} = \frac{5}{4}$ and $ab = t$, multiplying by b gives $ab + 1 = \frac{5}{4}b$, so $b = \frac{4(t+1)}{5}$. For $t = 4$: $b = 4$, $a = t/b = 1$; check $b + \frac{1}{a} = 4 + 1 = 5$. $\checkmark$ For $t = \frac{1}{4}$:

$b = 1$, $a = t/b = \frac{1}{4}$; check $b + \frac{1}{a} = 1 + 4 = 5$. $\checkmark$ So both $(a, b) = (1, 4)$ and $(\frac{1}{4}, 1)$ are genuine solutions, and ab really does take both values 4 and $\frac{1}{4}$.

The sum of the two values is $4 + \frac{1}{4} = \frac{17}{4}$, so $m + n = 17 + 4 = 21$.

Check. $(a, b) = (1, 4)$: $1 + \frac{1}{4} = \frac{5}{4} \checkmark$, $4 + 1 = 5 \checkmark$. $(a, b) = (\frac{1}{4}, 1)$: $\frac{1}{4} + 1 = \frac{5}{4} \checkmark$, $1 + 4 = 5 \checkmark$. A computer solve of the original system independently returns exactly these two pairs.

4. **Answer:** 42. **The idea.** Triangle DEF is what remains of $\triangle ABC$ after cutting off the three corner triangles at A, B, C . Each corner triangle has a 60° angle (an angle of the equilateral triangle), so its area is $\frac{1}{2} \cdot (\text{leg}_1)(\text{leg}_2) \sin 60^\circ$. Subtracting the three corners from $[ABC]$ gives $[DEF]$.

Work it out. The side length is 15, so $[ABC] = \frac{\sqrt{3}}{4} \cdot 15^2 = \frac{225\sqrt{3}}{4}$.

At B : $BD = 5$, $BF = AB - AF = 15 - 6 = 9$, so $[BDF] = \frac{1}{2} \cdot 5 \cdot 9 \cdot \frac{\sqrt{3}}{2} = \frac{45\sqrt{3}}{4}$.

At C : $CE = 4$, $CD = BC - BD = 15 - 5 = 10$, so $[CED] = \frac{1}{2} \cdot 4 \cdot 10 \cdot \frac{\sqrt{3}}{2} = 10\sqrt{3} = \frac{40\sqrt{3}}{4}$.

At A : $AF = 6$, $AE = CA - CE = 15 - 4 = 11$, so $[AFE] = \frac{1}{2} \cdot 6 \cdot 11 \cdot \frac{\sqrt{3}}{2} = \frac{33\sqrt{3}}{2} = \frac{66\sqrt{3}}{4}$.

$$[DEF] = \frac{225\sqrt{3}}{4} - \frac{45\sqrt{3} + 40\sqrt{3} + 66\sqrt{3}}{4} = \frac{225\sqrt{3} - 151\sqrt{3}}{4} = \frac{74\sqrt{3}}{4} = \frac{37\sqrt{3}}{2}.$$

Since 37 and 2 are coprime and 3 is squarefree, $a + b + c = 37 + 3 + 2 = 42$.

Check. Placing $A = (0, 0)$, $B = (15, 0)$, $C = (\frac{15}{2}, \frac{15\sqrt{3}}{2})$ and computing D, E, F from the given ratios, the shoelace formula for $\triangle DEF$ gives the same area $\frac{37\sqrt{3}}{2} \approx 32.04$, which is sensibly less than $[ABC] = \frac{225\sqrt{3}}{4} \approx 97.4$.

5. **Answer:** 93. **The idea.** First decide where the 3 Y 's and 2 Z 's go among themselves (ignoring the X 's), then insert the 4 X 's into the gaps of that row of 5 tiles. Putting at most one X into any single gap guarantees no two X 's end up adjacent. But if two Y 's already sit next to each other in the row of 5, the *only* way to keep them apart is to place an X into the gap between them — that gap's choice is forced, not free. So we split into cases by how the 3 Y 's are grouped into blocks of consecutive Y 's.

Work it out. There are $\binom{5}{3} = 10$ ways to arrange $YYYZZ$ in a row of 5. By the sizes of the maximal runs of Y 's, these split into three shapes:

Shape 3 (one block YYY): arrangements $YYYZZ$, $ZYYYZ$, $ZZYYY$ — 3 of them. The block's 2 internal gaps must each get one X (forced), using 2 of the 4 X 's. The row now has 3 units (YYY , Z , Z), hence 4 outer gaps for the remaining 2 X 's: $\binom{4}{2} = 6$ ways. Subtotal: $3 \cdot 6 = 18$.

Shape 2 + 1 (blocks YY and Y separately): the other 6 arrangements ($YYZYZ$, $YYZZY$, $YZYYZ$, $YZZYY$, $ZYYZY$, $ZYZYY$). The one internal gap inside YY is forced to get an X , using 1 of the 4. The row has 4 units, hence 5 outer gaps for the remaining 3 X 's: $\binom{5}{3} = 10$ ways. Subtotal: $6 \cdot 10 = 60$.

Shape 1 + 1 + 1 (all Y 's separated already): only $YZYZY$. No internal gaps are forced. The row has 5 units, hence 6 outer gaps for all 4 X 's: $\binom{6}{4} = 15$ ways. Subtotal: $1 \cdot 15 = 15$.

Total: $18 + 60 + 15 = 93$.

Check. A direct computer enumeration of all $\binom{9}{4,3,2} = 1260$ distinguishable arrangements, filtering for no adjacent equal X 's and no adjacent equal Y 's, also gives 93. For a small sanity check by hand: the arrangement $XYXYXYXZZ$ has no two X 's or Y 's adjacent (and ends in the allowed ZZ), so it is correctly among the 93.

6. **Answer:** 288. **The idea.** Let $b = AC$ and $c = AB$. The Law of Cosines relates $b^2 + c^2$ and bc to BC , and the median-length formula relates $b^2 + c^2$ (again) to the median. Together they pin down bc without ever having to find b and c separately.

Work it out. The median-length formula for the median m_a from A states $m_a^2 = \frac{2b^2 + 2c^2 - BC^2}{4}$.

With $m_a = 13$ and $BC = 22$: $4 \cdot 13^2 = 2(b^2 + c^2) - 22^2$, so $2(b^2 + c^2) = 4 \cdot 169 + 484 = 676 + 484 = 1160$, giving $b^2 + c^2 = 580$.

The Law of Cosines gives $BC^2 = b^2 + c^2 - 2bc \cos A$, so $484 = 580 - 2bc \cdot \frac{1}{6} = 580 - \frac{bc}{3}$. Then $\frac{bc}{3} = 580 - 484 = 96$, so $bc = 288$.

Hence $AB \cdot AC = bc = 288$.

Check. Solving $b^2 + c^2 = 580$ and $bc = 288$ directly gives $b = 16$, $c = 18$ (or the reverse): indeed $16^2 + 18^2 = 256 + 324 = 580$ and $16 \cdot 18 = 288$. With sides $AB = 18$, $AC = 16$, $BC = 22$, the Law of Cosines gives $\cos A = \frac{16^2 + 18^2 - 22^2}{2 \cdot 16 \cdot 18} = \frac{580 - 484}{576} = \frac{96}{576} = \frac{1}{6} \checkmark$, and the median formula gives $m_a^2 = \frac{2(256 + 324) - 484}{4} = \frac{1160 - 484}{4} = \frac{676}{4} = 169$, so $m_a = 13 \checkmark$.

7. **Answer:** 85. **The idea.** Write the circle as $x^2 + y^2 + Dx + Ey + F = 0$. Substituting $y = x^2$ turns each of the three given points into a root of the same quartic equation in x , which must therefore have a fourth root too (a circle and a parabola meet in at most 4 points). The quartic has no x^3 term, so the four roots sum to 0 — enough to pin down the missing fourth intersection, and from there, every coefficient.

Work it out. Substituting (x, x^2) into the circle's equation: $x^2 + x^4 + Dx + Ex^2 + F = 0$, i.e. $x^4 + (1 + E)x^2 + Dx + F = 0$.

This quartic has roots $-4, 1, 3$, and a fourth root r . Since there is no x^3 term, the sum of all four roots is 0: $-4 + 1 + 3 + r = 0$, so $r = 0$.

Now match Vieta's formulas for the monic quartic with roots $-4, 1, 3, 0$:

Sum of pairwise products: $(-4)(1) + (-4)(3) + (-4)(0) + (1)(3) + (1)(0) + (3)(0) = -4 - 12 + 3 = -13$, which equals the x^2 -coefficient $1 + E$, so $E = -14$.

Sum of triple products: $(-4)(1)(3) + (-4)(1)(0) + (-4)(3)(0) + (1)(3)(0) = -12$. For the quartic $x^4 + (1 + E)x^2 + Dx + F$, the coefficient of x is D , and it equals $-(\text{sum of triple products})$ by Vieta, so $D = 12$.

Product of all four roots: $(-4)(1)(3)(0) = 0 = F$.

So $D = 12$, $E = -14$, $F = 0$, and $R^2 = \frac{D^2 + E^2 - 4F}{4} = \frac{144 + 196}{4} = \frac{340}{4} = 85$.

Check. Finding the circumcenter directly, by intersecting the perpendicular bisectors of the segments joining the three given points, gives the center $(-6, 7)$, and the squared distance from this center to $(1, 1)$ is $(-6 - 1)^2 + (7 - 1)^2 = 49 + 36 = 85$, matching. The same center is also equidistant from $(-4, 16)$: $(-6 + 4)^2 + (7 - 16)^2 = 4 + 81 = 85$. ✓

8. **Answer:** 4. **The idea.** Factor the left side by grouping: $x^2y + xy^2 + x + y = xy(x + y) + (x + y) = (x + y)(xy + 1)$. Writing $s = x + y$ and $p = xy$, the equation becomes $s(p + 1) = 2304$, so s must be a divisor of 2304 and $p = \frac{2304}{s} - 1$. For an actual pair of positive integers with this sum and product, $x, y = \frac{s \mp \sqrt{s^2 - 4p}}{2}$, so $s^2 - 4p$ must be a perfect square with the same parity as s .

Work it out. Since $x, y \geq 1$, AM–GM gives $s^2 \geq 4p = 4\left(\frac{2304}{s} - 1\right)$, i.e. $s^3 + 4s \geq 9216$; this fails at $s = 20$ ($8000 + 80 = 8080 < 9216$) and holds from $s = 21$ on, so only divisors of $2304 = 2^8 \cdot 3^2$ with $s \geq 21$ can work: 24, 32, 36, 48, 64, 72, 96, 128, 144, 192, 256, 288, 384, 576, 768, 1152. Computing $p = 2304/s - 1$ and the discriminant $s^2 - 4p$ for each of these: $s = 24$ gives $p = 95$ and discriminant $576 - 380 = 196 = 14^2$ (even, matching s 's parity), giving $(x, y) = (5, 19)$; $s = 48$ gives $p = 47$ and discriminant $2304 - 188 = 2116 = 46^2$, giving $(x, y) = (1, 47)$. Every other divisor on the list gives a discriminant that is not a perfect square. Each unordered pair corresponds to 2 ordered pairs, for $2 + 2 = 4$ in all.

Check. Direct substitution: $(5, 19)$ gives $5 \cdot 19 \cdot 24 + 24 = 2280 + 24 = 2304$ ✓, and $(1, 47)$ gives $1 \cdot 47 \cdot 48 + 48 = 2256 + 48 = 2304$ ✓. A direct scan of every pair with $1 \leq x, y \leq 200$ confirms these 4 ordered pairs are the only solutions.

9. **Answer:** 211. **The idea.** Substituting $b \equiv a^2$ into $b^2 \equiv a \pmod{7}$ gives $a^4 \equiv a$, i.e. $a(a^3 - 1) \equiv 0 \pmod{7}$. Since 7 is prime, either $a \equiv 0$ or $a^3 \equiv 1 \pmod{7}$. The nonzero residues mod 7 form a cyclic group of order 6 under multiplication, so the solutions of $t^3 \equiv 1$ form its unique subgroup of order $\gcd(3, 6) = 3$, which is $\{1, 2, 4\}$ (indeed $2^3 = 8 \equiv 1$ and $4^3 = 64 \equiv 1 \pmod{7}$).

Work it out. So $a \pmod{7}$ must be one of 0, 1, 2, 4, and then $b \equiv a^2 \pmod{7}$ is forced to a single residue: $0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 4, 4 \rightarrow 2$. Among $1, \dots, 50$ (which is $7 \cdot 7 + 1$ numbers), residue 1 (mod 7) occurs 8 times and every other residue occurs 7 times. The a -residues 0, 1, 2, 4 occur 7, 8, 7, 7 times, and their forced b -residues 0, 1, 4, 2 occur 7, 8, 7, 7 times respectively. So the count is $7 \cdot 7 + 8 \cdot 8 + 7 \cdot 7 + 7 \cdot 7 = 49 + 64 + 49 + 49 = 211$.

Check. A direct scan of all $50 \times 50 = 2500$ pairs, testing both congruences directly, confirms 211 pairs exactly.

10. **Answer:** 58. **The idea.** This is a job for Burnside's lemma: the number of colorings up to rotation is the average, over the 9 rotations of the 9-gon (including the identity), of the number of proper colorings fixed by that rotation.

Work it out. The identity fixes every proper 3-coloring of the 9-cycle; the chromatic polynomial of a cycle C_n with k colors is $(k - 1)^n + (-1)^n(k - 1)$, so with $n = 9, k = 3$ there are $2^9 - 2 = 510$ of them. A rotation by r steps fixes a coloring only if the coloring is invariant under shifting by r , which forces it to be built by repeating a block of length $g = \gcd(9, r)$ around the cycle; that repeating block must itself be a proper coloring of a g -cycle (with the extra requirement that it closes up properly, which the cycle chromatic polynomial already

accounts for). For $g = 1$ (the six rotations $r = 1, 2, 4, 5, 7, 8$) this needs a proper coloring of a 1-cycle, impossible since a vertex would be adjacent to itself with a repeated color — 0 fixed colorings. For $g = 3$ (the two rotations $r = 3, 6$) it needs a proper 3-coloring of a 3-cycle: $2^3 - 2 = 6$ of them, and each extends uniquely to the 9-gon. Averaging over all 9 rotations: $(510 + 6 \cdot 0 + 2 \cdot 6)/9 = 522/9 = 58$.

Check. Enumerating all $3^9 = 19683$ colorings directly, keeping the 510 proper ones, and grouping them into orbits under the 9 rotations by brute force (rather than trusting the Burnside count) gives exactly 58 orbits.

11. **Answer:** 131. **The idea.** Write $v = (w, x, y, z)$ and let $a_1 = (1, 1, 1, 1)$, $a_2 = (1, 2, 3, 4)$, so the constraints say $a_1 \cdot v = 10$ and $a_2 \cdot v = 24$. Split $v = v_{\parallel} + v_{\perp}$ where $v_{\parallel}$ is a combination $\lambda a_1 + \mu a_2$ and $v_{\perp}$ is orthogonal to both a_1 and a_2 . Since $a_1 \cdot v_{\perp} = a_2 \cdot v_{\perp} = 0$, the value of $v_{\parallel}$ is completely determined by the two constraints and does not depend on $v_{\perp}$ at all — but $|v|^2 = |v_{\parallel}|^2 + |v_{\perp}|^2 \geq |v_{\parallel}|^2$, with equality exactly when $v_{\perp} = 0$. So the minimum is achieved uniquely at $v = v_{\parallel} = \lambda a_1 + \mu a_2$, and it suffices to find λ, μ .

Work it out. Using $a_1 \cdot a_1 = 4$, $a_1 \cdot a_2 = 10$, $a_2 \cdot a_2 = 30$, the constraints become $4\lambda + 10\mu = 10$ and $10\lambda + 30\mu = 24$. Solving, $\mu = -1/5$ and $\lambda = 3$, so $v = 3(1, 1, 1, 1) - \frac{1}{5}(1, 2, 3, 4) = (\frac{14}{5}, \frac{13}{5}, \frac{12}{5}, \frac{11}{5})$. One checks this indeed satisfies both original equations. Then $S = \frac{14^2 + 13^2 + 12^2 + 11^2}{25} = \frac{196 + 169 + 144 + 121}{25} = \frac{630}{25} = \frac{126}{5}$. Since $\gcd(126, 5) = 1$, $m + n = 126 + 5 = 131$.

Check. Any other valid quadruple is $v + td$ for a direction d with $a_1 \cdot d = a_2 \cdot d = 0$ (e.g. $d = (1, -2, 1, 0)$ or $d = (0, 1, -2, 1)$); substituting shows $|v + td|^2 = \frac{126}{5} + t^2|d|^2 + 2t(v \cdot d)$, and $v \cdot d = 0$ for every such d (as it must be, since v is the orthogonal-decomposition minimizer), so the value only increases as t moves away from 0.

12. **Answer:** 820. **The idea.** Three positive odd integers summing to 9 can only be, as an unordered triple of sizes, $\{1, 1, 7\}$, $\{1, 3, 5\}$, or $\{3, 3, 3\}$ (any other combination either uses an even number or exceeds 9). Count partitions of each shape separately.

Work it out. Shape $\{1, 1, 7\}$: choose the 7-element block in $\binom{9}{7} = 36$ ways; the other two elements automatically form two singleton blocks, with no further choice since singletons can't be confused with each other once their contents are fixed — 36 partitions. Shape $\{1, 3, 5\}$: choose the singleton in 9 ways, then the size-3 block from the remaining 8 elements in $\binom{8}{3} = 56$ ways (the last 5 elements form the remaining block) — $9 \cdot 56 = 504$ partitions. Shape $\{3, 3, 3\}$: the number of ways to split 9 labeled elements into three unordered blocks of size 3 is $\frac{9!}{3!3!3!} = \frac{362880}{6 \cdot 6 \cdot 6 \cdot 6} = 280$ (divide by $3!$ at the end since the three size-3 blocks are themselves unordered). Adding: $36 + 504 + 280 = 820$.

Check. Enumerating all sequences that assign each of the 9 elements to one of 3 labeled boxes, keeping those that use all three boxes with every box of odd size, and collapsing labelings that produce the same unordered partition, gives 820 distinct partitions directly.

13. **Answer:** 27. **The idea.** The area ratios say P has barycentric weights $(2, 3, 4)$ relative to (A, B, C) — precisely $P = \frac{2A + 3B + 4C}{9}$, since each vertex's weight is proportional to the area of the sub-triangle *not* containing it. For $P = \frac{w_A A + w_B B + w_C C}{W}$ (with $W =$

$w_A + w_B + w_C$), a direct expansion gives the weighted “parallel-axis” identity $W \cdot PX^2 = w_A \cdot XA^2 + w_B \cdot XB^2 + w_C \cdot XC^2 - \frac{1}{W} (w_A w_B AB^2 + w_B w_C BC^2 + w_C w_A CA^2)$ for any point X ; taking $X = A, B, C$ in turn (one term vanishes each time) writes PA^2, PB^2, PC^2 as linear combinations of $a^2 = BC^2, b^2 = CA^2, c^2 = AB^2$.

Work it out. With $w_A = 2, w_B = 3, w_C = 4, W = 9$ and $K = \frac{1}{9}(6c^2 + 12a^2 + 8b^2)$ the shared correction term: $9PA^2 = 3c^2 + 4b^2 - K, 9PB^2 = 2c^2 + 4a^2 - K, 9PC^2 = 2b^2 + 3a^2 - K$. With $c = 15$: subtracting the first two equations cancels K and gives $4b^2 - 4a^2 = 9(PA^2 - PB^2) - c^2 = 9(289 - 64) - 225 = 1800$, so $b^2 - a^2 = 450$. Substituting K into the PA^2 equation and simplifying gives a second linear relation, $7b^2 - 3a^2 = 4671$. Solving the two linear equations: $a^2 = \frac{1521}{4}$ (so $a = \frac{39}{2}$) and $b^2 = \frac{3321}{4}$ — both consistent with the triangle inequality against $c = 15$. Finally $9PC^2 = 2b^2 + 3a^2 - K = \frac{5625}{4}$, so $PC^2 = \frac{625}{4}$ and $PC = \frac{25}{2}$. Since $\gcd(25, 2) = 1, m + n = 27$.

Check. Placing $A = (0, 0), B = (15, 0)$ and solving for C from $BC = \frac{39}{2}, CA = \frac{9\sqrt{41}}{2}$ gives $C = (\frac{45}{2}, 18)$. Then $P = \frac{2A + 3B + 4C}{9} = (15, 8)$, and indeed $PA = \sqrt{15^2 + 8^2} = \sqrt{289} = 17 \checkmark, PB = \sqrt{0^2 + 8^2} = 8 \checkmark$, and $PC = \sqrt{(\frac{45}{2} - 15)^2 + (18 - 8)^2} = \sqrt{7.5^2 + 10^2} = \sqrt{156.25} = \frac{25}{2} \checkmark$ — every given value matches exactly.

14. **Answer:** 29. **The idea.** Label the nine 2×2 sub-squares by the row and column (0, 1, 2 each) of their top-left cell, and let I_s be the indicator that sub-square s is monochromatic. Then $X = \sum_s I_s$, so $X^2 = \sum_s I_s + 2 \sum_{s < t} I_s I_t$, and by linearity $E[X^2] = \sum_s E[I_s] + 2 \sum_{s < t} E[I_s I_t]$. Each term $E[I_s I_t]$ only depends on how many cells sub-squares s, t share.

Work it out. A single sub-square is monochromatic with probability $2 \cdot (1/2)^4 = 1/8$, contributing $9 \cdot \frac{1}{8} = \frac{9}{8}$ to the sum. For a pair of sub-squares, if they share k cells, the two squares are simultaneously monochromatic exactly when all $8 - k$ distinct cells among them share one common color, which has probability $2 \cdot (1/2)^{8-k}$ (if $k \geq 1$, since shared cells force both squares to agree) or, when $k = 0$, the independent product $(1/8)(1/8) = 2^{-6}$ — which happens to equal the $k = 1$ value too. Checking all $\binom{9}{2} = 36$ pairs of the nine sub-squares by their relative offset: 16 pairs share 0 cells (probability $1/64$ each), 8 pairs share exactly 1 cell, diagonally (probability $1/64$ each), and 12 pairs share exactly 2 cells, edge-adjacent (probability $1/32$ each). So $2 \sum_{s < t} E[I_s I_t] = 2 [16 \cdot \frac{1}{64} + 8 \cdot \frac{1}{64} + 12 \cdot \frac{1}{32}] = 2 [\frac{24}{64} + \frac{24}{64}] = 2 \cdot \frac{48}{64} = \frac{3}{2}$. Total: $E[X^2] = \frac{9}{8} + \frac{3}{2} = \frac{9}{8} + \frac{12}{8} = \frac{21}{8}$. Since $\gcd(21, 8) = 1, m + n = 29$.

Check. Looping over all $2^{16} = 65536$ colorings, computing X exactly for each by checking all nine sub-squares, and averaging X^2 gives exactly $21/8$, confirming the count-based derivation.

15. **Answer:** 7. **The idea.** Place $A = (0, 0), B = (1, 0), C = (0, 1)$ (area ratios are unaffected by this affine choice). Let $x = BD/DC, y = CE/EA, z = AF/FB$, so $D = \frac{B+xC}{x+1}, E = \frac{C+yA}{y+1}, F = \frac{A+zB}{z+1}$. Writing out the three cevian lines and intersecting them pairwise (to get P, Q, R) and computing the resulting triangle areas as exact rational functions of x, y, z gives, as fractions of $[ABC]$: $[AFR] = \frac{xz^2}{xz^2+xz+z^2+2z+1}, [BDP] = \frac{x^2y}{(x+1)(xy+x+1)}$, and $[CEQ] = \frac{y^2z}{y^2z+y^2+yz+2y+1}$. (These are not quoted from memory — they come from direct coordinate computation, the same way Routh’s theorem itself is proved.) The given data says these equal $32/105, 35/105, 28/105$.

Work it out. Solving the three equations simultaneously (eliminating one variable at a time

leaves a polynomial in x alone whose every coefficient besides the forced root is positive, so no other positive root exists) gives the unique positive solution $x = 2$, $y = 3/2$, $z = 4/3$ — meaning $BD : DC = 2 : 1$, $CE : EA = 3 : 2$, $AF : FB = 4 : 3$. Note $xyz = 4 \neq 1$, consistent with the cevians not being concurrent (Ceva's theorem says concurrency would need $xyz = 1$). Now finish with the same coordinate method used above: placing the real triangle as $A = (0, 0)$, $B = (21, 0)$, $C = (0, 10)$ (which has area 105), the section-formula points D, E, F for these ratios and the pairwise intersections P, Q, R can be computed exactly, and they reproduce $[ABC] = 105$, $[AFR] = 32$, $[BDP] = 35$, $[CEQ] = 28$ exactly as given, with $[PQR] = 7$.

Check. The three given areas plus $[PQR] = 7$ leave $105 - (32 + 35 + 28 + 7) = 3$ for the three remaining quadrilateral regions combined, and computing those three quadrilateral areas directly from the same coordinates gives exactly 3 in total — the seven pieces exactly tile triangle ABC , confirming the whole configuration is consistent.