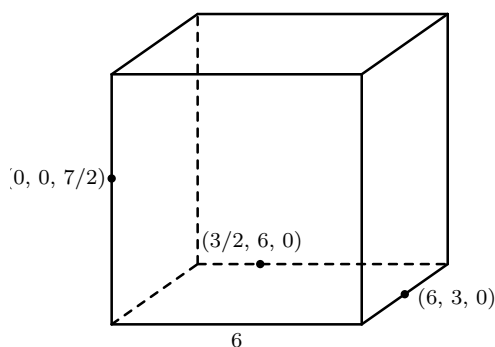


MathCastle AIME

Practice Contest 1

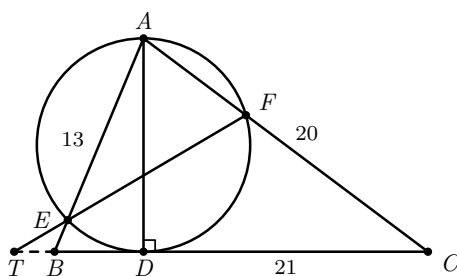
15 questions • 180 minutes • 1 point per correct answer.

1. For how many 4-element subsets of $\{1, 2, 3, \dots, 20\}$ is the mean of the four elements equal to their median? (The median of four numbers is the average of the two middle numbers.)
2. Integers b and c are chosen with $1 \leq b \leq 20$ and $1 \leq c \leq 20$. For how many ordered pairs (b, c) do both quadratics $x^2 + bx + c$ and $x^2 + cx + b$ have real roots (a repeated root counts as real)?
3. Twelve seats around a round table are numbered 1 to 12 in order, so seat 12 is next to seat 1. Each seat is given a red or a blue cushion. A *block* is a maximal set of consecutive seats (going around the table, possibly passing from seat 12 to seat 1) whose cushions all have the same colour; if all twelve cushions have the same colour, the whole table is a single block of 12 seats. How many of the 2^{12} colourings have every block made of an odd number of seats?
4. How many ordered pairs (x, y) of integers with $|x| \leq 1000$ satisfy $x^2 + y^2 = x^3$?
5. A solid cube of edge length 6 occupies the region $0 \leq x \leq 6, 0 \leq y \leq 6, 0 \leq z \leq 6$. A plane passes through the three points $(6, 3, 0)$, $(\frac{3}{2}, 6, 0)$ and $(0, 0, \frac{7}{2})$. The intersection of the plane with the solid cube is a polygon whose area is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

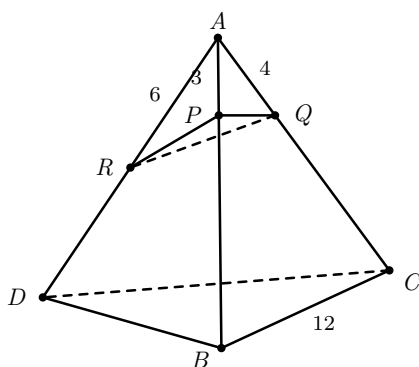


6. How many ordered pairs (A, B) are there, where A and B are disjoint subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ (either or both may be empty), such that no element of A differs by exactly 1 from an element of B ?
7. The number
$$\left(1 + 4 \cos^2 \frac{\pi}{11}\right) \left(1 + 4 \cos^2 \frac{2\pi}{11}\right) \left(1 + 4 \cos^2 \frac{3\pi}{11}\right) \left(1 + 4 \cos^2 \frac{4\pi}{11}\right) \left(1 + 4 \cos^2 \frac{5\pi}{11}\right)$$
is a positive integer. Find it.

8. Multiplying a number by 30 can only increase its number of positive divisors. For how many positive integers $n \leq 1000$ does $30n$ have exactly four times as many positive divisors as n has?
9. A *lattice point* is a point whose coordinates are both integers. For how many ordered pairs (x, y) of integers with $1 \leq x \leq 100$ and $1 \leq y \leq 100$ does the segment from $(0, 0)$ to (x, y) pass through exactly three lattice points other than its two endpoints?
10. Triangle ABC has $AB = 13$, $BC = 21$ and $CA = 20$. Let D be the foot of the altitude from A to $\overline{BC}$. The circle with diameter $\overline{AD}$ meets side $\overline{AB}$ again at E and side $\overline{AC}$ again at F . Lines EF and BC meet at a point T . The length TD can be written as $\frac{m}{n}$ in lowest terms. Find $m + n$.



11. Regular tetrahedron $ABCD$ has every edge of length 12. Points P , Q and R lie on edges $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ with $AP = 3$, $AQ = 4$ and $AR = 6$. The plane through P , Q , R cuts off a small tetrahedron $APQR$. The square of the distance from A to the plane PQR can be written as $\frac{m}{n}$ in lowest terms. Find $m + n$.



12. The six complex numbers z that satisfy $(z - 1)^6 = 64z^6$ are the vertices of a convex hexagon in the complex plane. The area of this hexagon can be written as $\frac{a\sqrt{b}}{c}$, where a , b , c are positive integers, a and c are relatively prime, and b is not divisible by the square of any prime. Find $a + b + c$.

13. How many ordered pairs (a, b) of integers with $|a| \leq 20$ and $|b| \leq 20$ have the property that the polynomial $x^4 + ax + b$ can be written as the product of two polynomials, each of degree at least 1, with integer coefficients?
14. For a permutation $\sigma = (a_1, a_2, \dots, a_8)$ of $1, 2, \dots, 8$, a *descent* is an index i with $1 \leq i \leq 7$ and $a_i > a_{i+1}$; let $d(\sigma)$ be the number of descents of σ . For example, $d(3, 1, 2, 8, 7, 4, 5, 6) = 3$. Let N be the sum of $2^{d(\sigma)}$ over all $8! = 40320$ permutations σ . Find the remainder when N is divided by 1000.
15. Let N be the number of ordered 10-tuples $(x_1, x_2, \dots, x_{10})$ with every $x_i \in \{1, 2, 3\}$ such that $x_1x_2 + x_2x_3 + x_3x_4 + \dots + x_9x_{10} + x_{10}x_1$ is divisible by 3. Find the remainder when N is divided by 1000.

Answer Key — AIME Practice Contest 1

1. 525
2. 183
3. 640
4. 64
5. 281
6. 577
7. 89
8. 68
9. 399
10. 91
11. 323
12. 34
13. 170
14. 835
15. 845

Solutions — AIME Practice Contest 1

1. **Answer:** 525. **The idea.** Write the subset as $a < b < c < d$. The mean is $\frac{a+b+c+d}{4}$ and the median is $\frac{b+c}{2}$, so the condition is $a + d = b + c$: the outer pair and the inner pair have the same sum. Now turn it around: any two different pairs with the same sum are automatically nested like this. So we only need to count, for each possible sum, how many pairs have that sum.

Work it out. Take two different pairs $\{x, y\}$ and $\{u, v\}$ from $\{1, \dots, 20\}$ with $x + y = u + v = s$. They cannot share an element (the other elements would then be equal too), so together they form a 4-element set. Its smallest element and its largest element lie in the same pair (the partner of the smallest is s minus the smallest, which is the largest), so the set is $a < b < c < d$ with $a + d = b + c$. Conversely a set with $a + d = b + c$ comes from exactly one such split, since the other two splits have $a + b < c + d$ and $a + c < b + d$. So the answer is

$$\sum_s \binom{p_s}{2}, \text{ where } p_s \text{ is the number of pairs in } \{1, \dots, 20\} \text{ with sum } s.$$

For $s = 3, 4, \dots, 20$ the pairs are $\{1, s-1\}, \{2, s-2\}, \dots$, so $p_s = \lfloor \frac{s-1}{2} \rfloor$: the values 1, 1, 2, 2, $\dots$, 9, 9. For $s = 21$, $p_{21} = 10$. Replacing each number x by $21 - x$ turns sum s into $42 - s$, so $s = 22, \dots, 39$ repeat the values 9, 9, $\dots$, 1, 1.

Hence the total is $4 \left[\binom{2}{2} + \binom{3}{2} + \dots + \binom{9}{2} \right] + \binom{10}{2} = 4 \binom{10}{3} + 45 = 480 + 45 = 525$, using the hockey-stick identity $\binom{2}{2} + \dots + \binom{9}{2} = \binom{10}{3} = 120$.

Check. Try $\{1, \dots, 6\}$. By the formula, the sums 3, $\dots$, 11 have $p_s = 1, 1, 2, 2, 3, 2, 2, 1, 1$, giving $1 + 1 + 3 + 1 + 1 = 7$. Directly, the sets with $a + d = b + c$ are $\{1, 2, 3, 4\}$, $\{1, 2, 4, 5\}$, $\{1, 3, 4, 6\}$, $\{1, 2, 5, 6\}$, $\{2, 3, 4, 5\}$, $\{2, 3, 5, 6\}$ and $\{3, 4, 5, 6\}$: seven of them. A computer check over all $\binom{20}{4} = 4845$ subsets also gives 525.

2. **Answer:** 183. **The idea.** The two conditions are $b^2 \geq 4c$ and $c^2 \geq 4b$: the region between two parabolas that cross at $(4, 4)$. First find where the region lives, then notice that for larger b one of the two conditions is automatically true inside the box, so the count becomes a simple column-by-column sum.

Work it out. Both conditions force $b, c \geq 4$. Indeed, if $b \leq 3$ then $c \leq b^2/4 \leq 9/4$, so $c \leq 2$ (and $b = 1$ leaves no c at all); then $c^2 \leq 4 < 4b$ because $b \geq 2$. So no pair has $b \leq 3$, and by symmetry none has $c \leq 3$.

For a fixed b , the conditions say $2\sqrt{b} \leq c \leq b^2/4$ (and $c \leq 20$). Column by column:

$b = 4$: $c = 4$ only (1). $b = 5$: $c \in \{5, 6\}$ (2). $b = 6$: $5 \leq c \leq 9$ (5). $b = 7$: $6 \leq c \leq 12$ (7). $b = 8$: $6 \leq c \leq 16$ (11). These give 26.

For $b \geq 9$ we have $b^2/4 \geq 81/4 > 20$, so the only condition is $c \geq 2\sqrt{b}$, i.e. $c \geq \lceil 2\sqrt{b} \rceil$: this is 6 for $b = 9$, 7 for $b = 10, 11, 12$, 8 for $b = 13, \dots, 16$ and 9 for $b = 17, \dots, 20$. The column sizes are 15, then 14 three times, 13 four times and 12 four times: $15 + 42 + 52 + 48 = 157$.

Total: $26 + 157 = 183$.

Check. The region is symmetric in b and c . On the diagonal $b = c$ the condition is $b^2 \geq 4b$, i.e. $b \geq 4$: 17 pairs. So the off-diagonal pairs number $183 - 17 = 166$, which is even, as the symmetry $(b, c) \leftrightarrow (c, b)$ requires. Counting only the pairs with $b < c$ (there $c^2 \geq 4b$ is

automatic, since $c > b \geq 4$ gives $c^2 > 4c > 4b$, so only $b^2 \geq 4c$ needs testing, i.e. $b < c \leq b^2/4$) gives $0 + 1 + 3 + 5 + 8 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 83$ for $b = 4, \dots, 19$, and $2 \cdot 83 + 17 = 183$.

3. **Answer:** 640. **The idea.** A colouring is decided by where the blocks start and by the colour of one seat: the blocks must alternate in colour. So we count the ways to cut the circle of 12 labelled seats into arcs of odd length. Look at the arc containing seat 1: once it is placed, the rest of the table is a straight row, and splitting a row into odd pieces is a Fibonacci count.

Work it out. Let $c(m)$ be the number of ways to write m as an ordered sum of odd positive integers (a row of m seats cut into odd blocks). Either the first part is 1, leaving $c(m-1)$ ways, or it is at least 3, and subtracting 2 from it gives a composition of $m-2$: $c(m) = c(m-1) + c(m-2)$. With $c(1) = c(2) = 1$ this is the Fibonacci sequence: $c(m) = F_m$, namely 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 for $m = 1, \dots, 11$.

The whole table cannot be one block (12 is even), so there are at least two blocks. Suppose the block containing seat 1 has odd length $\ell \leq 11$. Seat 1 can be in any of ℓ positions inside it, which fixes the block; the remaining $12 - \ell$ seats form a row between its two ends, to be cut into odd blocks in $F_{12-\ell}$ ways. The number of ways to cut the circle is

$$\sum_{\ell \text{ odd}} \ell F_{12-\ell} = 1 \cdot 89 + 3 \cdot 34 + 5 \cdot 13 + 7 \cdot 5 + 9 \cdot 2 + 11 \cdot 1 = 89 + 102 + 65 + 35 + 18 + 11 = 320.$$

An even number of odd blocks is needed to make 12, so the blocks can indeed alternate red, blue, red, blue around the table, and there are exactly 2 ways to colour them (choose the colour of seat 1). The answer is $2 \cdot 320 = 640$.

Check. Try 4 seats. The formula gives $1 \cdot F_3 + 3 \cdot F_1 = 2 + 3 = 5$ ways to cut: all four seats as single blocks, or one single seat and a block of three (4 ways). So 10 colourings: RBRB, BRBR and the 8 with exactly one seat differing from the other three. Listing all 16 colourings of 4 seats confirms 10. (In general the cut count is $L_n - 2$ for even n , where L_n is the Lucas number; $L_{12} - 2 = 322 - 2 = 320$.)

4. **Answer:** 64. **The idea.** Move x^2 across: $y^2 = x^3 - x^2 = x^2(x-1)$. When $x \neq 0$ this says $x-1 = (\frac{y}{x})^2$, so the integer $x-1$ is the square of a rational number, and therefore the square of an integer. That turns the equation into a one-parameter family.

Work it out. If $x = 0$ the equation reads $y^2 = 0$: the single pair $(0, 0)$.

If $x < 0$ then $x^2(x-1) < 0$, which cannot equal y^2 : no pairs.

If $x \geq 1$, then $x-1 = (\frac{y}{x})^2$. An integer that is the square of a fraction $\frac{p}{q}$ in lowest terms must have $q = 1$ (otherwise $\frac{p^2}{q^2}$ is again in lowest terms with denominator $q^2 > 1$), so $x-1 = t^2$ for an integer $t \geq 0$, and $y = \pm xt = \pm t(t^2 + 1)$. Conversely every $t \geq 0$ works: $x = t^2 + 1$, $y = \pm t(t^2 + 1)$ gives $y^2 = t^2 x^2 = (x-1)x^2$.

The bound $x = t^2 + 1 \leq 1000$ means $t^2 \leq 999$, i.e. $t \leq 31$ (since $31^2 = 961$ and $32^2 = 1024$). For $t = 0$ we get $x = 1$, $y = 0$: one pair. For each $t = 1, \dots, 31$ we get two pairs (the two signs of y): 62 pairs.

Total: $1 + 1 + 62 = 64$.

Check. $t = 2$ gives $(5, \pm 10)$: $25 + 100 = 125 = 5^3$. The largest, $t = 31$, gives $(962, \pm 29822)$, and $962^2 + 29822^2 = 962^2(1 + 961) = 962^3$. A computer search over $|x| \leq 1000$ (testing whether $x^3 - x^2$ is a perfect square) finds the same 64 pairs. The answer is 064.

5. **Answer:** 281. **The idea.** Rather than finding the polygon's vertices and splitting it into triangles, project it straight down onto the base $z = 0$. The shadow is easy to describe, and a flat region's area is multiplied by $\cos \theta$ under projection, where θ is the angle between the plane and the base, i.e. between their normals.

Work it out. Try a plane $px + qy + rz = d$. The points $(6, 3, 0)$ and $(\frac{3}{2}, 6, 0)$ give $6p + 3q = \frac{3}{2}p + 6q$, so $\frac{9}{2}p = 3q$ and $q = \frac{3}{2}p$; taking $p = 2$, $q = 3$ gives $d = 12 + 9 = 21$, and then $(0, 0, \frac{7}{2})$ gives $r = 6$. The plane is $2x + 3y + 6z = 21$.

Above a base point (x, y) the plane has height $z = \frac{21 - 2x - 3y}{6} \leq \frac{21}{6} < 6$. So it passes through the cube exactly above the part of the base square where $2x + 3y \leq 21$, and never leaves through the top face. That part is the square with one corner cut off: the line $2x + 3y = 21$ meets the edge $x = 6$ at $y = 3$ and the edge $y = 6$ at $x = \frac{3}{2}$, cutting off a right triangle with legs 3 and $\frac{9}{2}$ at the corner $(6, 6)$. The shadow is a pentagon of area $36 - \frac{1}{2} \cdot 3 \cdot \frac{9}{2} = 36 - \frac{27}{4} = \frac{117}{4}$.

The normal of the plane is $(2, 3, 6)$, of length $\sqrt{4 + 9 + 36} = 7$, and the normal of the base is $(0, 0, 1)$, so $\cos \theta = \frac{6}{7}$. The cross-section therefore has area $\frac{117}{4} \cdot \frac{7}{6} = \frac{273}{8}$. Since $\gcd(273, 8) = 1$, the answer is $273 + 8 = 281$.

Check. The section's vertices are $(0, 0, \frac{7}{2})$, $(6, 0, \frac{3}{2})$, $(6, 3, 0)$, $(\frac{3}{2}, 6, 0)$, $(0, 6, \frac{1}{2})$. Adding the cross products of consecutive position vectors and halving the length of the total gives the area directly: the total is $\frac{39}{4}(2, 3, 6)$, of length $\frac{273}{4}$, and half of that is $\frac{273}{8}$.

6. **Answer:** 577. **The idea.** A pair of disjoint sets is the same thing as a labelling of $1, 2, \dots, 7$ with three labels: A , B , or N (in neither). The condition only involves neighbours i and $i + 1$: labels A and B may never sit next to each other. Reading the labels from left to right and remembering the last label gives a recursion.

Work it out. For a row $1, \dots, n$ let a_n be the number of good labellings, x_n the number ending in N and y_n the number ending in A (by symmetry also the number ending in B), so $a_n = x_n + 2y_n$. A label N may follow anything: $x_n = a_{n-1}$. A label A may follow N or A : $y_n = x_{n-1} + y_{n-1} = a_{n-2} + y_{n-1}$.

Then $a_n = a_{n-1} + 2y_n = a_{n-1} + 2a_{n-2} + 2y_{n-1}$, and $2y_{n-1} = a_{n-1} - x_{n-1} = a_{n-1} - a_{n-2}$. So $a_n = 2a_{n-1} + a_{n-2}$.

Start: $a_1 = 3$, and $a_2 = 9 - 2 = 7$ (all label pairs except AB and BA). Then $a_3 = 17$, $a_4 = 41$, $a_5 = 99$, $a_6 = 239$, $a_7 = 2 \cdot 239 + 99 = 577$.

Check. For $n = 3$, count directly by the middle label. Middle N : the ends are free, $3 \cdot 3 = 9$. Middle A : each end is A or N , $2 \cdot 2 = 4$; middle B : likewise 4. Total 17, as the recursion says. (These numbers satisfy $a_n^2 - 2q^2 = \pm 1$ for suitable q , e.g. $577^2 - 2 \cdot 408^2 = 1$: they are the numerators of the continued-fraction approximations of $\sqrt{2}$.)

7. **Answer:** 89. **The idea.** Write each factor as a product of two linear factors in roots of unity. Since $1 + 4\cos^2 \theta = 3 + 2\cos 2\theta$, choose u with $u + \frac{1}{u} = 3$; with $\zeta = e^{2i\theta}$ we get $3 + 2\cos 2\theta = u + \frac{1}{u} + \zeta + \frac{1}{\zeta} = \frac{(u + \zeta)(u + \zeta^{-1})}{u}$. Over all 11th roots of unity, $\prod(u + \zeta)$ is just $u^{11} + 1$.

Work it out. First double the product. Since $\cos^2 \frac{k\pi}{11} = \cos^2 \frac{(11-k)\pi}{11}$, the product Q of $1 + 4\cos^2 \frac{k\pi}{11}$ over $k = 1, \dots, 10$ is the square of the requested number P .

For $k = 1, \dots, 10$, $\zeta_k = e^{2ik\pi/11}$ runs over the ten 11th roots of unity other than 1. For $\zeta = 1$ the expression $\frac{(u+\zeta)(u+\zeta^{-1})}{u}$ equals $\frac{(u+1)^2}{u} = u + 2 + \frac{1}{u} = 5$. Over all eleven roots, $\prod_{\zeta} (u+\zeta) = u^{11} + 1$ (the roots of $x^{11} = 1$ are the ζ , so $\prod (x - \zeta) = x^{11} - 1$; put $x = -u$ and multiply by -1), and the same holds for the ζ^{-1} , which are the same roots in another order. So

$$5Q = \frac{(u^{11} + 1)^2}{u^{11}} = u^{11} + 2 + u^{-11}.$$

Let $t_n = u^n + u^{-n}$. Then $t_0 = 2$, $t_1 = 3$ and $t_{n+1} = 3t_n - t_{n-1}$ (multiply $u^n + u^{-n}$ by $u + u^{-1}$). This gives 2, 3, 7, 18, 47, 123, 322, 843, 2207, 5778, 15127, 39603, so $t_{11} = 39603$ and $5Q = 39605$, $Q = 7921 = 89^2$. Every factor is positive, so $P = 89$.

Check. Numerically, $\cos^2 \frac{\pi}{11} \approx 0.9206$, $\cos^2 \frac{2\pi}{11} \approx 0.7077$, $\cos^2 \frac{3\pi}{11} \approx 0.4288$, $\cos^2 \frac{4\pi}{11} \approx 0.1726$, $\cos^2 \frac{5\pi}{11} \approx 0.0203$; the factors are about 4.683, 3.831, 2.715, 1.690, 1.081, with product about 89.0. (Curiously, 89 is the Fibonacci number F_{11} ; the same computation with 11 replaced by any odd n gives F_n .)

8. **Answer:** 68. **The idea.** Only the exponents of 2, 3 and 5 change when we multiply by 30, and each of them goes up by exactly one. So the ratio of divisor counts is a product of three factors of the form $\frac{e+2}{e+1}$, and these factors are small: $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$. Getting a product of exactly 4 is very restrictive.

Work it out. Write $n = 2^a 3^b 5^c m$ with $\gcd(m, 30) = 1$. Then $\tau(n) = (a+1)(b+1)(c+1)\tau(m)$ and $\tau(30n) = (a+2)(b+2)(c+2)\tau(m)$, so $\frac{\tau(30n)}{\tau(n)} = f(a)f(b)f(c)$, where $f(e) = \frac{e+2}{e+1}$. Here $f(0) = 2$, $f(1) = \frac{3}{2}$, $f(2) = \frac{4}{3}$, and f keeps decreasing toward 1. If none of a, b, c is 0, the product is at most $(\frac{3}{2})^3 = \frac{27}{8} < 4$. If two or three of them are 0, the product is $4f(e) > 4$ or 8. So exactly one exponent is 0, and the other two factors multiply to 2. Neither of those two can be 2 (the other factor would have to be 1, impossible), so both lie in $\{\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots\}$. Two factors of $\frac{3}{2}$ give $\frac{9}{4} \neq 2$, and two factors at most $\frac{4}{3}$ give at most $\frac{16}{9} < 2$; so one factor is $\frac{3}{2}$ and the other is $\frac{4}{3}$, i.e. the two exponents are 1 and 2. So $\{a, b, c\} = \{0, 1, 2\}$, and $n = Km$ where K is one of $3 \cdot 5^2 = 75$, $3^2 \cdot 5 = 45$, $2 \cdot 5^2 = 50$, $2^2 \cdot 5 = 20$, $2 \cdot 3^2 = 18$, $2^2 \cdot 3 = 12$, and m is coprime to 30 with $Km \leq 1000$. Numbers coprime to 30 come 8 to every block of 30: 1, 7, 11, 13, 17, 19, 23, 29 (plus 30 each time). $K = 75$: $m \leq 13$, giving 1, 7, 11, 13: 4. $K = 45$: $m \leq 22$: 6. $K = 50$: $m \leq 20$: 6. $K = 20$: $m \leq 50$: $8 + 6 = 14$ (the extra ones are 31, 37, 41, 43, 47, 49). $K = 18$: $m \leq 55$: $8 + 7 = 15$ (add 53). $K = 12$: $m \leq 83$: $16 + 7 = 23$ (the extra ones are 61, 67, 71, 73, 77, 79, 83). These six families do not overlap (the exponents of 2, 3, 5 differ), so the total is $4 + 6 + 6 + 14 + 15 + 23 = 68$.

Check. Take $n = 12 = 2^2 \cdot 3$: $\tau(12) = 6$ and $\tau(360) = 24 = 4 \cdot 6$. And $n = 60$ (all three primes present) gives $\tau(60) = 12$, $\tau(1800) = 36$, only three times as many, as the bound $\frac{27}{8}$ predicts. The answer is 068.

9. **Answer:** 399. **The idea.** The lattice points on the segment from $(0, 0)$ to (x, y) are evenly spaced: if $g = \gcd(x, y)$, they are exactly the points $\frac{k}{g}(x, y)$ for $k = 0, 1, \dots, g$. So there are $g - 1$ of them strictly between the endpoints, and we need $g = 4$. Dividing by 4 leaves a count of relatively prime pairs in a 25×25 square, which Euler's totient function handles.

Work it out. *The lattice points.* Write $x = ga$, $y = gb$ with $\gcd(a, b) = 1$. A point of the segment is (sa, sb) with $0 \leq s \leq g$ real. If sa and sb are integers, then choosing integers u, v with $ua + vb = 1$ (possible since $\gcd(a, b) = 1$) shows $s = u(sa) + v(sb)$ is an integer. Conversely

every integer s gives a lattice point. So the lattice points are $s = 0, 1, \dots, g$, and $g - 1$ of them lie strictly between the endpoints.

The count. We need $\gcd(x, y) = 4$, i.e. $x = 4a$, $y = 4b$ with $1 \leq a, b \leq 25$ and $\gcd(a, b) = 1$. Sort these pairs by $m = \max(a, b)$. For $m = 1$ there is only $(1, 1)$. For $m \geq 2$ the two coordinates cannot both equal m (then the gcd would be m), so one of them is m and the other is a number below m relatively prime to m : $2\varphi(m)$ pairs. Hence the count is $1 + 2 \sum_{m=2}^{25} \varphi(m) = 2 \sum_{m=1}^{25} \varphi(m) - 1$.

The values $\varphi(1), \dots, \varphi(25)$ are 1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, 12, 10, 22, 8, 20, in groups of five summing to 10, 22, 40, 56, 72, so $\sum_{m=1}^{25} \varphi(m) = 200$ and the answer is $2 \cdot 200 - 1 = 399$.

Check. Shrink the square to $1 \leq x, y \leq 8$. The formula gives $2(\varphi(1) + \varphi(2)) - 1 = 3$ pairs: $(4, 4)$, $(4, 8)$, $(8, 4)$. Indeed the segment to $(4, 8)$ passes through $(1, 2)$, $(2, 4)$, $(3, 6)$, while the one to $(8, 8)$ passes through seven lattice points. A computer counting $\gcd(x, y) - 1$ for all 10000 pairs also finds 399. The answer is 399.

10. **Answer:** 91. **The idea.** Two powers of a point do all the work. The circle ω with diameter AD is tangent to BC at D (its radius AD is perpendicular to BC). From A : B, C, F, E lie on one circle. From T : TD^2 equals the power of T with respect to ω and also, through that second circle, $TB \cdot TC$.

Work it out. Since AD is a diameter, $\angle AED = 90^\circ$, so in right triangle ADB the point E is the foot of the altitude from D and $AE \cdot AB = AD^2$. Likewise $AF \cdot AC = AD^2$. Hence $AE \cdot AB = AF \cdot AC$, and by the converse of the power of a point, B, C, F, E are concyclic. Now T lies on line EF and on line BC . Its power with respect to ω is $TE \cdot TF = TD^2$ (tangent at D), and its power with respect to circle $BCFE$ is $TE \cdot TF = TB \cdot TC$. So $TD^2 = TB \cdot TC$. Put $B = (0, 0)$, $C = (21, 0)$, $A = (x, y)$. From $x^2 + y^2 = 169$ and $(x - 21)^2 + y^2 = 400$ we get $-42x + 441 = 231$, so $x = 5$, $y = 12$, and $D = (5, 0)$. Let $T = (t, 0)$. Then $(t - 5)^2 = t(t - 21)$, i.e. $-10t + 25 = -21t$, so $t = -\frac{25}{11}$ (T lies beyond B). Therefore $TD = 5 + \frac{25}{11} = \frac{80}{11}$ and $m + n = 91$.

Check. Directly: $AE = \frac{AD^2}{AB} = \frac{144}{13}$, so $E = A + \frac{144}{169}(B - A) = (\frac{125}{169}, \frac{300}{169})$; $AF = \frac{144}{20} = \frac{36}{5}$, so $F = A + \frac{36}{100}(C - A) = (\frac{269}{25}, \frac{192}{25})$. The line through these meets $y = 0$ at $x = -\frac{25}{11}$, agreeing. Also $TB \cdot TC = \frac{25}{11} \cdot \frac{256}{11} = \frac{6400}{121} = (\frac{80}{11})^2$. The answer is 091.

11. **Answer:** 323. **The idea.** Compute the volume of $APQR$ two ways. Scaling from the big tetrahedron is easy because the three edges at A are just shrunk; the area of PQR comes from its three side lengths. Then height $= 3V/\text{area}$.

Work it out. A regular tetrahedron of edge s has volume $\frac{s^3}{6\sqrt{2}}$, so $V(ABCD) = \frac{1728}{6\sqrt{2}} = 144\sqrt{2}$. Tetrahedra sharing the trihedral angle at A have volumes proportional to the products of the edge lengths along it, so $V(APQR) = \frac{3 \cdot 4 \cdot 6}{12^3} \cdot 144\sqrt{2} = \frac{1}{24} \cdot 144\sqrt{2} = 6\sqrt{2}$. Every face angle at A is 60° , so by the law of cosines $PQ^2 = 9 + 16 - 12 = 13$, $QR^2 = 16 + 36 - 24 = 28$, $RP^2 = 36 + 9 - 18 = 27$. Heron's formula in squared-side form: $16K^2 = 2(pq + qr + rp) - (p^2 + q^2 + r^2)$ with $p, q, r = 13, 28, 27$: $16K^2 = 2(364 + 756 + 351) - (169 + 784 + 729) = 2942 - 1682 = 1260$, so $K^2 = \frac{315}{4}$. The distance h from A to the plane satisfies $V = \frac{1}{3}Kh$, so $h^2 = \frac{9V^2}{K^2} = \frac{9 \cdot 72}{315/4} = \frac{2592}{315} = \frac{288}{35}$. Thus $m + n = 288 + 35 = 323$.

Check. Sanity: if $AP = AQ = AR = 12$ the same method gives $h^2 = \frac{9 \cdot (144\sqrt{2})^2}{(36\sqrt{3})^2} = 96$, the

square of the true height $12\sqrt{2/3}$ of the tetrahedron. A coordinate computation with A at the origin also gives $h^2 = \frac{288}{35}$. The answer is 323.

12. **Answer:** 34. **The idea.** $z = 0$ is not a solution, so we may divide by z^6 : $\left(\frac{z-1}{z}\right)^6 = 64 = 2^6$. Hence $\frac{z-1}{z} = 2w$ for a sixth root of unity w , and $z = \frac{1}{1-2w}$. The six vertices come out explicitly, they are symmetric about the real axis, and the shoelace formula finishes.

Work it out. The sixth roots of unity are ± 1 and $\pm \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

$$w = 1: z = \frac{1}{-1} = -1. \quad w = -1: z = \frac{1}{3}.$$

$$w = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i: 1 - 2w = \mp\sqrt{3}i, \text{ so } z = \pm \frac{i}{\sqrt{3}} = \pm \frac{\sqrt{3}}{3}i.$$

$$w = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i: 1 - 2w = 2 \mp \sqrt{3}i, \text{ so } z = \frac{2 \pm \sqrt{3}i}{7}.$$

All six satisfy $|z - 1| = 2|z|$, a circle (centre $-\frac{1}{3}$, radius $\frac{2}{3}$), so they are in convex position; going around the circle counterclockwise from $\frac{1}{3}$ they are $\frac{1}{3}$, $\frac{2+\sqrt{3}i}{7}$, $\frac{\sqrt{3}}{3}i$, -1 , $-\frac{\sqrt{3}}{3}i$, $\frac{2-\sqrt{3}i}{7}$. The real axis cuts the hexagon into two mirror-image halves. The upper half is the quadrilateral with vertices $(\frac{1}{3}, 0)$, $(\frac{2}{7}, \frac{\sqrt{3}}{7})$, $(0, \frac{\sqrt{3}}{3})$, $(-1, 0)$ in counterclockwise order; its shoelace terms $x_i y_{i+1} - x_{i+1} y_i$ are

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{7} - 0 = \frac{\sqrt{3}}{21}, \quad \frac{2}{7} \cdot \frac{\sqrt{3}}{3} - 0 = \frac{2\sqrt{3}}{21}, \quad 0 - (-1) \cdot \frac{\sqrt{3}}{3} = \frac{7\sqrt{3}}{21}, \quad 0,$$

summing to $\frac{10\sqrt{3}}{21}$. So the upper half has area $\frac{5\sqrt{3}}{21}$ and the hexagon has area $\frac{10\sqrt{3}}{21}$. The answer is $10 + 3 + 21 = 34$.

Check. Each vertex really lies on the circle: $|\frac{1}{3} + \frac{1}{3}| = \frac{2}{3}$, $|-1 + \frac{1}{3}| = \frac{2}{3}$, $|\frac{1}{3} + \frac{\sqrt{3}}{3}i| = \sqrt{\frac{1}{9} + \frac{1}{3}} = \frac{2}{3}$, and $|\frac{2}{7} + \frac{1}{3} + \frac{\sqrt{3}}{7}i|^2 = \frac{169}{441} + \frac{27}{441} = \frac{196}{441} = \frac{4}{9}$. The area $\frac{10\sqrt{3}}{21} \approx 0.825$ is less than the disc's area $\frac{4\pi}{9} \approx 1.396$, as it must be. A computer solving the degree-6 equation and applying the shoelace formula agrees. The answer is 034.

13. **Answer:** 170. **The idea.** A monic integer quartic that factors over the integers can be written with monic factors (flip signs if both leading coefficients are -1). So either it has a linear factor, i.e. an integer root, or it splits as two monic quadratics. The first case gives a handful of lines of pairs (a, b) ; the second forces a rigid shape because $x^4 + ax + b$ has no x^3 and no x^2 term. Count each and remove the overlap.

Work it out. *Integer roots.* If s is a root, $b = -s^4 - as$. Since $|as| \leq 20|s|$ and $|b| \leq 20$, we need $s^4 \leq 20|s| + 20$, so $|s| \leq 2$. The lines are: $s = 0$: $b = 0$ (41 pairs); $s = 1$: $b = -a - 1$ ($a = -20..19$, 40 pairs); $s = -1$: $b = a - 1$ (40 pairs); $s = 2$: $b = -2a - 16$ ($a = -18..2$, 21 pairs); $s = -2$: $b = 2a - 16$ ($a = -2..18$, 21 pairs). The ten pairwise intersections are $(\pm 1, 0)$, $(\pm 8, 0)$, $(0, -1)$, $(0, -16)$, $(\pm 5, -6)$, $(\pm 15, 14)$, all distinct and in the box, so the lines cover $163 - 10 = 153$ pairs.

Two quadratics. Write $x^4 + ax + b = (x^2 + px + q)(x^2 - px + r)$ (the x^3 coefficients must cancel). The x^2 coefficient gives $q + r - p^2 = 0$, so $r = p^2 - q$, and then $a = p(r - q) = p(p^2 - 2q)$ and $b = qr = q(p^2 - q)$. For $|p| \geq 4$, $|a| \leq 20$ forces $|p^2 - 2q| \leq 5$, so q is near $p^2/2$ and $b = q(p^2 - q) \geq 60$; hence $|p| \leq 3$. Replacing p by $-p$ just swaps the factors, so take $p \geq 0$. $p = 0$: $a = 0$, $b = -q^2 \in \{0, -1, -4, -9, -16\}$; new (not on a line): $(0, -4)$, $(0, -9)$: 2. $p = 1$: $a = 1 - 2q$, $b = q(1 - q)$ for $q = -4..5$: the pairs $(\pm 9, -20)$, $(\pm 7, -12)$, $(\pm 5, -6)$, $(\pm 3, -2)$, $(\pm 1, 0)$; new: 6. $p = 2$: $a = 8 - 4q$, $b = q(4 - q)$ for $q = -2..6$: $(\pm 16, -12)$, $(\pm 12, -5)$, $(\pm 8, 0)$,

$(\pm 4, 3)$, $(0, 4)$; here $(\pm 4, 3)$ lie on $b = \mp a - 1$ and $(\pm 8, 0)$ on $b = 0$, so new: 5 (including $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$). $p = 3$: $a = 3(9 - 2q)$ with $q = 2..7$, $b = q(9 - q)$: $(\pm 15, 14)$, $(\pm 9, 18)$, $(\pm 3, 20)$; new: 4. Total: $153 + 2 + 6 + 5 + 4 = 170$.

Check. A spot check: $(a, b) = (-9, 18)$ gives $x^4 - 9x + 18 = (x^2 + 3x + 3)(x^2 - 3x + 6)$, which has no rational root, so it really needed the quadratic case. A computer factoring all 41^2 polynomials finds exactly 170 reducible ones. The answer is 170.

14. **Answer:** 835. **The idea.** A power of 2 counts yes/no choices, so read $2^{\text{something}}$ as “the number of ways to place optional dividers”. Reversing a permutation turns descents into ascents, so N is also the sum of $2^{(\text{number of ascents})}$. Now put a divider in each of the 7 gaps of a permutation where there is a descent (compulsory), and decide freely at each ascent: 2^{ascents} ways. The dividers cut the permutation into increasing blocks, i.e. into an ordered list of groups, which is a ranking of 8 runners in which ties are allowed. Those rankings are easy to count by a recursion.

Work it out. *Symmetry.* The reversal $(a_1, \dots, a_8) \mapsto (a_8, \dots, a_1)$ is a bijection of the permutations, and a descent at gap i becomes an ascent at gap $8 - i$. Every one of the 7 gaps is a descent or an ascent, so $N = \sum_{\sigma} 2^{d(\sigma)} = \sum_{\sigma} 2^{7-d(\sigma)}$, the sum of $2^{(\text{ascents})}$.

Dividers. Count pairs (σ, B) where B is a set of gaps of σ containing every descent; for each σ there are $2^{7-d(\sigma)}$ choices of B , so there are N pairs. Cutting σ at the gaps in B gives an ordered list of nonempty blocks, each increasing (a block contains no descent). Conversely, from an ordered list of nonempty disjoint blocks covering $\{1, \dots, 8\}$, writing each block in increasing order and putting the dividers between consecutive blocks rebuilds the only pair (σ, B) that produces it. So $N = a_8$, where a_n is the number of ways to split $\{1, \dots, n\}$ into an ordered list of nonempty groups (a ranking with ties).

Recursion. Choosing the first group (of size k) and then ranking the rest gives $a_n = \sum_{k=1}^n \binom{n}{k} a_{n-k}$ with $a_0 = 1$. So $a_1 = 1$, $a_2 = 2 + 1 = 3$, $a_3 = 3 \cdot 3 + 3 + 1 = 13$, $a_4 = 4 \cdot 13 + 6 \cdot 3 + 4 + 1 = 75$, $a_5 = 5 \cdot 75 + 10 \cdot 13 + 10 \cdot 3 + 5 + 1 = 541$, $a_6 = 6 \cdot 541 + 15 \cdot 75 + 20 \cdot 13 + 15 \cdot 3 + 6 + 1 = 4683$, $a_7 = 7 \cdot 4683 + 21 \cdot 541 + 35 \cdot 75 + 35 \cdot 13 + 21 \cdot 3 + 7 + 1 = 47293$, and $a_8 = 8 \cdot 47293 + 28 \cdot 4683 + 56 \cdot 541 + 70 \cdot 75 + 56 \cdot 13 + 28 \cdot 3 + 8 + 1 = 378344 + 131124 + 30296 + 5250 + 728 + 84 + 8 + 1 = 545835$. So $N = 545835$, and the remainder is 835.

Check. For $n = 3$: 123 has no descent, 132, 213, 231, 312 have one, and 321 has two, so the sum is $1 + 4 \cdot 2 + 4 = 13 = a_3$. A second route for $n = 8$: the numbers of permutations with $0, 1, \dots, 7$ descents (the Eulerian numbers, built by inserting 8 into permutations of $1..7$) are 1, 247, 4293, 15619, 15619, 4293, 247, 1, and $1 + 2 \cdot 247 + 4 \cdot 4293 + 8 \cdot 15619 + 16 \cdot 15619 + 32 \cdot 4293 + 64 \cdot 247 + 128 = 545835$. A computer summing $2^{d(\sigma)}$ over all 40320 permutations agrees. The answer is 835.

15. **Answer:** 845. **The idea.** Only residues mod 3 matter, so think of the entries as $0, 1, -1$ (for $3, 1, 2$). Detect “divisible by 3” with $\omega = e^{2\pi i/3}$: the average $\frac{1}{3}(1 + \omega^S + \omega^{2S})$ is 1 when $3 \mid S$ and 0 otherwise. The sum $\sum \omega^S$ over all tuples is a product of factors $\omega^{x_i x_{i+1}}$ around a ring, so it is the trace of the 10th power of the 3×3 matrix $M = (\omega^{ab})$, and this matrix, the discrete Fourier matrix, has a tiny power: $M^4 = 9I$.

Work it out. Let $S(x) = \sum_i x_i x_{i+1}$ (indices mod 10). Then $N = \frac{1}{3} \sum_x (1 + \omega^{S(x)} + \omega^{2S(x)}) = \frac{1}{3} (3^{10} + T + \bar{T})$, where $T = \sum_x \omega^{S(x)}$ (the third sum is the complex conjugate of the second, since $\omega^2 = \bar{\omega}$).

Index rows and columns by $a, b \in \{0, 1, -1\}$ and put $M_{ab} = \omega^{ab}$. Since $\omega^{S(x)} = M_{x_1 x_2} M_{x_2 x_3} \cdots M_{x_{10} x_1}$, summing over all tuples gives $T = \text{tr}(M^{10})$.

Now $(M^2)_{ab} = \sum_c \omega^{c(a+b)}$, which is 3 if $a + b \equiv 0 \pmod{3}$ and $1 + \omega + \omega^2 = 0$ otherwise. So $M^2 = 3P$, where P is the permutation matrix of $a \mapsto -a$ (it fixes 0 and swaps 1 and -1). As $P^2 = I$, $M^4 = 9I$ and $M^{10} = (M^4)^2 M^2 = 81 \cdot 3P = 243P$. The trace of P counts its fixed points: just $a = 0$. So $T = 243$, which is real, and $N = \frac{1}{3}(59049 + 243 + 243) = 19683 + 162 = 19845$. The remainder is 845.

Check. The same formula for a ring of $n = 4$ gives $M^4 = 9I$, $T = 27$ and $N = \frac{1}{3}(81 + 54) = 45$; for $n = 2$ it gives $T = \text{tr}(3P) = 3$ and $N = \frac{1}{3}(9 + 6) = 5$, and directly, $2x_1 x_2$ is divisible by 3 exactly when $x_1 = 3$ or $x_2 = 3$: $9 - 4 = 5$ pairs. A computer check of all $3^4 = 81$ and all $3^{10} = 59049$ tuples gives 45 and 19845. The answer is 845.